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Semi-analytical investigation of modified Boussinesq-Burger equations

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Abstract

In this paper, a new analysis of nonlinear modified Boussinesq-Burger equation is revisited via optimal perturbation iteration technique. We first consider artificial parameters and perturbation theory and combine them to deal with nonlinear partial differential equations. After that, the recommended theory is employed to get new semi-analytical solutions of nonlinear partial differential equations. As will be seen from the results, this technique needs no discretization or linearization and can be directly applied to many nonlinear differential equations.

Keywords: *Optimal perturbation iteration techniques, nonlinear partial differential equations.*

Değiştirilmiş Boussinesq-Burger denklemlerinin yarı analitik incelemesi

Öz

Bu çalışmada değiştirilmiş Boussinesq-Burger denklemlerinin optimal perturbasyon iterasyon yöntemi ile yarı analitik incelemesi yapılmıştır. Öncelikle önerilen metodun inşası için yapay parametreler ve perturbasyon teorisi ele alınmış ve bunlar birleştirilerek lineer olmayan kısmi diferansiyel denklemler için bir çözüm metodu geliştirilmiştir. Daha sonra ise elde edilen algoritmalar ile ele alınan problem yarı analitik olarak çözülmüştür. Sonuçlardan da anlaşılabileceği üzere bu teknik birçok lineer olmayan diferansiyel denkleme herhangi bir lineerizasyon gerektirmeden rahatlıkla uygulanabilmektedir.

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Anahtar kelimeler *Optimal perturbasyon iterasyon yöntemi, lineer olmayan kısmi diferansiyel denklemler.*

1. Introduction

The solitons and their equations are very prominent subjects in the field of engineering and mathematics. Over the last decades, many iterations have been made to analyze various nonlinear soliton equations. The exact analytical solutions of nonlinear wave equations help to understand the behavior and characteristics of nonlinear soliton equations. For this reason, seeking exact solutions of nonlinear wave equations is an important and interesting subject. Up to now, there have been many methods to solve these types of equations, such as Bäcklund transformation method [1], Adomian decomposition method [2], Darboux transformation method [3], Chebyshev spectral collocation method [4], tanh–coth method [5], Taylor collocation method [6], homogeneous balance method [7], variational iteration method [8]. Differential transform [9] and homotopy perturbation methods [10] are also very useful methods for solving ordinary and partial differential equations.

Over the last 5 years, there have been many attempts via many researchers for solving ODEs and PDEs. Among them, semi analytical techniques are much more considered as a direct solution to many types of ODEs and PDEs. One of them is optimal perturbation iteration method (OPIM) and it is used for handling many different and well-known nonlinear equations such as Bratu [11], delay [12], heat transfer [13] and Lane-Emden type equations [14]. This technique is also proved to be very efficient for nonlinear PDEs such as Burgers' [15], generalized regularized long wave equations [16]. Fractional differential equations can also be analyzed by using OPIM as in [17]. In those paper, one can see that, OPIM uses only algorithms for solving nonlinear equations. Fundamental idea is to decompose the equation in its nonlinear and linear part, then struggling with the nonlinear part by manipulating the artificial parameter and perturbation theory. After solving algorithm for the equations, we solve the algebraical equations arised from OPIAs. After obtaining the unknown parameters, we just substitute them to get the semi-analytical solutions. More information about techniques related on soliton equations can also be seen in [18-20]

The classical Boussinesq-Burger equations can be given as

$$\begin{aligned} u_t - \frac{1}{2}v_x + 2uu_x &= 0, \\ v_t - \frac{1}{2}u_{xxx} + 2(uv)_x &= 0, \quad 0 \le x \le 1, \end{aligned} \tag{1}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \frac{ck}{2} + \frac{ck}{2} tanh\left(\frac{-kx-\ln b}{2}\right), \\ v(x,0) &= \frac{-k^2}{8}\operatorname{sech}^2\left(\frac{kx+\ln b}{2}\right). \end{aligned} \tag{2}$$

The exact solution of these equations can be given as [21]

$$u(x,t) = \frac{ck}{2} + \frac{ck}{2} tanh\left(\frac{ck^2t-kx-\ln b}{2}\right), \tag{3}$$

$$v(x,t) = \frac{-k^2}{8}\operatorname{sech}^2\left(\frac{kx-ck^2t+\ln b}{2}\right). \tag{4}$$

The Boussinesq - Burger equations arise in the study of fluid flow and describe the propagation of shallow water waves. Here x and t respectively represent the normalized space and time, $u(x,t)$ is the horizontal velocity field and $v(x,t)$ denotes the height of the water surface above a horizontal level at the bottom [17]. In this paper, we will deal with the modified form of the above equations such as

$$\begin{aligned} u_t - \frac{1}{2}v_x + 2uu_x + u &= 0, \\ v_t - \frac{1}{2}u_{xxx} + 2(uv)_x + v &= 0, \quad 0 \le x \le 1, \end{aligned} \tag{5}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \frac{ck}{2} + \frac{ck}{2}tanh\left(\frac{-kx^2+x-\ln b}{2}\right), \\ v(x,0) &= \frac{-k^2}{8}\operatorname{sech}^2\left(\frac{kx^2+\ln b}{2}\right). \end{aligned} \tag{6}$$

2. OPIM algorithms for general optimal nonlinear differential equations

In order to perform OPIM fort he aforementioned equations, one can follow the steps below.

First, the Eqs. (1) can be taken as:

$$F_1(u_x,u_t,v_x,\varepsilon) = u_t - \frac{1}{2}v_x + 2\varepsilon uu_x + u = 0, \tag{7}$$
$$F_2(u_{xxx},u_x,v_t,v_x,\varepsilon) = v_t - \frac{1}{2}u_{xxx} + 2\varepsilon(uv)_x + v = 0 \tag{8}$$

where ε is the perturbation parameter. Without processing whole of the system, we can give the following general procedure. All Fin (5) will be split up as

$$F = S + R. \tag{9}$$

We do this decomposition fort he sake of more comfortable computations. First part of the equation (9) is easier part of the problem. One can readily solve that part. Additionally, R second part and one need to solve this part to obtain a new algorithm for the problems. Now, we use the theory of perturbation theory. Taking the straightforward expansion of perturbation series, one can write the following equality

$$u_{n+1} = u_n + \varepsilon(u_c)_n\ . \tag{10}$$

Here $(u_c)_n$ is a correction term and can be found by solving the first order OPIA problem. Now using the equation (10) - (7) and (8) we can get the OPIM algorithms. Replacing those equations and expanding them in a series, we have

$$R_1 + R_{1_{u_t}}((u_c)_n)_t\varepsilon + R_{1_{v_x}}((v_c)_n)_x\varepsilon + R_{1_{u_x}}((u_c)_n)_x\varepsilon + R_{1_\varepsilon}\varepsilon = -S_1, \tag{11}$$

$$R_2 + R_{2_{u_t}}((u_c)_n)_t\varepsilon + R_{2_{v_x}}((v_c)_n)_x\varepsilon + R_{2_{u_x}}((u_c)_n)_x\varepsilon + R_{2_\varepsilon}\varepsilon = -S_2. \quad (12)$$

Note that all computations are at $\varepsilon = 0$. Doing the mandatory computations for the (3), one can have

$$((u_c)_n)_t = \left(\frac{1}{2}v_n\right)_x - 2(u_n)_x(u_n) + (u_n), \quad (13)$$

$$((v_c)_n)_t = \left(\frac{1}{2}u_n\right)_x - 2((uv)_n)_x + (v_n). \quad (14)$$

The above expressions are called as OPIAs for the modified Boussinesq-Burger equations. After finding those OPIAs, we can begin to iterate with selecting appropriate $u_0$and v_0 (initial functions). There is no general theorem for those functions, and they can be elected by using ICs. By solving the algorithm (13) - (14), the first correction terms are obtained. However, generally first order approximations are not good enough for most of the problems. Therefore, we need to iterate more. Besides that, solutions can be healed by using the following parameters:

$$\begin{aligned} u_{n+1} &= u_n + P_n(u_c)_n \\ v_{n+1} &= v_n + P_n(v_c)_n \end{aligned}. \quad (15)$$

$P_0, P_1, P_2, \ldots$ can be seen as convergence-control constants. By using those parameters, we can easily control the convergence of the solutions. Generalizing with $n = 0,1, \ldots$, we get

$$\begin{aligned} u_1 &= u(x,t;P_0) = u_0 + P_0(u_c)_0 \\ v_1 &= v(x,t;P_0,P_1) = v_1 + P_1(v_c)_1 \\ &\vdots \\ u_m(x,t;P_0,\ldots,P_{m-1}) &= u_{m-1} + P_{m-1}(u_c)_{m-1} \\ v_m(x,t;P_0,\ldots,P_{m-1}) &= v_{m-1} + P_{m-1}(v_c)_{m-1} \end{aligned}. \quad (16)$$

After substituting u_m, v_m into the Eq. (6), the whole problem will be:

$$\begin{aligned} Re_1(x,t;P_0,\ldots,P_{m-1}) &= F_1(u_x,u_t,v_x,\varepsilon), \\ Re_2(x,t;P_0,\ldots,P_{m-1}) &= F_2(u_{xxx},u_x,v_t,v_x,\varepsilon) \end{aligned}. \quad (17)$$

Apparently, when $Re_{1,2}(x,t;P_0,\ldots,P_{m-1}) = 0$ we say that $u_m(x,t;P_0,\ldots,P_{m-1})$ and $v_m(x,t;P_0,\ldots,P_{m-1})$ are the exact analytical solutions for the problem. But we do not usually encounter such a situation especially for nonlinear ODEs and PDEs. Besides that, we can try to minimize the following:

$$J(P_0,\ldots,P_{m-1}) = \int_0^T \int_a^b Re_i^{\,2}\,(x,t;P_0,\ldots,P_{m-1})dxdt\,. \quad (18)$$

Here a, b, T are chosen from the physical domain of the problem. Lastly, $P_0, P_1, \ldots$ are found optimally by implementing many types of different techniques. Mostly used of them are collocation or the method of least squares. After finding $P_0, P_1, \ldots$ and placing them into the Eq. (17), the OPIM approximate solution of order *m* can be easily obtained For much more detailed data, we refer readers to [11-14].

3. Illustration

In this section, we try to show the effectiveness of the proposed method. To do this we will solve NPDE with initial conditions

$$u_0 = \frac{-1}{2} - \frac{1}{2}\tanh\left(\frac{x^2+x}{2}\right), \tag{19}$$
$$v_0 = \frac{-1}{8}\text{sech}^2\left(\frac{-x}{2}\right). \tag{20}$$

Then progressing as mentioned in the previous section, one can reach the following second order approximate OPIM solutions

$$u_2 = \frac{-1}{2} - \frac{1}{2}\tan h\left(\frac{x^2+x}{2}\right) - 6P_0 t\coth\left(\frac{x^2+x}{2}\right) - \left(2t\cosh\left[\frac{x^2+x}{2}\right] - t\cosh\left[\frac{x^2+x}{2}\right]P_0 - 2t^2\sinh[x^2+x]P_0\right)P_{1,} \tag{21}$$

$$v_2 = \frac{-1}{8}\text{sech}^2\left(\frac{-x^2+x}{2}\right) - 2P_0 t\sinh x - \left(4t\sinh\left[\frac{x^2+x}{2}\right] - 3t^2\cosh\left[\frac{x}{2}\right]P_0 - 3t^2\sinh[x^2+x]P_0\right)P_1 \tag{22}$$

and more of them can be reached by iterating. To get convergence-control parameters, we need to compute the residuals of both of the solutions. Putting $a = 0, b = 1, T = 10$ fort he Eq. (18) and solving the equality

$$\frac{\partial J}{\partial P_0} = \frac{\partial J}{\partial P_1} = \frac{\partial J}{\partial P_2} = \ldots = 0$$

we will get $P_0 = 1.80256$, $P_1 = 1.99044$ and $P_1 = -1.90033$. Substituting these constans into the corresponding approximate solutions, we have the third order OPIM solutions. Figure 1 and Figure 2 give absolute errors for approximate results in the 9th and 10th order solutions.

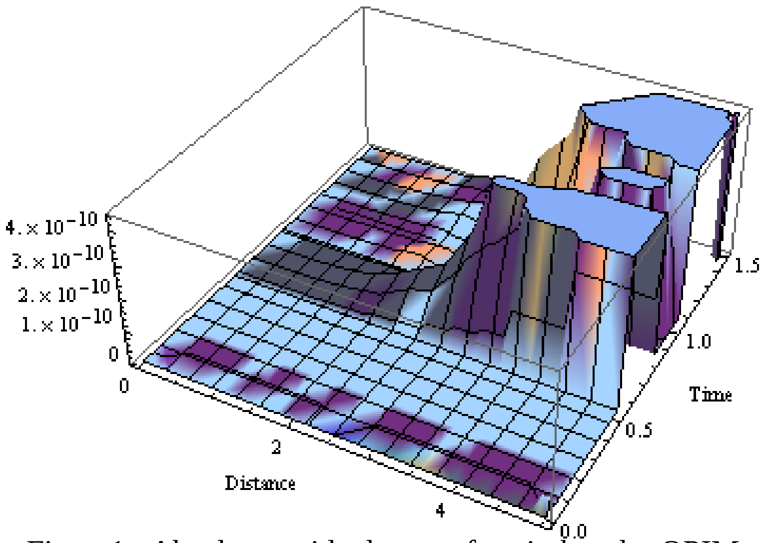

Figure1. Absolute residual errors for ninth order OPIM solution.

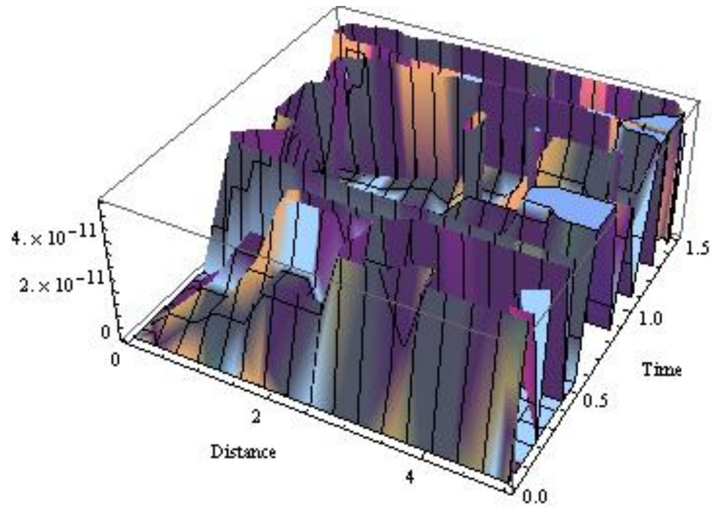

Figure 2. Absolute residual errors for tenth order OPIM solution.

4. Conclusion

In this research, the nonlinear modified Boussinesq-Burger have been studied with the aid of newly proposed optimal perturbation iteration method. Like in many other papers stated in the introduction part, OPIM results are efficient for NPDEs. The graphics of absolute errors are sketched by using the numerical solutions in order to approve he efficiency of the proposed scheme. As can be seen from those figures, absolute errors are not too much for higher order approximations. So, one can deduce that OPIM provides a simple way to control the convergence region for strong nonlinearity problem like Boussinesq-Burger equations.

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