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T_0 CONVERGENCE APPROACH SPACES

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Abstract. In previous papers, several T_0-objects in set-based topological category have been introduced and compared. In this paper, we give the characterization of general $\overline{T_0}$ (resp. T_0, and T_0') convergence approach spaces as well as show how these notions are linked to each other.

1. Introduction

In 1989, Colebunders and Lowen [16] introduced convergence approach space to satisfy the categorical properties such as Cartesian closedness which are failed in approach space [17].

Classical T_0 separation of topology plays a vital role not only in mathematics such as to get an alternative characterization of locally semi-simple coverings in terms of light morphisms in algebraic topology [13] but also in computer science where this concept correspond to access the values through observations [26]. In addition to that, T_0 axiom has been used to build topological models in denotational semantics of programming language and lambda calculus where Hausdorff topologies fail to build such models [24, 25]. Furthermore, it has been used to characterize digital line in digital topology and to construct cellular complex in image processing and computer graphs [10, 14, 15].

Due to huge importance of T_0 separation, this concept has been extended to topological categories by several mathematicians such as Brümmer [8] in 1971, Marny [21] in 1973, Hoffmann [11] in 1974, Harvey [9] in 1977 and Baran [2] in 1991. Moreover, in 1991, Weck-Schwarz [27] and in 1995, Baran [3] analyzed the relationship among these various generalization of T_0 objects. One of the main reason to extend T_0 separation was to define T_2 objects in arbitrary topological categories [5].

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The main object of this paper is to characterize each of T_0, $\overline{T_0}$ and T_0' convergence approach spaces and show how these are related to each other.

2. Preliminaries

Let $\mathcal{E}$ and $\mathcal{B}$ be two categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is called topological functor if (i) $\mathcal{U}$ is concrete (i.e., faithful and amnestic) (ii) $\mathcal{U}$ consists of small fibers and (iii) every $\mathcal{U}$-source has a unique initial lift [1, 22, 23].

Note that topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is called normalized if subterminals have a unique structure.

Let X be a set, $A \subseteq X$, $F(X)$ be the set of all filters and $\mathcal{A}$ be collection of subsets of X. The stack of $\mathcal{A}$ and the indicator map $\theta_A : X \to [0,\infty]$ are defined by $[\mathcal{A}] = \{B \subseteq X | \exists A \in \mathcal{A} : A \subseteq B\}$ and

$$\theta_A(x) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

respectively.

Definition 1. *(cf.* [16, 18, 20]*) A map $\lambda : F(X) \longrightarrow [0,\infty]^X$ is called a convergence approach structure on X if it satisfies the followings:*

(i) $\forall x \in X : \lambda[x](x) = 0,$
(ii) $\forall \alpha, \beta \in F(X) : \alpha \subset \beta \Rightarrow \lambda\beta \leq \lambda\alpha,$
(iii) $\forall \alpha, \beta \in F(X) : \lambda(\alpha \cap \beta) = \sup\{\lambda(\alpha), \lambda(\beta)\}.$

The pair (X, λ) is called a convergence approach space.

Definition 2. *(cf.* [16, 18, 20]*) Let (X, λ) and (X', λ') be convergence approach spaces. The map $f : (X, \lambda) \longrightarrow (X', \lambda')$ is called a contraction map if it satisfies for all $\alpha \in F(X) : \lambda'(f(\alpha)) \circ f \leq \lambda\alpha$.*

The category whose objects are convergence approach spaces and morphisms are contraction maps is denoted by **CApp** and it is a Cartesian closed topological category over **Set** [16, 18, 20].

Definition 3. *(cf.* [16, 18, 20]*) Let X be a non-empty set and (X_i, λ_i) be the class of convergence approach spaces.*

(i) *A source $\{f_i : X \to (X_i, \lambda_i)\}$ in* **CApp** *has initial lift if and only if for all $\alpha \in F(X)$, $\lambda\alpha = \sup_{i \in I} \lambda_i(f_i(\alpha)) \circ f_i$, where $f_i(\alpha)$ is a filter generated by $\{f_i(A_i), i \in I\}$, i.e., $f_i(\alpha) = \{A_i \subset X_i : \exists B \in \alpha$ such that $f_i(B) \subset A_i\}$.*
(ii) *A sink $\{f_i : (X_i, \lambda_i) \to X\}$ in* **CApp** *has final lift if and only if for all $\alpha \in F(X)$ and $x \in X$,*

$$\lambda(\alpha)(x) = \begin{cases} 0, & \alpha = [x] \\ \inf\limits_{i \in I} \inf\limits_{y \in f_i^{-1}(x)} \inf\limits_{\substack{\beta \in F(X_i) \\ \subset \alpha}} \lambda_i(\beta)(y), & \alpha \neq [x] \end{cases}$$

(iii) *The discrete structure* (X, λ_{dis}) *on* X *in* **CApp** *is defined by for all* $\alpha \in F(X)$ *and* $x \in X$,

$$\lambda_{dis}(\alpha) = \begin{cases} \theta_{\{x\}}, & \alpha = [x] \\ \infty, & \alpha \neq [x] \end{cases}$$

(iv) *The indiscrete structure* (X, λ_{ind}) *on* X *in* **CApp** *is defined by for all* $\alpha \in F(X)$ *and* $x \in X$,

$$\lambda_{ind}(\alpha)(x) = 0$$

3. T_0 Convergence Approach Spaces

Let B be a nonempty set, $B^2 \coprod B^2$ be the coproduct of B^2 and $B^2 \vee_{\triangle} B^2$ be two distinct copies of B^2 identified along the diagonal [2]. Let $q : B^2 \coprod B^2 \to B^2 \vee_{\triangle} B^2$ be the quotient map. A point (x, y) in $B^2 \vee_{\triangle} B^2$ is denoted by $(x, y)_1$ (resp. $(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_{\triangle} B^2$. Note that $(x, x)_1 = (x, x)_2 = (x, x)$.

Definition 4. *(cf.* [2]*) A map* $A : B^2 \vee_{\triangle} B^2 \to B^3$ *is called a principle axis map if*

$$A((x, y)_i) = \begin{cases} (x, y, x), & i = 1 \\ (x, x, y), & i = 2 \end{cases}$$

Definition 5. *(cf.* [2]*) A map* $\nabla : B^2 \vee_{\triangle} B^2 \to B^2$ *is called a folding map if* $\nabla((x, y)_i) = (x, y)$ *for* $i = 1, 2$.

Definition 6. *(cf.* [2, 21]*) Let* $\mathcal{U} : \mathcal{E} \to$ **Set** *be topological in the sense of* [1, 22] *and* X *be an object in* $\mathcal{E}$ *with* $\mathcal{U}(X) = B$.

(i) X *is* $\overline{T_0}$ *iff initial lift of the* $\mathcal{U}$*-source* $\{A : B^2 \vee_{\triangle} B^2 \to \mathcal{U}(X^3) = B^3$ *and* $\nabla : B^2 \vee_{\triangle} B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ *is discrete, where* $\mathcal{D}$ *is a discrete functor which is left adjoint to* $\mathcal{U}$.

(ii) X *is* T_0' *iff initial lift of the* $\mathcal{U}$*-source* $\{id : B^2 \vee_{\triangle} B^2 \to \mathcal{U}(B^2 \vee_{\triangle} B^2)'^2 \vee_{\triangle} B^2$ *and* $\nabla : B^2 \vee_{\triangle} B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ *is discrete, where* $(B^2 \vee_{\triangle} B^2)'$ *is the final lift of* $\mathcal{U}$*-sink* $\{q \circ i_1, q \circ i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_{\triangle} B^2\}$ *and* $i_k : B^2 \to B^2 \coprod B^2$ *are the canonical injections for* $k = 1, 2$.

(iii) X *is* T_0 *iff* X *doesn't contain an indiscrete subspace with (at least) two points.*

Theorem 7. *A convergence approach space* (X, λ) *is* $\overline{T_0}$ *iff for all* $x, y \in X$ *with* $x \neq y$, $\lambda([x])(y) = \infty$ *or* $\lambda([y])(x) = \infty$.

Proof. Let (X, λ) be $\overline{T_0}$ for all $x, y \in X$ with $x \neq y$. Note that $[(x, y)_1] \in F(X^2 \vee_{\triangle} X^2)$, $(x, y)_2 \in X^2 \vee_{\triangle} X^2$ and

$$\lambda_{dis}([\nabla(x, y)_1])(\nabla(x, y)_2) = \lambda_{dis}([(x, y)])(x, y) = 0,$$

$$\lambda([\pi_1 A(x, y)_1](\pi_1 A(x, y)_2) = \lambda([x])(x) = 0,$$

$$\lambda([\pi_2 A(x,y)_1](\pi_2 A(x,y)_2) = \lambda([y])(x)$$

and

$$\lambda([\pi_3 A(x,y)_1](\pi_3 A(x,y)_2) = \lambda([x])(y),$$

where $\pi_i : X^3 \to X$ are the projection maps, $i = 1, 2, 3$. Since (X, λ) is $\overline{T_0}$, by Definition 3 (i),

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}([\nabla(x,y)_1])(\nabla(x,y)_2), \lambda([\pi_1 A(x,y)_1])(\pi_1 A(x,y)_2), \\ &\quad \lambda([\pi_2 A(x,y)_1])(\pi_2 A(x,y)_2), \lambda([\pi_3 A(x,y)_1])(\pi_3 A(x,y)_2)\} \\ &= \sup\{0, \lambda([x])(y), \lambda([y])(x)\} = \sup\{\lambda([x])(y), \lambda([y])(x)\} \end{aligned}$$

and consequently, $\lambda([x])(y) = \infty$ or $\lambda([y])(x) = \infty$.

Conversely, let $\overline{\lambda}$ be an initial convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $A : X^2 \vee_\triangle X^2 \to (X^3, \lambda^3)$ and $\nabla : X^2 \vee_\triangle X^2 \to (X^2, \lambda_{dis})$, where λ_{dis} is discrete convergence approach structure on X^2 and λ^3 is the product convergence approach structure on X^3 induced by $\pi_i : X^3 \to X$ the projection maps for $i = 1, 2, 3$. Suppose $\alpha \in F(X^2 \vee_\triangle X^2)$ and $v \in X^2 \vee_\triangle X^2$ with $\nabla v = (x, y)$. By Definition 1, we show that

$$\overline{\lambda}(\alpha) = \begin{cases} \theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v] \end{cases}$$

where $\theta_{\{v\}}$ is the indicator of $\{v\}$. Let w be any point in $X^2 \vee_\triangle X^2$. Note that

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \begin{cases} \theta_{\{(x,y)\}}\nabla w, & \nabla\alpha = [(x,y)] \\ \infty, & \nabla\alpha \neq [(x,y)] \end{cases}$$

$$= \begin{cases} 0, & \nabla\alpha = [(x,y)] \text{ and } \nabla w = (x,y) \\ \infty, & \nabla\alpha = [(x,y)] \text{ and } \nabla w \neq (x,y) \\ \infty, & \nabla\alpha \neq [(x,y)] \text{ and } \nabla w \neq (x,y) \end{cases}$$

Case I: If $x = y$, then $\nabla w = (x, x)$ implies $w = (x, x)_1 = (x, x)_2 = v$ and $\nabla\alpha = [(x, x)]$ implies $\alpha = [(x, x)_i] = [(x, x)]$ for $i = 1, 2$. By Definition 3 (i), $\overline{\lambda}(\nabla\alpha)(\nabla w) = \overline{\lambda}([(x, x)])(x, x) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

Suppose that $x \neq y$. $\nabla w = (x, y)$ implies $w = (x, y)_1$ or $u = (x, y)_2$ and $\nabla\alpha = [(x, y)]$ implies $\alpha = [(x, y)_1]$, $[(x, y)_2]$, $[\{(x, y)_1, (x, y)_2\}]$ or $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$. Firstly, we show that the case $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$ with $\alpha \neq [\emptyset]$ and $\alpha \neq [\{(x, y)_1, (x, y)_2\}]$ cannot occur. To end this, if $[\emptyset] \neq \alpha \neq [\{(x, y)_1, (x, y)_2\}]$, then $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$ iff $\alpha = [(x, y)_1]$ or $\alpha = [(x, y)_2]$. Clearly, if $\alpha = [(x, y)_1]$ or $[(x, y)_2]$, then $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$. Conversely, if $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$ with $[\emptyset] \neq \alpha \neq [\{(x, y)_1, (x, y)_2\}]$, then there exists $V \in \alpha$ such that $V \neq \{(x, y)_1, (x, y)_2\}$ and $V \neq \emptyset$. Since V and $\{(x, y)_1, (x, y)_2\}$ are in α and α is a filter, it follows that

$V \cap \{(x,y)_1, (x,y)_2\} = \{(x,y)_1\}$ or $\{(x,y)_2\}$ is in α, i.e., $\alpha = [(x,y)_1]$ or $[(x,y)_2]$. Hence, we must have $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1, (x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ and $w = (x,y)_i$, $i = 1,2$, then $\overline{\lambda}([(x,y)_i])((x,y)_i) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

If $\alpha = [(x,y)_2]$ and $w = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[(x,y)_2])(\nabla(x,y)_1) = \lambda_{dis}([(x,y)])(x,y) = 0,$$

$$\lambda(\pi_1 A\alpha)(\pi_1 Aw) = \lambda([\pi_1 A(x,y)_2])(\pi_1 A(x,y)_1)) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 A\alpha)(\pi_2 Aw) = \lambda([\pi_2 A(x,y)_2])(\pi_2 A(x,y)_1) = \lambda([x])(y)$$

and

$$\lambda(\pi_3 A\alpha)(\pi_3 Aw) = \lambda([\pi_3 A(x,y)_2])(\pi_3 A(x,y)_1) = \lambda([y])(x),$$

by Definition 3 (i),

$$\begin{aligned}
\overline{\lambda}(\alpha)(w) &= \overline{\lambda}([(x,y)_2])((x,y)_1) \\
&= \sup\{\lambda_{dis}([\nabla(x,y)_2])(\nabla(x,y)_1), \lambda([\pi_1 A(x,y)_2])(\pi_1 A(x,y)_1), \\
&\quad \lambda([\pi_2 A(x,y)_2])(\pi_2 A(x,y)_1), \lambda([\pi_3 A(x,y)_2])(\pi_3 A(x,y)_1)\} \\
&= \sup\{0, \lambda([y])(x), \lambda([x])(y)\} = \sup\{\lambda([y])(x), \lambda([x])(y)\} = \infty
\end{aligned}$$

since by the assumption $\lambda([y])(x) = \infty$ or $\lambda([x])(y) = \infty$.

If $\alpha = [\{(x,y)_1, (x,y)_2\}]$ and $w = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[\{(x,y)_1, (x,y)_2\}])(\nabla(x,y)_1) = \lambda_{dis}([x])(x) = 0,$$

$$\lambda(\pi_1 A\alpha)(\pi_1 Aw) = \lambda([\{\pi_1 A(x,y)_1, \pi_1 A(x,y)_2\}])(\pi_1 A(x,y)_1) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 A\alpha)(\pi_2 Aw) = \lambda([\{\pi_2 A(x,y)_1, \pi_2 A(x,y)_2\}])(\pi_2 A(x,y)_1) = \lambda([\{x,y\}])(y)$$

and

$$\lambda(\pi_3 A\alpha)(\pi_3 Aw) = \lambda([\{\pi_3 A(x,y)_1, \pi_3 A(x,y)_2\}])(\pi_3 A(x,y)_1) = \lambda([\{x,y\}])(x).$$

Note that $[\{x,y\}] \subset [y]$ and $[\{x,y\}] \subset [x]$. Since λ is a convergence approach structure, we get $\lambda([y])(x) \leq \lambda([\{x,y\}])(x)$ and $\lambda([x])(y) \leq \lambda([\{x,y\}])(y)$. The assumption $\lambda([y])(x) = \infty$ (resp. $\lambda([x])(y) = \infty$) implies $\lambda([\{x,y\}])(x) = \infty$ (resp. $\lambda([\{x,y\}])(y) = \infty$).

By Definition 3 (i),

$$\begin{aligned}
\overline{\lambda}(\alpha)(w) &= \overline{\lambda}([\{(x,y)_1, (x,y)_2\}])((x,y)_1) \\
&= \sup\{\lambda_{dis}([\{\nabla(x,y)_1, \nabla(x,y)_2\}])(\nabla(x,y)_1), \lambda([\{\pi_1 A(x,y)_1, \pi_1 A(x,y)_2\}]) \\
&\quad (\pi_1 A(x,y)_1), \lambda([\{\pi_2 A(x,y)_1, \pi_2 A(x,y)_2\}])(\pi_2 A(x,y)_1), \lambda([\{\pi_3 A(x,y)_1, \\
&\quad \pi_3 A(x,y)_2\}])(\pi_3 A(x,y)_1)\} = \sup\{0, \infty\} = \infty.
\end{aligned}$$

For the cases $\alpha = [(x,y)_1]$ or $[\{(x,y)_1, (x,y)_2\}]$ and $w = (x,y)_2$, it can be done analogously to the above argument.

Case II: Let $(z,z) = \nabla w \neq (x,y)$ for some $z \in X$ and $\nabla\alpha = [(x,y)]$. It follows that $w = (z,z)_1 = (z,z)_2$ and $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1, (x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ or $[\{(x,y)_1,(x,y)_2\}]$ for $i = 1,2$ and $w = (z,z)_1 = (z,z)_2$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}([(x,y)])(z,z) = \infty$ since λ_{dis} is a discrete convergence approach structure and $(x,y) \neq (z,z)$. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} \\ &= \sup\{\infty, \lambda(\pi_1 A\alpha)(z,z), \lambda(\pi_2 A\alpha)(z,z), \lambda(\pi_3 A\alpha)(z,z)\} = \infty.\end{aligned}$$

Case III: Suppose $\nabla w \neq (x,y)$ and $\nabla\alpha \neq [(x,y)]$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \infty$ since λ_{dis} is a discrete convergence approach structure, and consequently

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} \\ &= \sup\{\infty, \lambda(\pi_1 A\alpha)(\pi_1 Aw), \lambda(\pi_2 A\alpha)(\pi_2 Aw), \lambda(\pi_3 A\alpha)(\pi_3 Aw)\} = \infty.\end{aligned}$$

Therefore, for all $\alpha \in F(X^2 \vee_\triangle X^2)$ and $\forall v \in X^2 \vee_\triangle X^2$, we get

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v]\end{cases}$$

i.e., by Definition 3 (iii), $\overline{\lambda}$ is discrete convergence approach structure on $X^2 \vee_\triangle X^2$ and by Definition 6 (i), (X,λ) is $\overline{T_0}$. □

Let X be a non-empty set and $\alpha, \beta \in F(X)$. We denote by $\alpha \cup \beta$ the smallest filter containing both α and β, i.e., $\alpha \cup \beta$ is the filter generated by the set $\{V \cap W : V \in \alpha, W \in \beta\}$.

Lemma 8. *Let $(X_j, \lambda_j)_{j \in I}$ be a class of* **CApp** *objects and $X = \coprod_{j\in I} X_j$, the coproduct of $\{X_j\}_{j\in I}$. The coproduct convergence approach structure λ on X with respect to the family of canonical injections $i_j : (X_j, \lambda_j) \to X = \coprod_{j\in I} X_j$ is defined by*

$$\lambda(\alpha)(x_k) = \begin{cases}0, & \text{if } \alpha = [x_k] \\ \lambda_k(\alpha \cup [X_k])(x_k), & \text{if } i_k(\beta) \subset \alpha \text{ for some } k \in I \text{ and } \beta_k \in F(X_k) \\ \infty, & \text{if } i_k(\beta) \not\subset \alpha \text{ for all } k \in I \text{ and } \beta_k \in F(X_k)\end{cases}$$

Proof. Let $\alpha \in F(X)$ with $\alpha \neq [x]$ for all $x \in X = \coprod_{j\in I} X_j$. By definition 3 (iii), $\lambda(\alpha)(x_k) = \inf\{\lambda_k(\beta_k)(x_k) : \beta_k \in F(X_k)$ for some $k \in I$ such that $i_k(\beta_k) \subset \alpha\}$. If $i_k(\beta_k) \subset \alpha$ for some $k \in I$ and $\beta_k \in F(X_k)$, then such k can be at most one and for this k, $\alpha \cup [X_k]$ is the greatest element $\beta_k \in F(X_k)$ such that $i_k(\beta_k) \subset \alpha$, i.e., $i_k(\alpha \cup [X_k]) = \alpha$. Hence, $\lambda(\alpha)(x_k) = \lambda_k(\alpha \cup [X_k])(x_k)$. □

Theorem 9. *Every convergence approach space is T_0'.*

Proof. Let (X,λ) be a convergence approach space. We show that (X,λ) is T_0'. Let $\overline{\lambda}$ be an initial convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $\nabla : X^2 \vee_\triangle X^2 \to (X^2, \lambda_{dis})$ and $id : X^2 \vee_\triangle X^2 \to (X^2 \vee_\triangle X^2, \lambda^*)$, where λ_{dis} is discrete convergence approach structure on X^2 and λ^* is the final convergence approach structure on $X^2 \vee_\triangle X^2$ induced by $q \circ i_k : X^2 \to X^2 \vee_\triangle X^2$ for $k = 1, 2$ and let $v \in X^2 \vee_\triangle X^2$ with $\nabla v = (x, y)$. Suppose $\alpha \in F(X^2 \vee_\triangle X^2)$ and $w \in X^2 \vee_\triangle X^2$. Note that

$$\begin{aligned}\lambda_{dis}(\nabla\alpha)(\nabla w) &= \begin{cases}\theta_{\{(x,y)\}}\nabla w, & \nabla\alpha = [(x,y)]\\ \infty, & \nabla\alpha \neq [(x,y)]\end{cases}\\ &= \begin{cases}0, & \nabla\alpha = [(x,y)] \text{ and } \nabla w = (x,y)\\ \infty, & \nabla\alpha = [(x,y)] \text{ and } \nabla w \neq (x,y)\\ \infty, & \nabla\alpha \neq [(x,y)] \text{ and } \nabla w \neq (x,y)\end{cases}\end{aligned}$$

Case I: If $x = y$, then $\nabla w = (x, x)$ implies $w = (x, x)_1 = (x, x)_2 = (x, x) = v$ and $\nabla\alpha = [(x, x)]$ implies $\alpha = [(x, x)_i] = [(x, x)]$ for $i = 1, 2$. By Definition 3 (i), $\overline{\lambda}(\alpha)(w) = \overline{\lambda}([(x, x)_i])(x, x)_i = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

Let $x \neq y$. $\nabla\alpha = [(x, y)]$ implies $\alpha = [(x, y)_1]$, $[(x, y)_2]$, $[\{(x, y)_1, (x, y)_2\}]$ or $\alpha \supset [\{(x, y)_1, (x, y)_2\}]$ and $\nabla w = (x, y)$ implies $w = (x, y)_1$ or $w = (x, y)_2$. By using the similar argument given in the proof of Theorem 7, we must have $\alpha = [(x, y)_1]$, $[(x, y)_2]$ or $[\{(x, y)_1, (x, y)_2\}]$.

If $\alpha = [(x, y)_j]$ and $w = (x, y)_j$ for $j = 1, 2$, then $\overline{\lambda}([(x, y)_j])((x, y)_j) = 0$ since $\overline{\lambda}$ is a convergence approach structure on $X^2 \vee_\triangle X^2$.

If $\alpha = [(x, y)_1]$ and $w = (x, y)_2$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}(\nabla[(x, y)_1])(\nabla(x, y)_2) = \lambda_{dis}([(x, y)])(x, y) = 0,$$
$$\lambda^*(id\alpha)(idw) = \lambda^*(\alpha)(w) = \lambda^*([(x, y)_1])((x, y)_2).$$

Since $i_2\beta \not\subset \alpha = [(x, y)_1]$ for all $\beta \in F(X^2)$, by Lemma 8,

$$\lambda^*(\alpha)(w) = \lambda^*([(x, y)_1])((x, y)_2) = \infty.$$

Hence, by Definition 3 (i),

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \overline{\lambda}([(x, y)_1])((x, y)_2)\\ &= \sup\{\lambda_{dis}([\nabla(x, y)_1])(\nabla(x, y)_2), \lambda^*(id[(x, y)_1])(id(x, y)_2)\}\\ &= \sup\{0, \infty\} = \infty.\end{aligned}$$

Suppose $\alpha = [\{(x, y)_1, (x, y)_2\}]$ and $w = (x, y)_2$.

In particular,

$$\lambda^*(id\alpha)(idw) = \lambda^*(\alpha)(w) = \lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2).$$

Since λ^* is a final convergence approach structure on $X^2 \vee_\triangle X^2$ and $[\{(x, y)_1, (x, y)_2\}] \subset [(x, y)_1]$, we get $\lambda^*([(x, y)_1])((x, y)_2) \leq \lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2)$. By the same statement used above, $\lambda^*([(x, y)_1])((x, y)_2) = \infty$, and consequently,

$$\lambda^*([\{(x, y)_1, (x, y)_2\}])((x, y)_2) = \infty.$$

By Definition 3 (i),

$\overline{\lambda}(\alpha)(w) = \overline{\lambda}([\{(x, y)_1, (x, y)_2\}])((x, y)_1)$

$= \sup\{\lambda_{dis}(\nabla[\{(x,y)_1,(x,y)_2\}])(\nabla(x,y)_2), \lambda^*(id[\{(x,y)_1,(x,y)_2\}]),(id(x,y)_2)\}$
$= \sup\{0,\infty\} = \infty.$

For the cases $\alpha = [(x,y)_2]$ (resp. $[\{(x,y)_1,(x,y)_2\}]$) and $w = (x,y)_1$, by Lemma 8 and the argument used above, we get $\overline{\lambda}(\alpha)(w) = \infty$.

Case II: Let $(z,z) = \nabla w \neq (x,y)$ for some $z \in X$ and $\nabla\alpha = [(x,y)]$. It follows that $w = (z,z)_1 = (z,z)_2$ and $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ (resp. $[\{(x,y)_i,(x,y)_j\}]$) for $i,j = 1,2$ with $i \neq j$ and $w = (z,z)_1 = (z,z)_2 = (z,z)$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \lambda_{dis}([(x,y)])(z,z) = \infty$ since λ_{dis} is a discrete convergence approach structure and $(x,y) \neq (z,z) = \nabla w$. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda^*(id\alpha)(idw)\}\\ &= \sup\{\infty, \lambda^*(\alpha)(w)\} = \infty.\end{aligned}$$

Case III: Suppose $\nabla w \neq (x,y)$ and $\nabla\alpha \neq [(x,y)]$, then $\lambda_{dis}(\nabla\alpha)(\nabla w) = \infty$ since λ_{dis} is a discrete convergence approach structure, and consequently

$$\begin{aligned}\overline{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla w), \lambda^*(id\alpha)(idw)\}\\ &= \sup\{\infty, \lambda^*(\alpha)(w)\} = \infty.\end{aligned}$$

Therefore, for all $\alpha \in F(X^2 \vee_\triangle X^2)$,

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v]\\ \infty, & \alpha \neq [v]\end{cases}$$

, i.e., by Definition 3 (iii), $\overline{\lambda}(\alpha)$ is discrete convergence approach structure over $X^2 \vee_\triangle X^2$. By Definition 6 (ii), (X,λ) is T_0'. □

Theorem 10. *A convergence approach space (X,λ) is T_0 iff for all $x,y \in X$ with $x \neq y$, $\lambda([y])(x) > 0$ or $\lambda([x])(y) > 0$.*

Proof. The proof is the same as the proof of [12, 19]. □

Example 11. *Let X be a set with $|X| \geq 2$. By Theorems 7, 9 and 10, every indiscrete convergence approach space, i.e., for all $\alpha \in F(X)$ and for all $x \in X$, $\lambda(\alpha)(x) = 0$ is T_0' but neither $\overline{T_0}$ nor T_0.*

Example 12. *Let X be a non-empty set, $F(X)$ be the set of all filters and $\lambda : F(X) \to [0,\infty]^X$ be a map defined as follows: For all $\alpha \in F(X)$ and $u \in X$,*

$$\lambda(\alpha)(u) = \begin{cases}0, & \alpha = [u]\\ 1, & \alpha \neq [u]\end{cases}$$

Clearly, (X,λ) is a convergence approach space. By Theorems 7, 9 and 10, (X,λ) is T_0 (resp. T_0') but not $\overline{T_0}$.

Remark 13. (I) *In* **Top** *(category of topological spaces and continuous maps), $\overline{T_0}$, T_0' and T_0 are equivalent and reduce to classical T_0 axiom (i.e., for each distinct points x and y, there exists a neighborhood of x doesn't contain y or vice versa)* [4].

(II) *For any arbitrary topological category,*
  (i) *$\overline{T_0}$ implies T_0' but converse is not true in general* [3].
  (ii) *There is no relation between T_0 and each of $\overline{T_0}$ and T_0'* [3].
    (a) *$\overline{T_0}$ could be only discrete objects such as in ∞***pqsMet** *(extended pseudo-quasi-semi metric spaces and non-expansive maps)* [7].
    (b) *$\overline{T_0}$ could be all objects, e.g., in* **Born** *(bornological spaces and bounded maps)* [3].
    (c) *In category* **Born**, $T_0 \implies \overline{T_0} = T_0'$ [3].
    (d) *In category* **Lim** *of limit spaces and filter convergence maps, $\overline{T_0} = T_0 \implies T_0'$* [3].
    (e) *In category* **SUConv** *of semi-uniform convergence spaces and uniformly continuous maps, $\overline{T_0} \implies T_0 \implies T_0'$* [6].

(III) *In convergence approach space (X, λ), by Theorems 7, 9 and 10, $\overline{T_0} \implies T_0 \implies T_0'$ but converse of each implication is not true in general by Examples 11 and 12.*

References

[1] Adamek, j., Herrlich, H. and Strecker. G. E., Abstract and Concrete Categories, Pure and Applied Mathematics, John Wiley & Sons, New York, 1990.
[2] Baran, M., Separation properties, *Indian J. pure appl. Math*, 23 (1991), 333–341.
[3] Baran, M. and Altindiş, H., T_0 objects in topological categories, *J. Univ. Kuwait (Sci.)*, 22 (1995), 123–127.
[4] Baran, M., Separation Properties in Topological Categories, *Math. Balkanica.*, 10 (1996), 39–48.
[5] Baran, M. and Altindiş, H., T_2 objects in Topological Categories, *Acta Mathematica Hungarica*, 71(1-2) (1996), 41–48.
[6] Baran, M., Kula, S. and Erciyes, A., T_0 and T_1 semiuniform convergence spaces, *Filomat*, 27.4 (2013), 537–546.
[7] Baran, T. M., T_0 and T_1 Pseudo-Quasi-Semi Metric Spaces, *Ph.D.* Thesis, Erciyes University, 2018.
[8] Brummer, G. C., A categorical study of initiality in Uniform topology, Ph.D. Thesis, University of Cape Town, 1971.
[9] Harvey, J., T_0-separation in topological categories, *Quaestiones Mathematicae*, 2.(1-3) (1977), 1971–1990.
[10] Hermann, G. T., On topology as applied to image analysis, *Computer vision, graphics and Image processing*, 52.3 (1990), 409–415.
[11] Hoffmann, R. E., (E, M)-Universally topological functors, *Habilitationsschnift, University of Dusseldorf*, 1974.
[12] Jager, G., A note on neighbourhoods for approach spaces, *Hacettepe Journal of Mathematics and Statistics*, 41.2 (2012), 283–290.
[13] Janelidze, G., Light morphisms for generalized T_0-reflections, *Topology and its Applications*, 156.12 (2009), 2109–2115.

[14] Kovalevsky, V. A., Finite topology as applied to image analysis, *Computer vision, graphics and Image processing*, 46.2 (1989), 141–161.
[15] Kovalevsky, V. A. and Kopperman, R., Some topology-based image processing algorithms, *Annals of the New York Academy of Sciences*, 728.1 (1994), 174–182.
[16] Lowen, E. and Lowen, R., Topological quasitopos hulls of categories containing topological and metric objects, *Cahiers de topologie et géométrie différentielle catégoriques*, 30.3 (1989), 213–228
[17] Lowen, R., Approach spaces A common Supercategory of TOP and Met, *Mathematische Nachrichten*, 141.1 (1989), 183–226.
[18] Lowen, R., Approach spaces: The missing link in the Topology-Uniformity-Metric triad, Oxford University Press, 1997.
[19] Lowen, R. and Sioen, M., A note on separation in AP, *Applied general topology*, 4.2 (2003), 475–486.
[20] Lowen, R., Index Analysis: Approach theory at work, Springer, 2015.
[21] Marny, T., Rechts-Bikategoriestrukturen in topologischen Kategorien, Ph. D. Thesis, Free University of Berlin, 1973.
[22] Preuss, G., Theory of topological structures: an approach to categorical topology, D. Reidel Publ. Co., Dordrecht, 1988.
[23] Preuss, G., Foundations of topology: an approach to convenient topology, Kluwer Academic Publishers, Dordrecht, 2002.
[24] Salibra, A., A Continuum of theories of lambda calculus without semantics, Proceedings of 16th Annual IEEE Symposium on logic in computer science, 2001.
[25] Stoy, J. E., Denotational Semantics: The Scott-Strachey approach to programing language, MIT press, 1977.
[26] Taylor, P., Sober spaces and continuations, *Theory and Applications of Categories*, 10.12 (2002), 248–300.
[27] Weck-Schwarz, S., T_0-objects and separated objects in topological categories, *Quaestiones Mathematicae*, 14.3 (1991), 315–325.

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