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RESULTS ON QUASI-STATISTICAL LIMIT AND QUASI-STATISTICAL CLUSTER POINTS

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Abstract. In this paper we introduce the concepts of quasi-statistical limit point and quasi-statistical cluster point of a sequence. We give some inclusion results concerning these concepts. We also give the relationship between the Knopp core and quasi-statistical core of a sequence. Finally we state some theorems which deal with quasi-summability and quasi-statistical convergence of a sequence under some assumptions.

1. Introduction

The convergence of sequences has many generalizations with the aim of providing deeper insights into summability theory. One of the most important generalizations is statistical convergence [1], [2], [12], [14]. It is quite effective especially when the classical limit does not exist since it is broader than ordinary convergence. Therefore concept of convergence has been studied by many authors [6], [7], [8], [9], [13]. It has also been used in number theory, trigonometric series and approximation theory [15], [16]. In [10] Ganichev and Kadets have introduced the concept of quasi-statistical filter. Then by using the filter Özgüç and Yurdakadim have defined the quasi-statistical convergence and have studied the relationship between statistical convergence and quasi-statistical convergence in [11]. The statistical analogues of limit points, limit superior, limit inferior and core of a sequence have been obtained by Fridy and Orhan [3], [4], [5].

In this study we introduce the concepts of quasi-statistical limit point and quasi-statistical cluster point of a sequence. We give some inclusion results concerning these concepts. We also give the relationship between the Knopp core and quasi-statistical core of a sequence. Finally we state some theorems which deal with quasi-summability and quasi-statistical convergence of a sequence under some assumptions.

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Now let us recall the basic notations and definitions which we need throughout the paper.

If K is a set of positive integers, $|K|$ will denote the cardinality of K. The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \ k \in K\}\right|,$$

if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \ge \varepsilon\}$ has natural density zero. In this case we write $st - \lim x = L$.

Throughout the paper we assume that $c := (c_n)$ is a sequence of positive real numbers such that

$$\lim_n c_n = \infty \text{ and } \limsup_n \frac{c_n}{n} < \infty. \tag{1}$$

We define the quasi-density of $E \subset \mathbb{N}$ corresponding to the sequence (c_n) by

$$\delta_c(E) := \lim_n \frac{1}{c_n} \left|\{k \le n : \ k \in E\}\right|$$

if it exists.

The sequence $x = (x_k)$ is called quasi-statistically convergent to L provided that for every $\varepsilon > 0$ the set $E_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \ge \varepsilon\}$ has quasi-density zero. In this case we write $st_q - \lim x = L$ or $x_k \to L \ (st_q)$.

The next result has been obtained in [11] to present the relationship between quasi-statistical convergence and statistical convergence.

Lemma 1. *If $x = (x_k)$ is quasi-statistically convergent to L then it is statistically convergent to L.*

An example has been given in order to show that the converse of Lemma 1 does not hold (see [11]).

The following result has also been given to relate the statistical convergence and quasi-statistical convergence.

Under the assumptions (1) and

$$d := \inf_n \frac{c_n}{n} > 0 \tag{2}$$

we immediately obtain that

"$x = (x_k)$ is statistically convergent to L if and only if x is quasi-statistically convergent to L."

By S_q, we denote the set of all quasi-statistically convergent sequences.

It is easy to see that every convergent sequence is quasi-statistically convergent, i.e., $c \subset S_q$ where c is the set of all convergent sequences.

If x is a sequence we write $\{x_k : k \in \mathbb{N}\}$ to denote the range of x. If $\{x_{k(j)} : j \in \mathbb{N}\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$.

In case $\delta_c(K) = 0$, $\{x\}_K$ is called a subsequence of quasi-density zero or a thin subsequence. On the other hand $\{x\}_K$ is called a nonthin subsequence of x if K does not have quasi density zero. Note that $\{x\}_K$ is a nonthin subsequence if either $\delta_c(K)$ is a positive number or does not exist.

The number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L.

Definition 2. *The number λ is a quasi-statistical limit point of the sequence x if there is a nonthin subsequence which converges to λ.*

Note that we will denote by Λ_x^c, L_x, the set of quasi-statistical limit points of x, and the set of ordinary limit points of x, respectively. It is clear that $\Lambda_x^c \subseteq L_x$ for any sequence x.

Proposition 3. *$\Lambda_x \subseteq \Lambda_x^c$ holds where Λ_x denotes the set of statistical limit points of x.*

Proof. Let $\lambda \in \Lambda_x$. Then there exists a subset M such that $\delta(M) \neq 0$ and $\{x\}_M$ converges to λ. One can write that

$$\frac{1}{n} \left|\{k \leq n : k \in M\}\right| \leq H \frac{1}{c_n} \left|\{k \leq n : k \in M\}\right|$$

where $H := \sup\limits_n \frac{c_n}{n}$ and it follows that $\delta_c(M) \neq 0$. This completes the proof. $\square$

It is known that under the conditions (1) and (2) quasi-statistical convergence coincides with statistical convergence. If we assume that the sequence $c = (c_n)$ satisfies the conditions (1) and (2), we have $\Lambda_x^c = \Lambda_x$.

Example 4. *Define x by*

$$x_k = \begin{cases} r_n & ; \quad k = n^2, n = 0, 1, 2, .. \\ k & ; \quad \textit{otherwise} \end{cases}$$

where $\{r_n\}_{n=1}^{\infty}$ is a sequence whose range is the set of all rational numbers. It is known that $\Lambda_x = \varnothing$, $L_x = \mathbb{R}$ (see Example 2 of [3]). Since $\Lambda_x^c \subseteq \Lambda_x$, we get that $\Lambda_x^c = \varnothing$.

Definition 5. *The number γ is called a quasi-statistical cluster point of the sequence x if the set $\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ does not have quasi-density zero for every $\varepsilon > 0$.*

We will denote the set of all quasi-statistical cluster points of x by Γ_x^c. It is clear that $\Gamma_x^c \subseteq L_x$ for every sequence x.

Proposition 6. *$\Gamma_x \subseteq \Gamma_x^c$ holds where Γ_x denotes the set of all statistical cluster points of x.*

Proof. Let $\gamma \in \Gamma_x$. Then $\delta(M := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$ for every $\varepsilon > 0$. One can write that

$$0 \neq \delta(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) = \frac{1}{n} |\{k \leq n : k \in M\}| \leq H \frac{1}{c_n} |\{k \leq n : k \in M\}|$$

and it follows that $\delta_c(M) \neq 0$. This completes the proof. □

Under the conditions (1) and (2), $\Gamma_x^c = \Gamma_x$ holds.

Following result presents the inclusion relationship between Γ_x^c and Λ_x^c.

Theorem 7. *For every sequence x, $\Lambda_x^c \subseteq \Gamma_x^c$ holds.*

Proof. Let $\gamma \in \Lambda_x^c$. Then $\lim\limits_j x_{k(j)} = \gamma$ and $\limsup\limits_n \frac{1}{c_n} |\{k(j) \leq n\}| = r > 0$ hold. Also the set $\{j : |x_{k(j)} - \gamma| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$ so

$$\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\} \supseteq \{k(j) : j \in \mathbb{N}\} - \text{finite set}.$$

Therefore

$$\frac{1}{c_n} |\{k \leq n : |x_k - \gamma| < \varepsilon\}| \geq \frac{1}{c_n} |\{k(j) : j \in \mathbb{N}\}| - \frac{1}{c_n} O(1) \geq \frac{r}{2}$$

for infinitely many n. Hence $\delta_c(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$ for every $\varepsilon > 0$ which completes the proof. □

It is known that Λ_x does not need to be closed but Γ_x and L_x are closed sets. In a similar proof to given by Fridy [3], one can also show the following.

Proposition 8. *For any sequence x, the set Γ_x^c is closed.*

Theorem 9. *If $\delta_c(\{k : x_k \neq y_k\}) = 0$ then $\Lambda_x^c = \Lambda_y^c$ and $\Gamma_x^c = \Gamma_y^c$.*

Proof. Assume that $\delta_c(\{k : x_k \neq y_k\}) = 0$ and let $\lambda \in \Lambda_x^c$, the nonthin sequence $\{x\}_K$ converges to λ. Note that $\delta_c(\{k : x_k = y_k\}) \neq 0$. Therefore the latter set yields a nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ which converges to λ. Hence $\Lambda_x^c \subseteq \Lambda_y^c$. By symmetry one can also get $\Lambda_x^c \supseteq \Lambda_y^c$. The second assertion can be proved in a similar way. □

The following theorem is easy to prove by using the same technique in Theorem 2 of [3]. Therefore we omit it.

Theorem 10. *If x is a number sequence then there exists a sequence y such that $L_y = \Gamma_x^c$ and $\delta_c(\{k : x_k \neq y_k\}) = 0$. Moreover, the range of y is a subset of the range of x.*

Another noteworthy and useful result concerning the quasi-statistical cluster points is as follows.

Theorem 11. *If x is a number sequence that has a bounded nonthin subsequence, then x has a quasi-statistical cluster point.*

Proof. For such x, the above theorem ensures that there exists a sequence y such that $L_y = \Gamma_x^c$ and $\delta_c(\{k : x_k \neq y_k\}) = 0$. Then y must have a bounded nonthin subsequence, so by the Bolzano-Weierstrass Theorem $L_y \neq \varnothing$ which implies $\Gamma_x^c \neq \varnothing$. □

Now we immediately get the following corollary.

Corollary 12. *If x is a bounded sequence, then x has a quasi-statistical cluster point.*

Theorem 13. *If x is a bounded sequence then it has a thin subsequence $\{x\}_K$ such that $\{x_k : k \in \mathbb{N} - K\} \cup \Gamma_x^c$ is a compact set.*

Proof. Again by the above results one can choose a bounded sequence y such that $L_y = \Gamma_x^c$, $\{y_k : k \in \mathbb{N}\} \subseteq \{x_k : k \in \mathbb{N}\}$, and $\delta_c(K) = 0$ where $K := \{k \in \mathbb{N} : x_k \neq y_k\}$. This implies

$$\{x_k : k \in \mathbb{N} - K\} \cup \Gamma_x^c = \{y_k : k \in \mathbb{N}\} \cup L_y$$

and the right-hand set is compact. □

2. Quasi-Statistical Limit Superior and Inferior

The aim of this section is to present quasi-statistical limit superior and inferior to obtain some quasi-statistical analogues of ordinary limit superior, inferior and statistical limit superior, inferior as in [4]. We also introduce quasi-statistical core of a sequence and prove some results.

Definition 14. *If x is a real number sequence then the quasi-statistical limit superior and inferior of x are defined by*

$$st_q - \limsup x = \begin{cases} \sup B_x^c & , \quad \text{if } B_x^c \neq \emptyset \\ -\infty & , \quad \text{if } B_x^c = \emptyset \end{cases},$$

$$st_q - \liminf x = \begin{cases} \inf A_x^c & , \quad \text{if } A_x^c \neq \emptyset \\ +\infty & , \quad \text{if } A_x^c = \emptyset \end{cases}$$

where $B_x^c = \{b \in \mathbb{R} : \delta_c(\{k : x_k > b\}) \neq 0\}$, $A_x^c = \{a \in \mathbb{R} : \delta_c(\{k : x_k < a\}) \neq 0\}$. We now give a simple example to understand the concepts just defined.

Example 15. *Let $c := (c_n)$ be the sequence of positive real numbers such that $\lim_n c_n = \infty$, and $\lim_n \frac{\sqrt{n}}{c_n} = \infty$. We can choose a subsequence $\{c_{n_p}\}$ such that $c_{n_p} > 1$ for each $p \in \mathbb{N}$.*

Consider the sequence $x = (x_k)$ defined by

$$x_k := \begin{cases} c_k & , \quad k \text{ is square and } c_k \in \{c_{n_p} : p \in \mathbb{N}\} \\ 2 & , \quad k \text{ is square and } c_k \notin \{c_{n_p} : p \in \mathbb{N}\} \\ 1 & , \quad k \text{ is odd and } k \text{ is not square} \\ 0 & , \quad k \text{ is even and } k \text{ is not square} \end{cases}.$$

One can easily see that $st_q - \limsup x = 1$, $st_q - \liminf x = 0$.

Definition 16. *The real number sequence x is said to be quasi-statistically bounded if there is a number M such that $\delta_c(\{k \in \mathbb{N} : |x_k| > M\}) = 0$.*

The sequence x in the above example is not quasi-statistically convergent but quasi-statistically bounded. Also note that quasi-statistical boundedness implies that $st_q - \limsup$, $st_q - \liminf$ are finite and if the sequence is quasi-statistically bounded then $st_q - \limsup x$ is the greatest element of the set of quasi-statistical cluster points and $st_q - \liminf x$ is the least element of this set.

Now we investigate the relationship between $st_q - \limsup x$ and $st - \limsup x$ and also the relationship between $st_q - \liminf x$ and $st - \liminf x$.

Remark 17. *Let $H < \infty$. Then*

$$st_q - \liminf x \leq st - \liminf x \leq \ st - \limsup x \leq st_q - \limsup x$$

holds for any real sequence.

Proof. Let $\alpha_2 = st_q - \liminf x$ and $\alpha_1 = st - \liminf x$. Then $\delta(\{k : x_k < \alpha_1 + \varepsilon\}) \neq 0$ and $\delta(\{k : x_k < \alpha_1 - \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. Since $H < \infty$, we have that $\delta(\{k : x_k < \alpha_1 + \varepsilon\}) \leq \delta_c(\{k : x_k < \alpha_1 + \varepsilon\})$ and this implies $\delta_c(\{k : x_k < \alpha_1 + \varepsilon\}) \neq 0$. Then

$$\alpha_1 + \varepsilon \in A_x^c = \{a \in \mathbb{R} : \delta_c(\{k : x_k < a\}) \neq 0\}$$

and it is known that $\inf A_x^c = \alpha_2$ which implies $\alpha_2 \leq \alpha_1 + \varepsilon$ for every $\varepsilon > 0$. ε is arbitrary and we obtain that $\alpha_1 = st - \liminf x \geq \alpha_2 = st_q - \liminf x$. Now let $\beta_2 = st_q - \limsup x$, $\beta_1 = st - \limsup x$. Then $\delta(\{k : x_k > \beta_1 - \varepsilon\}) \neq 0$ and $\delta(\{k : x_k > \beta_1 + \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. Since $H < \infty$ we have that $\delta(\{k : x_k > \beta_1 - \varepsilon\}) \leq \delta_c(\{k : x_k > \beta_1 - \varepsilon\})$ and this implies $\delta_c(\{k : x_k > \beta_1 - \varepsilon\}) \neq 0$. Then

$$\beta_1 - \varepsilon \in B_x^c = \{b \in \mathbb{R} : \delta(\{k : x_k > b\}) \neq 0\}$$

and it is known that $\sup B_x^c = \beta_2$ which implies $\beta_1 - \varepsilon \leq \beta_2$ for every $\varepsilon > 0$. ε is arbitrary and we obtain that $\beta_1 = st - \limsup x \leq \beta_2 = st_q - \limsup x$ which completes the proof. □

Knopp has introduced the concept of the core of a sequence and has proved the well known core theorem. In order to produce natural analogues of Knopp core and statistical core of a sequence, we can replace limit points and statistical cluster points with quasi-statistical cluster points in [4], [5].

Definition 18. *If x is a quasi-statistically bounded real sequence then the quasi-statistical core of x is the closed interval $st_q - core\,\{x\} = [st_q - \liminf x, st_q - \limsup x]$. In case x is not quasi-statistically bounded, $st_q - core\,\{x\}$ is defined accordingly as either $[st_q - \liminf x, \infty)$, $(-\infty, \infty)$ or $(-\infty, st_q - \limsup x]$.*

One can easily see from Remark 1 that

$$st - core\{x\} \subseteq st_q - core\,\{x\} \subseteq K - core\{x\}.$$

Recall that the sequence $x = (x_k)$ is said to be strongly quasi-summable to L if

$$\lim_n \frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| = 0.$$

The space of all strongly quasi-summable sequences is denoted by N_q.

$$N_q := \left\{ x : \text{for some } L,\ \lim_n \frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

Also the sequence $x = (x_k)$ is said to be quasi-summable to L if

$$\lim_n \frac{1}{c_n} \sum_{k=1}^{n} x_k = L.$$

Now we can give a result concerning with the quasi-summability and quasi-statistical limit superior.

Theorem 19. *Let the sequence x is bounded above, $\ell = \delta_c(\mathbb{N})$ and $\beta l = st_q - \limsup x$. If the sequence x is quasi-summable to $\beta \ell^2$ then x is quasi-statistically convergent to $\beta \ell$.*

Proof. Suppose that x is not quasi-statistically convergent to $\beta\ell$. Then $st_q - \liminf x < \beta l$, therefore there is a number $\mu < \beta l$ such that $\delta_c(\{k : x_k < \mu\}) \neq 0$. Let $K' := \{k : x_k < \mu\}$. Then by the definition of quasi-statistical limit superior, $\delta_c(\{k : x_k > \beta l + \varepsilon\}) = 0$, for every $\varepsilon > 0$. Now define

$$K'' := \{k : \mu \leq x_k \leq \beta l + \varepsilon\},\ K''' := \{k : x_k > \beta l + \varepsilon\},\ B := \sup x < \infty.$$

Since $\delta_c(K') \neq 0$, there are infinitely many n such that

$$\frac{1}{c_n} \left| K'_n \right| \geq r > 0,$$

for each such n we have

$$\begin{aligned}
\frac{1}{c_n} \sum_{k=1}^{n} x_k &= \frac{1}{c_n} \sum_{k=1,\ k \in K'_n}^{n} x_k + \frac{1}{c_n} \sum_{k=1, k \in K''_n}^{n} x_k + \frac{1}{c_n} \sum_{k=1, k \in K'''_n}^{n} x_k \\
&< \frac{\mu}{c_n} \left| K'_n \right| + \frac{\beta\ell + \varepsilon}{c_n} \left| K''_n \right| + \frac{B}{c_n} \left| K'''_n \right| \\
&= \frac{\mu}{c_n} \left| K'_n \right| + (\beta\ell + \varepsilon)(\ell - \frac{\left| K'_n \right|}{c_n}) + o(1) \\
&\leq (\mu - \beta\ell) \frac{\left| K'_n \right|}{c_n} + \beta\ell^2 + \varepsilon(\ell - \frac{\left| K'_n \right|}{c_n}) \\
&\leq \beta\ell^2 - (\beta\ell - \mu) \frac{\left| K'_n \right|}{c_n} + \varepsilon(\ell - r) + o(1)
\end{aligned}$$

$$\leq \beta\ell^2 - (\beta\ell - \mu)r + \varepsilon(\ell - r) + o(1)$$

Since ε is arbitrary it follows that $\liminf \frac{1}{c_n}\sum_{k=1}^{n} x_k < \beta\ell^2$. Hence x is not quasi-summable to $\beta\ell^2$ and this completes the proof. □

This theorem is a generalization of Theorem 5 in [3]. Using the symmetry, we also have the following for lower bounds.

Corollary 20. *Let the sequence x is bounded below, $\ell = \delta_c(\mathbb{N})$ and $\alpha l = st_q - \liminf x$. If the sequence x is quasi-summable to $\alpha\ell^2$ then x is quasi-statistically convergent to $\alpha\ell$.*

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