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Research Article

Decay of Fourier Transforms and Generalized Besov Spaces

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ABSTRACT. A characterization of the generalized Lipschitz and Besov spaces in terms of decay of Fourier transforms is given. In particular, necessary and sufficient conditions of Titchmarsh type are obtained. The method is based on two-sided estimate for the rate of approximation of a β-admissible family of multipliers operators in terms of decay properties of Fourier transforms.

Keywords: Generalized Besov spaces, Fourier transforms, Titchmarsh type theorem.

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1. INTRODUCTION

The study of decay of Fourier transform / Fourier coefficients is one of the classical topics in Fourier analysis. Classical inequalities as Hardy-Littlewood and Haurdorsff-Young (see [29]) give us the basic decay of Fourier transforms. Titchmarsh showed ([29]) that the decay of Fourier transform can be improved for univariate functions satisfying a Lipschitz condition defined by smoothness. His result reads as follows.

Theorem 1.1. ([29, Theorem 85]) *Let $f \in L^2$ and $\widehat{f}$ its Fourier transform. The following conditions are equivalent*

$$\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^2 dx = O(h^{2\alpha}) \quad \text{as } h \to 0^+ \quad (0 < \alpha < 1)$$

and

$$\int_{1/h \leq |x|} [\widehat{f}(x)]^2 dx = O(h^{2\alpha}) \quad \text{as } h \to 0^+.$$

Extensions of the Titchmarsh theorem were obtained by several authors ([19, 20, 21, 33]) and can be extended to higher dimensional Euclidean spaces ([7, 34]) replacing the majorant function $\varphi(h) = h^\alpha$ in the Lipschitz condition by a regularly varying one ([4, 16]). The problem concerning about Fourier series on $\mathbb{T}$ can be found in [24, 25] while for Fourier transforms in [31]. The problem in $L^p(\mathbb{R}^d)$ for Fourier series can be seen in [13, 18] and for Fourier transforms we suggest [6, 8, 13] and references quoted there.

In this paper, we provide a further extension of Theorem 1.1 for functions in $L^p(\mathbb{R}^d)$ and an abstract Lipschitz condition, see Theorem 1.3 below. In particular, for $p = 2$, $d = 1$ and $\varphi(t) = t^\alpha$, $t \in (0, \infty)$, $0 < \alpha < 1$, our achievement recovers Theorem 1.1, due Lemma 2.2. In order to present this generalized version of the result, we need to establish a two-sided estimate for the rate of approximation of an admissible family of multipliers operators in terms of decay

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properties of Fourier transforms. This extends the known results proved in [13] for $d \geq 2$ and for the combination of multivariate averages.

For $d \geq 1$ the Fourier transform $\widehat{f}$ of a function f, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\xi \cdot x} dx, \quad x \in \mathbb{R}^d.$$

We write $L^p(\mathbb{R}^d) := (L^p(\mathbb{R}^d), \|\cdot\|_p)$ for the usual Banach spaces of p-integrable functions ($1 \leq p \leq \infty$).

We deal with a family of multipliers operators ([23]) $\{T_t\}_{t>0}$ on $L^p(\mathbb{R}^d)$ with its multiplier family $\{\eta_t\}_{t>0}$ generated by dilations of a measurable function $\eta : (0, \infty) \longrightarrow \mathbb{R}$, i.e.,

$$T_t(f)\widehat{\ }(\xi) = \eta_t(|\xi|)\widehat{f}(\xi),$$

where $\eta_t(|\xi|) := \eta(t|\xi|)$, for all $\xi \in \mathbb{R}^d$ and $t > 0$. If there exists $\gamma > 0$ such that

$$[\min(1, ts)]^{2\gamma} \asymp |1 - \eta_t(s)|, \quad \text{for all } t > 0, \tag{1.1}$$

then we say that $\{T_t\}_{t>0}$ *is a γ-admissible family of multipliers operators on* $L^p(\mathbb{R}^d)$. A well-known admissible family of multipliers operators , on $L^p(\mathbb{R}^d)$ for $d \geq 2$, includes the classical spherical mean operator and its combinations (see [2, 9, 13] and references quoted there).

We will employ generalized Lipschitz (and Besov) classes defined in terms of the rate of approximation of an admissible family of multipliers operators. The main point of the definition resides on the majorant function (defined ahead) and not on the fractional choice of orders of admissibility for the families of multipliers operators above. Indeed, no new Lipschitz/Besov classes are given just by considering fractional orders admissible family of multipliers operators, due condition (1.1) and Marchaud-type inequalities (see [10, 22, 30] and references quoted there).

In order to state the main theorems of the paper, we need to introduce some more definition. A majorant function in this paper is always a nondecreasing measurable function $\varphi : (0, \infty) \longrightarrow \mathbb{R}_+$ such that

$$\lim_{t \to 0_+} \varphi(t) \to 0,$$

and

$$\int_0^t \frac{\varphi(u)}{u} du \lesssim \varphi(t) \quad \text{for all } t > 0. \tag{1.2}$$

We denote by M the collection of all majorant functions. For $\beta > 0$, we define the following subset of M

$$\Omega_\beta := \left\{ \varphi \in M : \int_t^\infty \frac{\varphi(u)}{u^{\beta+1}} du \lesssim \frac{\varphi(t)}{t^\beta}, \, t > 0 \right\}.$$

The family Ω_β can be defined in terms of the almost monotonicity property.

A function $\varphi : (0, \infty) \longrightarrow \mathbb{R}_+$ is β*-almost decreasing* ([4, p. 72]) if it satisfies the condition:

$$\frac{\varphi(u_2)}{u_2^\beta} \lesssim \frac{\varphi(u_1)}{u_1^\beta}, \quad \text{for any } u_1 \leq u_2.$$

For $\beta > 0$, we write

$$\Omega'_\beta := \{\varphi \in M : \text{there exists } 0 < \epsilon < \beta \text{ such that } \varphi \text{ is } (\beta - \epsilon)\text{-almost decreasing}\}.$$

$A(t) \asymp B(t)$ stands for $B(t) \lesssim A(t)$ and $A(t) \lesssim B(t)$, where $A(t) \lesssim B(t)$ means that $A(t) \leq c\, B(t)$, for some constant $c > 0$ not depending upon t.

Simple calculations and Bari-Stechkin Lemma ([1], see also [26, p.754]) are enough to prove that the classes Ω'_β and Ω_β coincide:

$$\Omega_\beta = \Omega'_\beta, \quad \text{for each } \beta > 0. \tag{1.3}$$

Obviously,

$$\bigcup_{0<\alpha<\beta} \Omega_\alpha = \Omega_\beta, \quad \text{for any } \beta > 0.$$

In fact, for any $0 < \alpha < \beta$ we have $\Omega_\alpha \subset \Omega_\beta$. In order to verify equality above, is enough to prove that for a given $\varphi \in \Omega_\beta$ there exists $0 < \alpha < \beta$ such that $\varphi \in \Omega_\alpha$. If $\varphi \in \Omega_\beta$, then (1.3) implies that φ is $(\beta - \epsilon)$-almost decreasing, for some $0 < \epsilon < \beta$. It means that for any $t \leq s$, it holds

$$\frac{\varphi(s)}{s^{\beta-\epsilon/2}} \lesssim \frac{\varphi(t)}{t^{\beta-\epsilon}s^{\epsilon/2}}.$$

Integrating both sides of inequality above, we obtain

$$\int_t^\infty \frac{\varphi(s)}{s^{\beta-\epsilon/2+1}}ds \lesssim \frac{\varphi(t)}{t^{\beta-\epsilon}} \int_t^\infty s^{-\epsilon/2-1}ds = 2/\epsilon \frac{\varphi(t)}{t^{\beta-\epsilon/2}}.$$

Thus, $\varphi \in \Omega_{\beta-\epsilon/2}$.

An interesting subclass of Ω_β is given via the following definition. A function $f : (0, \infty) \longrightarrow \mathbb{R}_+$ is *regularly varying* ([16]) with index $\alpha \in \mathbb{R}$ if for any $\lambda > 0$, it holds $f(\lambda x)/f(x) \to \lambda^\alpha$ as $x \to \infty$. We write RV_α for the set of all regularly varying functions with index α. It is not hard to see that if $\varphi \in \mathrm{RV}_\alpha$, then it can be represented as $\varphi(x) = x^\alpha \varsigma(x)$, $x \in (0, \infty)$, where ς is a regularly varying function with index zero (i.e., a slowly varying function). More than that the Representation Theorem ([4, p. 17]) gives a characterization for all regularly varying functions.

We observe that $\mathrm{RV}_\alpha \subsetneq \Omega_\beta$, for all $0 < \alpha < \beta$. This fact follows from basic theory of regularly varying functions, the needed details can be found in [4, p. 68–72]. Due to this, the following functions belong to Ω_β,

$$t^\alpha \ln(1+t), \ \ (t\ln(1+t))^\alpha, \quad t^\alpha \ln(\ln(e+t)), \ \ t^\alpha \exp\left[\frac{\ln t}{\ln(\ln t)}\right]$$

and

$$t^\alpha \exp[(\log t)^{\alpha_1}(\log_2 t)^{\alpha_2} \ldots (\log_n t)^{\alpha_n}],$$

where $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, n$, for all $0 < \alpha < \beta$. The usual majorant function employed in the Titchmarsh theorem $\varphi(t) = t^\alpha$, belongs to Ω_β if and only if $0 < \alpha < \beta$.

Definition 1.2. For $\varphi \in \Omega_{2\beta}$, we define the *generalized Lipschitz class* in $L^p(\mathbb{R}^d)$ by

$$\mathrm{Lip}\,(p, \beta, \varphi) = \left\{f \in L^p(\mathbb{R}^d) : \|T_t(f) - f\|_p = O(\varphi(t)) \text{ as } t \to 0^+\right\}, \quad 1 \leq p \leq \infty, \tag{1.4}$$

where $\{T_t\}_t$ is a β-admissible family of multipliers operators.

Necessary and sufficient conditions of Titchmarsh type for the generalized Lipschitz class read as follow.

Theorem 1.3. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $\varphi \in \Omega_{2\beta}$.*
(A) Let $1 < p \leq 2$ and $p \leq q \leq p'$. If $f \in \mathrm{Lip}\,(p, \beta, \varphi)$, then

$$\left(\int_{t\leq|\xi|\leq 2t} \left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad \text{as } t \to \infty. \tag{1.5}$$

(B) Let $2 \leq p < \infty$, $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ *and* $p' \leq q \leq p$. *If*

$$\left(\int_{t\leq|\xi|\leq 2t}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad \text{as } t\to\infty,$$

then $f \in Lip\,(p,\beta,\varphi)$.

In order to define the generalized Besov spaces, we need to restrict our majorant classes as follows. For $0 < q, \gamma < \infty$, we write

$$\Omega^q_\gamma := \left\{\varphi \in \Omega_\gamma : \int_0^1 \frac{1}{[\varphi(t^{-1})]^q}\frac{dt}{t} < \infty\right\}.$$

Definition 1.4. For $0 < q < \infty$ and $\varphi \in \Omega^q_{2\beta}$, we define the *generalized Besov space* $B^\varphi_{p,q}(\mathbb{R}^d)$ by

$$B^\varphi_{p,q}(\mathbb{R}^d) = \left\{f \in L^p(\mathbb{R}^d) : |f|_{B^\varphi_{p,q}} := \int_0^1 \left(\frac{\|T_t(f)-f\|_p}{\varphi(t)}\right)^q \frac{dt}{t} < \infty\right\}. \tag{1.6}$$

For $q = \infty$ and $\varphi \in \Omega_\gamma$,

$$B^\varphi_{p,\infty}(\mathbb{R}^d) := \left\{f \in L^p(\mathbb{R}^d) : |f|_{B^\varphi_{p,\infty}} := \sup_{t>0}\left\{\frac{\|T_t(f)-f\|_p}{\varphi(t)}\right\} < \infty\right\}.$$

As usual, if $q < \infty$, we endow $B^\varphi_{p,q}$ with the norm $\|\cdot\|_{B^\varphi_{p,q}} := \left(\|\cdot\|_p^q + |\cdot|_{B^\varphi_{p,q}}\right)^{1/q}$, otherwise $\|\cdot\|_{B^\varphi_{p,\infty}} := \|\cdot\|_p + |\cdot|_{B^\varphi_{p,\infty}}$. In particular, for $q = \infty$, these spaces are the generalized Lipschitz ones. The Besov spaces here seem to depend upon a majorant function and an admissible family of multipliers operators, but, as usual, that is not true. As a matter of fact, this is a topic of investigation [14].

The following gives us necessary and sufficient conditions in terms of decay properties of Fourier transforms for functions in the generalized Besov spaces.

Theorem 1.5. *Let* $\{T_t\}_{t>0}$ *be a* β*-admissible family of multipliers operators on* $L^p(\mathbb{R}^d)$ *and* $\varphi \in \Omega^q_{2\beta}$.
(A) Let $1 < p \leq 2$ *and* $p \leq q \leq p'$. *If* $f \in B^\varphi_{p,q}(\mathbb{R}^d)$, *then*

$$\int_0^\infty \int_{t\leq|\xi|\leq 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t} < \infty. \tag{1.7}$$

(B) Let $2 \leq p < \infty$, $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ *and* $p' \leq q \leq p$. *If*

$$\int_0^\infty \int_{t\leq|\xi|\leq 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t} < \infty, \tag{1.8}$$

then $f \in B^\varphi_{p,q}(\mathbb{R}^d)$.

For the particular choice $\varphi(t) = t^\alpha$, $0 < \alpha < \ell$ for some $\ell \in \mathbb{N}$, and the ℓ-th family of combinations of multivariate averages on $\mathbb{R}^d$, for $d \geq 2$, spaces $B^\varphi_{p,q}(\mathbb{R}^d) \cap \widehat{GM}^d_p$ became the ones characterized in [13, Section 7] ($\widehat{GM}^d_p$ is defined ahead).

The paper is organized as follows. In Section 2, we present a two-sided estimate for the rate of approximation of an β-admissible family of multipliers operators in terms of decay properties of Fourier transforms. This estimate plays a crucial role in the proof of Theorem 1.3, presented in this section. The inverse Fourier-Hankel transform of certain radial functions is applied in order to show the necessity of the condition concerning the majorant functions in order to prove Theorem 1.3. Section 3 is regarded to the proof of Theorem 1.5. Finally, in Section

4, we present the concept of general monotonicity of functions (GM_p^d class) and we outline how to make assumptions in Theorems 1.3 and 1.5 less restrictive. As a corollary, we prove a pointwise inequality for Fourier transforms of functions in $\widehat{GM}_p^d$, that is, a Riemann-Lebesgue type inequality.

2. Proof of Theorem 1.3

The rate of approximation of an admissible family of multipliers operators can be estimated in terms of decay properties of Fourier transforms as follows. For $d \geq 2$, the following result can be seen as a corollary of [13, Theorem 2.1, p. 1289] and the ideas of the proof are included below for completeness.

Proposition 2.1. *Let $\{T_t\}_{t>0}$ be a γ-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$.*

(A) Let $1 < p \leq 2$. If $p \leq q \leq p'$, then $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and

$$\left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{2\gamma}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} \lesssim \|T_t(f)-f\|_p.$$

(B) Let $2 \leq p < \infty$. If $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$ and $p' \leq q \leq p$, then

$$\|T_t(f)-f\|_p \lesssim \left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{2\gamma}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q}.$$

The proof of proposition above is a simple adaptation of the proof of [13, Theorem 2.1, p. 1289], since the main arguments completely fit here. An application of Pitt's inequality (see [3]) combined to the admissibility condition on the family of multipliers operators finishes the proof.

For $d \geq 2$, Theorem 2.1 in [13] is easily recovered from Proposition 2.1 for $\gamma = \ell$ a natural number and the combinations of multivariate averages family as the admissible one. The latter has a generalized version as follows. All the facts mentioned below can be found in [15]. Let $r > 0$, a real number. For each $t > 0$, we write

$$V_{r,t}(f)(x) := \frac{-2}{\binom{2r}{r}}\sum_{k=1}^{\infty}(-1)^k\binom{2r}{r-k}V_{kt}(f)(x), \quad f \in L^p(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $\{V_t\}_t$ is the usual family of spherical mean operator on $L^p(\mathbb{R}^d)$, and for r and s real numbers,

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}, \quad \text{for } s \notin \mathbb{Z}_-, \quad \binom{r}{0} = r \quad \text{and} \quad \binom{r}{s} = 0, \quad \text{for } s \in \mathbb{Z}_-.$$

The operator defined by (2.1) is bounded on $L^p(\mathbb{R}^d)$ and for $r = \ell$ a natural number the family $\{V_{r,t}\}_t$ becomes the combination of multivariate averages $\{V_{\ell,t}\}_t$ given in [9]. If $m_{r,t}$ stands for the multiplier of $V_{r,t}$, for each $t > 0$, then

$$1 - m_t^r(|\xi|) = 1 - m^r(t|\xi|) := \frac{2^{2r+1}\Gamma((m+1)/2)}{\binom{2r}{r}\Gamma(m/2)\Gamma(1/2)}\int_0^1 (\sin(t|\xi|s/2))^{2r}(1-s^2)^{(d-1)/2}ds, \quad \xi \in \mathbb{R}^d.$$

In this case, $\{V_{r,t}\}_t$ is a r-admissible family of multipliers operators, since

$$\min(1,s)^{2r} \asymp 1 - m_{r,t}(s) = 1 - m_r(ts), \quad s > 0.$$

Proof of Theorem 1.3 makes use of the next lemma.

Lemma 2.2. *Let $\varphi \in M$, $f \in L^p(\mathbb{R}^d)$ and $1 < p, q < \infty$. The following two conditions are equivalent:*

$$\left(\int_{1/t \le |\xi| \le 2/t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} \lesssim \varphi(t), \quad t > 0 \tag{2.2}$$

and

$$\left(\int_{1/t \le |\xi|} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} \lesssim \varphi(t), \quad t > 0. \tag{2.3}$$

Proof. It is easy to see that (2.3) implies (2.2). Assuming that (2.2) holds, we write the integral in the left-hand side of inequality (2.3) in terms of the radial part (see [32]) of the integrating function, as follows

$$I(t) := \int_{1/t}^{\infty} r^{dq(1-1/p-1/q)} \left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega \right) r^{(d-1)} dr, \quad t > 0,$$

where S^{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ centered at origin endowed with σ_{d-1} the induced Lebesgue measure (if $d = 1$ we skip this step). It is easily seen that

$$I(t) \lesssim \int_{1/t}^{\infty} r^{dq(1-1/p-1/q)} \left[\int_r^{2r} \left(\int_{S^{d-1}} |\widehat{f}(\rho\omega)|^q d\omega \right) d\rho \right] r^{(d-1)} \frac{dr}{r}.$$

If $r \le \rho \le 2r$, then $r^{dq(1-1/p-1/q)} \lesssim \rho^{dq(1-1/p-1/q)}$, and due to inequality (2.2) we arrive at

$$I(t) \lesssim \int_{1/t}^{\infty} \frac{[\varphi(r^{-1})]^q}{r} dr = \int_0^t \frac{[\varphi(u)]^q}{u} du.$$

In order to finish the proof, it is enough to observe that

$$\int_0^t \frac{[\varphi(u)]^q}{u} du \lesssim [\varphi(t)]^q \quad \text{and} \quad \int_0^t \frac{[\varphi(u)]}{u} du \lesssim \varphi(t), \quad t > 0,$$

are equivalent (see [26]) and the later is the condition (1.2) for $\varphi \in M$. □

Proof. of Theorem 1.3. The proof of part (A) is a trivial application of Proposition 2.1, part (A). In order to prove part (B), we apply Proposition 2.1, part (B), and we obtain

$$\|T_t(f) - f\|_p^q \lesssim \int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi.$$

Denoting by $I_q^\beta(f)$ the right-hand side of inequality above, we have

$$\|T_t(f) - f\|_p^q \lesssim I_q^\beta(f),$$

where

$$I_q^\beta(f) = \int_{|\xi| \ge 1/t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi + t^{2q\beta} \int_{|\xi| < 1/t} |\xi|^{2q\beta} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi.$$

Due to Lemma 2.2, the proof will be completed if the following holds

$$t^{2q\beta} \int_{|\xi| < 1/t} \left[|\xi|^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi = O\left(\varphi(t)\right)^q, \quad \text{as } t \to 0^+. \tag{2.4}$$

We first consider the case $d \geq 2$ and we employ an adaption of the Titchmarsh proof in [29, Theorem 84]. For $t > 0$, denote

$$I_q^{\beta<}(f) := \int_{|\xi|<1/t} \left[|\xi|^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi.$$

The following inequality holds

$$I_q^{\beta<}(f) \leq \int_{|\tau|<1/t} |\tau|^{2q\beta} h(\tau) |\tau|^{q(d-1)} d\tau,$$

where

$$h(\tau) := \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)} |\widehat{f}(\tau\omega)| \right]^q d\sigma_{d-1}(\omega), \quad -1/t < \tau < 1/t.$$

By writing

$$\int_{|\tau|<1/t} |\tau|^{2q\beta} h(\tau) |\tau|^{q(d-1)} d\tau := I_q^{\beta^-}(h,t) + I_q^{\beta^+}(h,t), \tag{2.5}$$

where

$$I_q^{\beta^-}(h,t) := \int_{-1/t}^{0} (-\tau)^{2q\beta} \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)} |\widehat{f}(\tau\omega)| (-\tau)^{(d-1)} \right]^q d\sigma_d(\omega) d\tau$$

and

$$I_q^{\beta^+}(h,t) := \int_{0}^{1/t} \tau^{2q\beta} \int_{S^{d-1}} \left[|\tau\omega|^{d(1-1/p-1/q)} |\widehat{f}(\tau\omega)| \tau^{(d-1)} \right]^q d\sigma_d(\omega) d\tau, \quad t > 0,$$

it is sufficient to show that both $I_q^{\beta^-}(h,t)$ and $I_q^{\beta^+}(h,t)$ are $O\left(t^{-2q\beta} \left(\varphi(t)\right)^q\right)$ as $t \to 0^+$.

We define

$$\phi_+(t) = \int_{1/t}^{+\infty} h(\tau) \tau^{q(d-1)} d\tau, \quad t > 0,$$

and observe that

$$\lim_{t\to 0^+} t^{2q\beta} \phi_+(t^{-1}) = 0. \tag{2.6}$$

In fact, we have

$$\lim_{t\to 0^+} t^{2q\beta} \phi_+(t^{-1}) \lesssim \lim_{t\to 0^+} \left(t^{2\beta}\varphi(t)\right)^q = \lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta}}\right)^q.$$

Equality (1.3) implies that there exists $0 < \epsilon < 2\beta$ such that φ is $(2\beta - \epsilon)$-almost decreasing. This leads us to

$$\lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta}}\right)^q = \lim_{t\to\infty} \left(\frac{\varphi(t^{-1})}{t^{2\beta-\epsilon}}\right)^q \frac{1}{t^{q\epsilon}} \lesssim (\varphi(1))^q \lim_{t\to\infty} \frac{1}{t^{q\epsilon}} = 0,$$

and (2.6) holds.

Note that $\phi_+'(\tau) = -h(\tau^{-1})\tau^{-q(d-1)/2}$, $0 < \tau < 1/t$, and

$$I_q^{\beta^+}(h,t) = \int_0^{1/t} -\tau^{2q\beta} \phi_+'(\tau^{-1}) d\tau, \quad t > 0,$$

thus integration by parts and (2.6) imply

$$\begin{aligned} I_q^{\beta^+}(h,t) &= \left(-\tau^{2q\beta}\phi_+(\tau^{-1})\right)_0^{1/t} + 2q\beta\int_0^{1/t}\tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau \\ &= -t^{-2q\beta}\phi_+(t) + 2q\beta\int_0^{1/t}\tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau \\ &\leq 2q\beta\int_0^{1/t}\tau^{2q\beta-1}\phi_+(\tau^{-1})d\tau, \quad t>0. \end{aligned}$$

Since $\phi_+((\cdot)^{-1})$ is a nondecreasing function on $(0,\infty)$, it follows

$$I_q^{\beta^+}(h,t) \leq 2q\beta\phi_+(t)\int_0^{1/t}\tau^{2q\beta-1}d\tau = \phi_+(t)t^{-2q\beta}, \quad t>0. \tag{2.7}$$

Handling $I_q^{\beta^-}(h,t)$ as above, by defining

$$\phi_-(t) = \int_{-\infty}^{-1/t} h(\tau)(-\tau)^{q(d-1)}d\tau, \quad t>0,$$

we get

$$I_q^{\beta^-}(h,t) \leq t^{-2q\beta}\phi_-(t) + 2q\beta\phi_-(t)\int_{-1/t}^{0}(-\tau)^{2q\beta-1}d\tau = 2t^{-2q\beta}\phi_-(t), \quad t>0. \tag{2.8}$$

Combining inequalities (2.5), (2.7) and (2.8) with our assumptions (i.e. $\phi_+(t) = O\left(\varphi(t)\right)^q$ and $\phi_-(t) = O\left(\varphi(t)\right)^q$, as $t\to 0^+$), we reach to

$$\|T_t(f)-f\|_p \lesssim \left(\int_{\mathbb{R}^d}\left[\min(1,t|\xi|)^{2\beta}|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)^{1/q} = O\left(\varphi(t)\right), \quad \text{as } t\to 0^+.$$

Thus, $f\in \mathrm{Lip}\,(p,\beta,\varphi)$.

For $d=1$, the same proof presented above can be rewritten with minor adjustments as follows. For $t>0$, denote

$$I_q^{\beta<}(f) := \int_{|\xi|<1/t}\left[|\xi|^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi = I_q^{\beta^-}(f,t) + I_q^{\beta^+}(f,t),$$

where

$$I_q^{\beta^-}(f,t) := \int_{-1/t}^{0}\left[\xi^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi$$

and

$$I_q^{\beta^+}(f,t) := \int_{0}^{1/t}\left[\xi^{2\beta}|\xi|^{(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi, \quad t>0,$$

it is sufficient to show that both $I_q^{\beta^-}(f,t)$ and $I_q^{\beta^+}(f,t)$ are $O\left(t^{-2q\beta}\left(\varphi(t)\right)^q\right)$ as $t\to 0^+$.

It is not hard to see that if

$$g(t) = \int_{|s|<1/t}|s|^{q(1-1/p-1/q)}|\widehat{f}(s)|^q ds, \quad t>0,$$

then

$$I_q^{\beta^-}(f,t) = \int_{-1/t}^{0}s^{2q\beta}g'(s^{-1})ds, \quad \text{and} \quad I_q^{\beta^+}(f,t) = \int_0^{1/t}s^{2q\beta}g'(s^{-1})ds, \quad t>0,$$

Also, we observe that the same reasoning applied in order to prove equality (2.6) fits here and we have

$$\lim_{t\to 0^+} t^{2q\beta} g(t^{-1}) = 0. \tag{2.9}$$

Thus integration by parts and (2.9) imply

$$\begin{aligned} I_q^{\beta^+}(f,t) &= -t^{-2q\beta} g(t) + 2q\beta \int_0^{1/t} s^{2q\beta-1} g(s^{-1}) d\, s \\ &\leq 2q\beta \int_0^{1/t} s^{2q\beta-1} g(s^{-1}) d\, s, \quad t>0. \end{aligned}$$

Since $g((\cdot)^{-1})$ is a nondecreasing function on $(0,\infty)$, it follows

$$I_q^{\beta^+}(f,t) \leq 2q\beta g(t) \int_0^{1/t} s^{2q\beta-1} d\, s = g(t) t^{-2q\beta}, \quad t>0. \tag{2.10}$$

Handling $I_q^{\beta^-}(f,t)$ similarly as above, we reach to

$$I_q^{\beta^-}(f,t) \leq t^{-2q\beta} g(t) + 2q\beta g(t) \int_{-1/t}^{0} (-s)^{2q\beta-1} d\, s = 2t^{-2q\beta} g(t), \quad t>0. \tag{2.11}$$

Combining inequalities (2.10) and (2.11) with our assumption ($g(t) = O\left(\varphi(t)\right)^q$ as $t \to 0^+$), we obtain

$$\|T_t(f) - f\|_p \lesssim \left(\int_{\mathbb{R}} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\,\xi \right)^{1/q} = O\left(\varphi(t)\right), \quad \text{as } t\to 0^+,$$

and therefore $f \in \mathrm{Lip}\,(p,\beta,\varphi)$. $\square$

Corollary 2.3. *If $\varphi \in \Omega_{2\beta}$, then $f \in Lip\,(2,\beta,\varphi)$ if and only if*

$$\left(\int_{t\leq|\xi|\leq 2t} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} = O\left(\varphi(t^{-1})\right), \quad as\ t\to\infty.$$

Remark 2.4. We have defined the class Ω_β by the collection of all $\varphi \in M$ satisfying the following

$$\int_t^\infty \frac{\varphi(u)}{u^{\beta+1}} du \lesssim \frac{\varphi(t)}{t^\beta}. \tag{2.12}$$

Inequality (2.12) is necessary in order to have Theorem 1.3, part (B), true. Let $\varphi \in M$ a function that does not fulfill (2.12), then Theorem 1.3, part (A), still holds true. However, the same does not hold for part (B).

We consider the case $d \geq 2$, similarly we can deal with $d=1$. Let $2 \leq p < \infty$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ in $L^p(\mathbb{R}^d)$ given in terms of the inverse Fourier-Hankel transform of $|\xi|^{-(2\beta+1/p')}$, $\xi \in \mathbb{R}\setminus\{0\}$, that is,

$$f(x) = \frac{\sigma_d}{(2\pi)^d} \int_0^\infty \frac{j_{d/2-1}(xs)}{|x|^{2\beta+1/p'}} s^{d-1} ds,$$

where σ_d is the volume of the unit sphere in $\mathbb{R}^d$ and $j_\alpha(\,\cdot\,)$ denotes the normalize Bessel function (see [11]).

If $\varphi(t) := t^{2\beta}$, then $\varphi \in M$ but φ does not meet condition (2.12). Also, it is clear that

$$\int_{1/t\leq|\xi|} |\widehat{f}(\xi)|^{p'} d\xi = 2 \int_{1/t}^{+\infty} \frac{1}{|\xi|^{2\beta p'+1}} d\xi = O([\varphi(t)]^{p'})$$

or, equivalently,

$$\left(\int_{1/t\le|\xi|\le 2/t} |\widehat{f}(\xi)|^{p'}\,d\xi\right)^{1/p'} = O(\varphi(t)).$$

It means that for $q = p'$, the function f fits into assumptions of Theorem 1.3, part (B). Also, we have

$$t^{2p'\beta}\int_{1/t<|\xi|} |\xi|^{2p'\beta}|\widehat{f}(\xi)|^{p'}\,d\,\xi = t^{2p'\beta}\int_{1/t<|\xi|} |\xi|^{-1}\,d\,\xi = +\infty, \quad \text{for all } t>0,$$

and therefore, $f \notin \mathrm{Lip}(p,\beta,\varphi)$.

3. Proof of Theorem 1.5

In this section, we only work with $d \ge 2$. For $d = 1$, the result was proved in [13] for the usual fractional moduli of smoothness ([5, 22]). If one wants to consider the admissible family of multipliers operators instead the fractional moduli of smoothness, for this case, with small adjustments the same proof presented in [13, p. 1310] fits here.

Proof. of Theorem 1.5.We rewrite the integral in the left-hand side of inequality (1.7), as $I_1 + I_2$, where

$$I_1 := \int_0^{1/2}\int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t}$$

and

$$I_2 := \int_{1/2}^{\infty}\int_{t\le|\xi|\le 2t}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{dt}{t}.$$

Since φ is non-decreasing, for any $t \le |\xi| \le 2t$ it holds $\varphi(t^{-1}/2) \le \varphi(|\xi|^{-1})$ and we have

$$I_1 \lesssim \int_0^{1/2}\frac{1}{[\varphi(t^{-1}/2)]^q}\left(\int_{t\le|\xi|\le 2t}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)\frac{ds}{s}.$$

The change of variables $t = s/2$ leads us to

$$\begin{aligned} I_1 &\lesssim \int_0^{1}\frac{1}{[\varphi(s^{-1})]^q}\left(\int_{s/2\le|\xi|\le s}\left[|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|\right]^q d\xi\right)\frac{dt}{t} \\ &\lesssim \|(\cdot)^{d(1-1/p-1/q)}\widehat{f}(\cdot)\|_q^q. \end{aligned}$$

For I_2, the change of variables $t = s^{-1}/2$ implies

$$I_2 = \int_0^{1}\int_{1/2s\le|\xi|\le 1/s}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\frac{ds}{s}.$$

We note that if $0 < s \leq 1$ and $1/2s \leq |\xi| \leq 1/s$, then $\varphi(s) \leq \varphi(|\xi|^{-1})$ and $s|\xi| \leq 1$. Combining these inequalities to Propositon 2.1, part (A), we have

$$\begin{aligned}
I_2 &\lesssim \int_0^1 \int_{1/2s\leq|\xi|\leq 1/s} \left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})} \right)^q d\xi \frac{ds}{s} \\
&\leq \int_0^1 \int_{1/2s\leq|\xi|\leq 1/s} \left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(s)} \right)^q d\xi \frac{ds}{s} \\
&\lesssim \int_0^1 \frac{\|T_s(f)-f\|_p^q}{[\varphi(s)]^q}\frac{ds}{s} = \int_0^1 \frac{\|T_s(f)-f\|_p^q}{[\varphi(s)]^q}\frac{ds}{s} \leq \|f\|^q_{B^{\varphi}_{p,q}}.
\end{aligned}$$

Thus the first part of the theorem is proved.

To prove the second part, with an application of Proposition 2.1, part (B), we arrive at

$$\frac{\|T_t(f)-f\|_p^q}{[\varphi(t)]^q} \lesssim \int_{\mathbb{R}^d} I_t(\xi)d\xi = \int_0^\infty I_{t,0}(r)r^{(d-1)}dr \quad \text{for all} \quad t>0, \tag{3.1}$$

where

$$I_t(\xi) := \frac{\min(1,t|\xi|)^{2q\beta}}{[\varphi(t)]^q}|\xi|^{dq(1-1/p-1/q)}|\widehat{f}(\xi)|^q, \quad \xi\in\mathbb{R}^d,$$

and $I_{t,0}$ denotes its radial part. Integrating both sides of inequality (3.1) and defining

$$J_1+J_2 := \int_0^1 \left(\int_0^1 I_0^t(r)r^{(d-1)}dr\right)\frac{dt}{t} + \int_0^1 \left(\int_1^{1/t} I_0^t(r)r^{(d-1)}dr\right)\frac{dt}{t}$$

and

$$J_3 := \int_0^1 \left(\int_{1/t}^\infty I_0^t(r)r^{(d-1)}dr\right)\frac{dt}{t},$$

we just need to conclude that $J_i < \infty$, $i=1,2,3$.

In order to estimate J_1, we apply the $(2\beta-\epsilon)$-almost decreasingness property to φ, to obtain

$$\begin{aligned}
J_1 &= \int_0^1 \frac{t^{2q\beta}}{[\varphi(t)]^q}\left[\int_0^1 r^{dq(1-1/p-1/q)+2q\beta}\left(\int_{S^{d-1}}|\widehat{f}(r\omega)|^q d\omega\right)r^{(d-1)}dr\right]\frac{dt}{t} \\
&\lesssim \int_0^1 t^{\epsilon q}\left[\int_0^1 r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}}|\widehat{f}(r\omega)|^q d\omega\right)r^{(d-1)}dr\right]\frac{dt}{t} \\
&\leq \|(\cdot)^{d(1-1/p-1/q)}\widehat{f}(\cdot)\|_q^q\int_0^1 t^{\epsilon q-1}dt < \infty.
\end{aligned}$$

Moving on to the estimate for J_2+J_3, we first write J_2 explicitly as follows

$$J_2 = \int_0^1 \frac{t^{2\beta q}}{[\varphi(t)]^q}\left[\int_1^{1/t} r^{dq(1-1/p-1/q)+2q\beta}\left(\int_{S^{d-1}}|\widehat{f}(r\omega)|^q d\omega\right)r^{(d-1)}dr\right]\frac{ds}{s}.$$

Since φ is $(2\beta-\epsilon)$-almost decreasing, we have

$$\frac{\varphi(r^{-1})}{r^{-2\beta+\epsilon}} \lesssim \frac{\varphi(t)}{t^{2\beta-\epsilon}}, \quad \text{for } 1\leq r\leq 1/t,$$

which leads us to

$$\frac{t^{2\beta}}{\varphi(t)} \lesssim \frac{r^{-2\beta+\epsilon}t^{\epsilon}}{\varphi(r^{-1})}, \quad \text{for } 1\leq r\leq 1/t.$$

Consequently,

$$J_2 \lesssim \int_0^1 t^{\epsilon q}\left[\int_1^{1/t} r^{dq(1-1/p-1/q)+q\epsilon}\left(\int_{S^{d-1}} \frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q} d\omega\right) r^{(d-1)} dr\right]\frac{ds}{s}.$$

Now, the change of variables $t = s^{-1}$ in the right-hand side of inequality above gives us

$$\begin{aligned} J_2 &\lesssim \int_1^{\infty} s^{-q\epsilon}\left[\int_1^{s} r^{dq(1-1/p-1/q)+q\epsilon}\left(\int_{S^{d-1}} \frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q} d\omega\right) r^{(d-1)} dr\right]\frac{ds}{s} \\ &\lesssim \int_1^{\infty} s^{-q\epsilon}\left\{\int_1^{s} r^{q\epsilon-1}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} \frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q} d\omega\right) u^{(d-1)} du\right] dr\right\}\frac{ds}{s}. \end{aligned}$$

For J_3, the change of variable $t^{-1} = s$ implies

$$\begin{aligned} J_3 &= \int_0^1 \frac{1}{[\varphi(t)]^q}\left[\int_{1/t}^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega\right) r^{(d-1)} dr\right]\frac{dt}{t} \\ &= \int_1^{\infty} \frac{1}{[\varphi(s^{-1})]^q}\left[\int_s^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} |\widehat{f}(r\omega)|^q d\omega\right) r^{(d-1)} dr\right]\frac{ds}{s}. \end{aligned}$$

Observing that, for all $1 \le s \le r < \infty$, the inequality $\varphi(r^{-1}) \le \varphi(s^{-1})$ holds, we obtain

$$\begin{aligned} J_3 &\lesssim \int_1^{\infty}\left[\int_s^{\infty} r^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} \frac{|\widehat{f}(r\omega)|^q}{[\varphi(r^{-1})]^q} d\omega\right) r^{(d-1)} dr\right]\frac{ds}{s} \\ &\lesssim \int_1^{\infty}\left\{\int_s^{\infty} r^{-1}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} \frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q} d\omega\right) u^{(d-1)} du\right] dr\right\}\frac{ds}{s}. \end{aligned}$$

Finally, taking in account the estimates for J_2 and J_3, Hardy's inequalities [23, p. 272] imply

$$\begin{aligned} J_2 + J_3 &\lesssim \int_0^{\infty}\left[\int_r^{2r} u^{dq(1-1/p-1/q)}\left(\int_{S^{d-1}} \frac{|\widehat{f}(u\,\omega)|^q}{[\varphi(u^{-1})]^q} d\omega\right) u^{(d-1)} du\right]\frac{dr}{r} \\ &= \int_0^{\infty}\int_{r\le|\xi|\le 2r}\left(\frac{|\xi|^{d(1-1/p-1/q)}|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^q d\xi\,\frac{dr}{r} < \infty, \end{aligned}$$

and $f \in B^{\varphi}_{p,q}(\mathbb{R}^d)$. The theorem is proved. □

We close this section with a direct consequence of Theorem 1.5.

Corollary 3.1. *If $\varphi \in \Omega^q_{2\beta}$, then $f \in B^{\varphi}_{2,2}(\mathbb{R}^d)$ if and only if*

$$\int_0^{\infty}\int_{t\le|\xi|\le 2t}\left(\frac{|\widehat{f}(\xi)|}{\varphi(|\xi|^{-1})}\right)^2 d\xi\,\frac{dt}{t} < \infty.$$

4. $\widehat{GM}^{d}_{p}$ CLASS: RIEMANN-LEBESGUE TYPE INEQUALITY AND FINAL REMARKS

From now on, we will work with GM-classes (general monotone classes) of functions. This concept was firstly introduced in [27], where also the main properties were established.

A locally bounded variation function $g : (0,\infty) \longrightarrow \mathbb{R}$, vanishing at infinity and such that for some $c > 0$ (only depending on g) satisfies

$$\int_t^{\infty} |d\,g(s)| \lesssim \int_{t/c}^{\infty} \frac{|g(s)|}{s} ds < \infty, \quad \text{for all } t > 0, \tag{4.1}$$

is called *general monotone* (see [17, 25, 28]) and we write $g \in GM$. In addition, if g satisfies the following condition

$$\int_0^1 s^{d-1}|g(s)|ds + \int_1^\infty s^{(d-1)/2}|dg(s)| < \infty,$$

for $d \geq 1$ an integer number, then we write $g \in GM^d$ (see [12, 13] and references quoted there for details).

In this section, we write f_0 for the radial part of a given f from $\mathbb{R}^d$. We consider the following collection of functions defined in terms of the inverse Fourier-Hankel transform:

(4.2)

$$\widehat{GM}_p^d := \left\{ f \in L^p(\mathbb{R}^d) : f \text{ is radial}, \ f_0(t) = \frac{\sigma_{d-1}}{(2\pi)^d} \int_0^\infty s^{d-1} F_0(s) j_{d/2-1}(ts) ds, \ F_0 \in GM^d \right\}.$$

For $d \geq 2$ and $1 \leq p < 2d/(d+1)$, the collection above contains all radial positive-definite functions $f(x) = f_0(|x|)$, $x \in \mathbb{R}^d$, such that its Fourier transforms F_0 lies in GM^d. For $d = 1$, the same conclusion holds if $p = 1$ (see [13, p. 1293] and [17] for more examples).

Conditions in Theorem 2.1 can be considerably relaxed if we consider the class $\widehat{GM}_p^d$ as showed in [13, Theorem 4.1]. Following the path designed by the authors in [13], conditions of Theorem 2.1 are extended as follows.

Proposition 4.1. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $f \in \widehat{GM}_p^d$.*

(A) Let $1 < p \leq q < \infty$. If $\widehat{f}$ is nonnegative, then

$$\left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} \widehat{f}(\xi) \right]^q d\xi \right)^{1/q} \lesssim \|T_t(f) - f\|_p. \tag{4.3}$$

(B) Let $1 < q \leq p < \infty$ with $2d/(d+1) < p$. If $|\cdot|^{d(1-1/p-1/q)} \widehat{f}(\cdot) \in L^q$, then

$$\|T_t(f) - f\|_p \lesssim \left(\int_{\mathbb{R}^d} \left[\min(1, t|\xi|)^{2\beta} |\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q}. \tag{4.4}$$

Due to [13, Theorem 4.1, p. 1293] is not hard to see that the basics facts (besides several calculations) needed in order to repeat that proof in here are the following: $[\min(1, t(\cdot))]^{2\beta} F_0(\cdot)$ must be in GM^d, $h := f - T_t(f)$ must be radial and its radial part given by $h_0(s) = [1 - \eta_t(s)] F_0(s)$, $s \in (0, \infty)$. It is clear that all these facts hold true under assumptions made in Proposition 4.1, that is why the details of the proof were omitted.

Proposition 4.2. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$, $1 < p \leq q < \infty$, and $\varphi \in \Omega_{2\beta}$. If $f \in Lip\,(p, \beta, \varphi) \cap \widehat{GM}_p^d$ and $\widehat{f}$ is nonnegative, then*

$$\left(\int_{t \leq |\xi| \leq 2t} \left[|\xi|^{d(1-1/p-1/q)} \widehat{f}(\xi) \right]^q d\xi \right)^{1/q} = O\left(\varphi(t^{-1}) \right). \tag{4.5}$$

Additionally, if $2d/(d+1) < q$, $f \in \widehat{GM}_q^d$, $|\cdot|^{d(1-1/p-1/q)} \widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$ and

$$\left(\int_{t \leq |\xi| \leq 2t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^p d\xi \right)^{1/p} = O\left(\varphi(t^{-1}) \right), \tag{4.6}$$

then $f \in Lip\,(q, \beta, \varphi)$.

The proof of (4.5) is a direct application of Theorem 4.1, part (A). While (4.6) follows from the proof of Theorem 1.3, but instead of applying Proposition 2.1, we need to apply Proposition 4.1, part (B). For $p = q$, the proposition above becomes the following.

Corollary 4.3. *Let $2d/(d+1) < p$ and $f \in \widehat{GM}_p^d$ such that $\widehat{f}$ is non-negative and $|\cdot|^{d(1-2/p)}\widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$. Then $f \in Lip\,(p, \beta, \varphi)$ if and only if*

$$\left(\int_{t\le|\xi|\le 2t}\left[|\xi|^{d(1-2/p)}\widehat{f}(\xi)\right]^p d\xi\right)^{1/p} = O\left(\varphi(t^{-1})\right).$$

Another consequence of Proposition 4.1 is a pointwise estimate for the Fourier transforms of functions in $\widehat{GM}_p^d$ satisfying the Lipschitz condition. The Riemann-Lebesgue type inequality is the content of the next result.

Corollary 4.4. *Let $1 < p \le q < \infty$ and $\varphi \in \Omega_{2\beta}$. If $f \in \widehat{GM}_p^d \cap Lip\,(p, \beta, \varphi)$ is such that $\widehat{f}$ is nonnegative, then*

$$\widehat{f}(\xi) = O\left(|\xi|^{-d/q'}\varphi(|\xi|^{-1})\right), \quad \textit{as } |\xi| \to \infty.$$

Proof. Observe that for $f \in \widehat{GM}_p^d$, if its Fourier transform $\widehat{f}$ is written as F_0, then it satisfies inequality (4.1) and it holds

$$F_0(t) \lesssim \int_{t/c}^{\infty}\frac{F_0(s)}{s}ds, \quad \text{for all } t > 0.$$

An application of Hölder inequality leads us to

$$F_0(t) \lesssim t^{-d/q'}\left(\int_{t/c}^{\infty} s^{qd-d-1}[F_0(s)]^q ds\right)^{1/q}, \quad \text{for all } t > 0.$$

Finally, Proposition (4.2) implies

$$\left(\int_{t/c}^{\infty} s^{qd-d-1}[F_0(s)]^q ds\right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad \text{as } t \to \infty,$$

and the proof follows. □

A version of Theorem 1.5 for $\widehat{GM}_p^d$ class also has a more relaxed condition version.

Proposition 4.5. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$, $\varphi \in \Omega_{2\beta}^q$ and $f \in \widehat{GM}_p^d$.*

(A) Let $1 < p \le q < \infty$. If $f \in B_{p,q}^{\varphi}(\mathbb{R}^d)$ is such that $\widehat{f}$ is nonnegative, then

$$\int_0^{\infty} t^{d(q-1)}\left(\frac{F_0(t)}{\varphi(t^{-1})}\right)^q \frac{dt}{t} < \infty.$$

(B) Let $1 < q \le p < \infty$ with $2d/(d+1) < p$. If $|\cdot|^{d(1-1/p-1/q)}\widehat{f}(\cdot) \in L^q$, and

$$\int_0^{\infty} t^{d(q-1)}\left(\frac{|F_0(t)|}{\varphi(t^{-1})}\right)^q \frac{dt}{t} < \infty,$$

then $f \in B_{p,q}^{\varphi}(\mathbb{R}^d)$.

The proof is a simple adaptation of the proofs of Theorem 1.5 above and Theorem 7.3 in [13, p. 1310]. For $p = q$, we obtain the following.

Corollary 4.6. *Let* $2d/(d+1) < p$, $f \in \widehat{GM}_p^d$ *such that* $\widehat{f}$ *is nonnegative and* $|\cdot|^{d(1-2/p)}\widehat{f}(\cdot) \in L^p(\mathbb{R}^d)$. *Then,* $f \in B^{\varphi}_{p,q}(\mathbb{R}^d)$ *if and only if*

$$\int_0^\infty t^{d(q-1)} \left(\frac{F_0(t)}{\varphi(t^{-1})}\right)^q \frac{dt}{t} < \infty.$$

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