Dynamical Analysis of a Nonlinear Financial System with Computational Algebra Methods

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ABSTRACT

This paper presents algebraic invariants and bifurcation analysis of a nonlinear financial system. We focus on the local stability of the equilibrium points of this system and find the suitable values of parameters for Hopf Bifurcation. Finally, we investigate the invariants to show the general behaviour.

Keywords: Stability, Invariants, Bifurcation Analysis

Doğrusal Olmayan Bir Finansal Sistemin Hesaplamalı Cebir Metotlarıyla Dinamik Analizi

ÖZET

Bu çalışmada doğrusal olmayan bir finansal sistemin çatallanma analizi ve cebirsel değişmezleri sunulmaktadır. Sistemin denge noktalarının kararlılık analizi ve Hopf çatallanmasının göründüğü parametre değerlerine odaklanılmıştır. Son olarak genel davranışın gösterilmesi için değişmezler incelenmiştir.

Anahtar Kelimeler: Kararlılık Analizi, Değişmezler, Çatallanma Analizi

I. INTRODUCTION

A nonlinear financial system is a mathematical system that models the savings, the investment and the financing tools between the accumulator and the party in need of these resources. Economists and researchers from many other disciplines have studied how mathematical methods can be used in finance to make financial decisions and to predict the future. The search for the causes and the results of the financial events has always been difficult to predict. The curiosity to predict the financial events before they occur has led to an increasing interest in the economic models of nonlinear dynamical systems which model financial events in recent years. However, the methods used to examine nonlinear financial systems are either conventional methods based on time series, decision trees and similar discrete approaches or methods applied by linearization of nonlinear systems such as linear stability analysis. Nowadays, it is possible to obtain new results by means of the new studies in computational algebra field and powerful methods such as algebraic invariant surfaces and Hopf bifurcation with our advanced computer technology. Recent powerful methods of computational algebra enable better understanding of the financial processes, forces, balances, risks and crises.

One of the most used nonlinear financial system in modeling financial dynamics is proposed by Ma et al. in 2001 with the chaotic dynamical system [1,2]

$$\dot{x} = z + (y - a)x = P(x, y, z)$$

$$\dot{y} = 1 - by - x^2 = Q(x, y, z) \tag{1}$$

$$\dot{z} = -x - cz = R(x, y, z)$$

where the nonnegative parameters a, b, and c denote the saving amount, the per-investment cost and the elasticity of demands of commercials which are positive, respectively. The variables x, y and z represent the interest rate, the investment demand and the price exponent, respectively.

Many generalizations of system (1) has been studied ever since. In 2012, Yu et al. have constructed the improved chaotic 4D finance system introducing a new variable to the system which demonstrates the average profit margin [3]. They have stabilized the hyperchaotic system to its unstable equilibrium by using speed feedback control and linear feedback control. Tacha et al. have replaced the x^2 term, in the second equation of the system (1), with $|x|$ in 2016, suggesting a more accurate economic point of view [4].

In this work, they have employed the bifurcation diagrams, Lyapunov exponents and phase portraits to observe the route to chaos through the mechanism of period doubling and crisis phenomena. The results have shown that when the parameter a is small the system fluctuates, leading to chaos. Similarly, the parameter c can be kept small to avoid chaos. On the contrary the parameter b can be kept large to avoid chaos otherwise there is not enough investment in the mechanism. Combining the work of Yu et al. and Tacha et al., Hajipour et al. have proposed a new system where they have replaced the x^2 term in the second equation of the improved chaotic 4D finance system in 2018 [5]. This is because of the fact that in the real economy world the interest rate is kept in a small positive value. They have developed an efficient adaptive sliding mode controller technique to stabilize the system.

On the other hand, time delayed generalization of system (1) have been discussed by Chen et al. in 2008 to observe complex dynamics such as periodic, quasi-periodic, and chaotic behaviors. In 2011, the effect of delayed feedbacks on the bifurcations of the system have been extensively investigated by Chen et al. [6]. The small-amplitude periodic solution emerging from a Hopf bifurcation and invariant algebraic surfaces for any parameter values of this system have been recently studied by M. R. Candido et al. by averaging theory [9]. In this paper we focus on stability analysis and invariant algebraic surfaces of this system that will lead to the new researches on this system.

II. LINEAR STABILITY ANALYSIS

Considering that the equations of system (1) are equal to zero, we obtain the following three equilibrium points:

$E_1(0, y_1^*, 0)$, $E_2(x_2^*, y_2^*, z_2^*)$, and $E_3(x_3^*, y_3^*, z_3^*)$ where $y_1^* = \frac{1}{b}$, $x_2^* = -\sqrt{1 - ab - \frac{b}{c}}$, $y_2^* = y_3^* = a + \frac{1}{c}$,

$z_2^* = \sqrt{\frac{1}{c^2} - \frac{b(1+ac)}{c^3}}$, $x_3^* = \sqrt{1 - ab - \frac{b}{c}}$, and $z_3^* = -\sqrt{\frac{1}{c^2} - \frac{b(1+ac)}{c^3}}$.

The Jacobian is obtained as the following matrix by linearizing system (1):

$$J_1 = \begin{pmatrix} y - a & x & 1 \\ -2x & -b & 0 \\ -1 & 0 & -c \end{pmatrix}. \tag{2}$$

At E_1, the eigenvalues of the Jacobian are obtained as $\lambda_1 = -b$ and

$$\lambda_{2,3} = \frac{1}{2b} - \frac{a+c}{2} \pm \frac{\sqrt{(b(a-c-2)-1)(b(a-c+2)-1)}}{2b}. \tag{3}$$

When the real part of all of these eigenvalues are negative, the solution of system (1) at this equilibrium E_1 is locally asymptotically stable. We consider that the characteristic polynomial at E_1 is given by $\lambda^3 + a_{11}\lambda^2 + a_{12}\lambda + a_{13}$ where $a_{11} = a - \frac{1}{b} + b + c$, $a_{12} = -\frac{c}{b} + bc + a(b+c)$ and $a_{13} = b - c + abc$. According to the Routh-Hurwitz criterion, the equilibrium point is asymptotically stable when $a_{11} > 0$, $a_{12} > 0$, $a_{13} - a_{11}a_{12} < 0$. The parameter conditions that satisfy the stability case with $a \geq 0, b \geq 0, c \geq 0$ are given as below:

i. $a = 0$ and $\frac{1}{b} < c$ and $b > 1$ and $b > c$

ii. $b = \frac{1}{a}$ and $\left(\left(a + c > \frac{1}{b} \text{ and } a > 0 \text{ and } a < 1 \right) \text{ or } (c > 0 \text{ and } a \geq 1) \right)$

iii. $a > 0$ and $\left(\left(\frac{1}{a} > b \text{ and } \frac{b}{1-ab} > c \text{ and } a + c > \frac{1}{b} \text{ and } \frac{1}{1+a} < b \right) \text{ or } (ab > 1 \text{ and } c \geq 0) \right)$

For example, when $a = 4$ and $b = 1.2$, system (1) has a stable equilibrium point E_1 $(0, 0.833, 0)$. This system has the following eigenvalues for $k = 6.0278 - 6.33\,c + c^2$:

i. $Eig1_1 = \{-1.2\}$
ii. $Eig1_2 = \{-1.583 - 0.5\,c - 0.5\sqrt{k}\}$
iii. $Eig1_3 = \{\, 0.5(-3.16 - c + \sqrt{k})\}.$

For the case $a = 4, b = 1.2$, the Routh-Hurwitz criterion indicates that system (1) is stable at E_1 for $c > -0.315789$. This result contains both complex conjugate and reals roots. When another subcases are introduced to the system such as $k \geq 0$ and real parts of $\lambda_{2,3}$ are negative, we have the following interval $-0.315789 < c \leq 1.16667$ or $c \geq 5.16667$. In both cases we assume that $c \in \mathbb{R}$.

The real parts of the eigenvalues that indicates the stability at this equilibrium point are plotted for $-10 < c < 10$ in Fig. 1. The first region $(-0.315789 < c \leq 1.16667)$ gives negative real roots, $1.16667 < c < 5.16667$ gives complex conjugate roots therefore real parts of $\lambda_{2,3}$ coincide in this interval in Fig.1., c ≥ 5.16667 gives negative real roots.
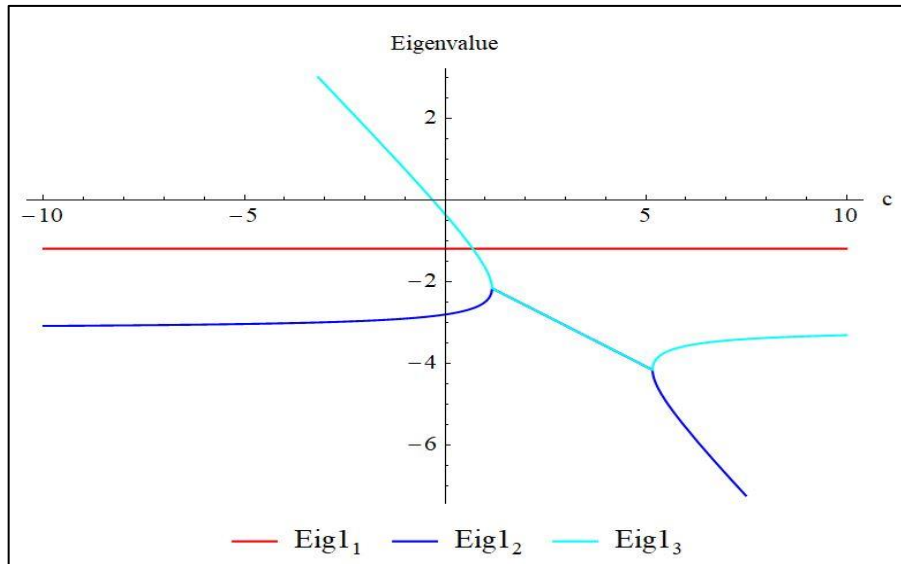
Figure 1. *Real parts of the eigenvalues of System (1) at E_1 depending on c.*
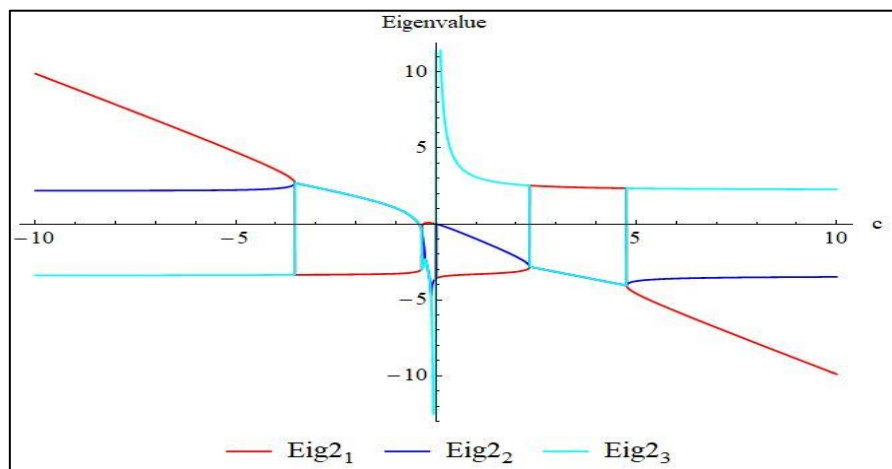
Figure 2. *Real parts of the eigenvalues of System (1) at E_2 depending on c for $-10 < c < 10$.*

It is not possible to determine the eigenvalues of the linearized system at $E_{2,3}$ by linear stability analysis. The characteristic polynomial at $E_{2,3}$ is

$$\lambda^3 + \left(b - \frac{1}{c} + c\right)\lambda^2 + \left(2 + b\left(c - \frac{3}{c} - 2a\right)\right)\lambda - 2(b(1 + ac) - c) = 0. \tag{4}$$

According to the Routh-Hurwitz criterion, all the roots of the characteristic polynomial of the linear system at $E_{2,3}$ have negative real parts when the following case is satisfied

$$\left(b - \frac{1}{c} + c\right)\left(2 + b\left(c - \frac{3}{c} - 2a\right)\right) + 2(b - c + abc) = 0. \tag{5}$$

Hence, system (1) is stable at $E_{2,3}$ when the parameters satisfy the given condition above.

The second equilibrium point E_2 has real and complex conjugate components for $a = 4$ and $b = 1.2$. The real parts of the eigenvalues at this equilibrium point are plotted in Fig. 2 for $-10 < c < 10$ to show the general behavior. When Routh-Hurwitz criteria is applied on this equilibrium point, values that makes the second equilibrium point stable are given as $-0.428976 < c < -0.315789$. The third equilibrium point has the values we omitted here. In Fig.3. the detailed behavior of these eigenvalues indicating this interval is given for $-1 < c < 0$.

For example, when $c = -0.35$, eigenvalues at the second equilibrium point E_2 and the third equilibrium point E_3 are equal and negative $\{-0.1501, -0.582, -2.9749\}$. Both equilibrium points are stable under this condition.
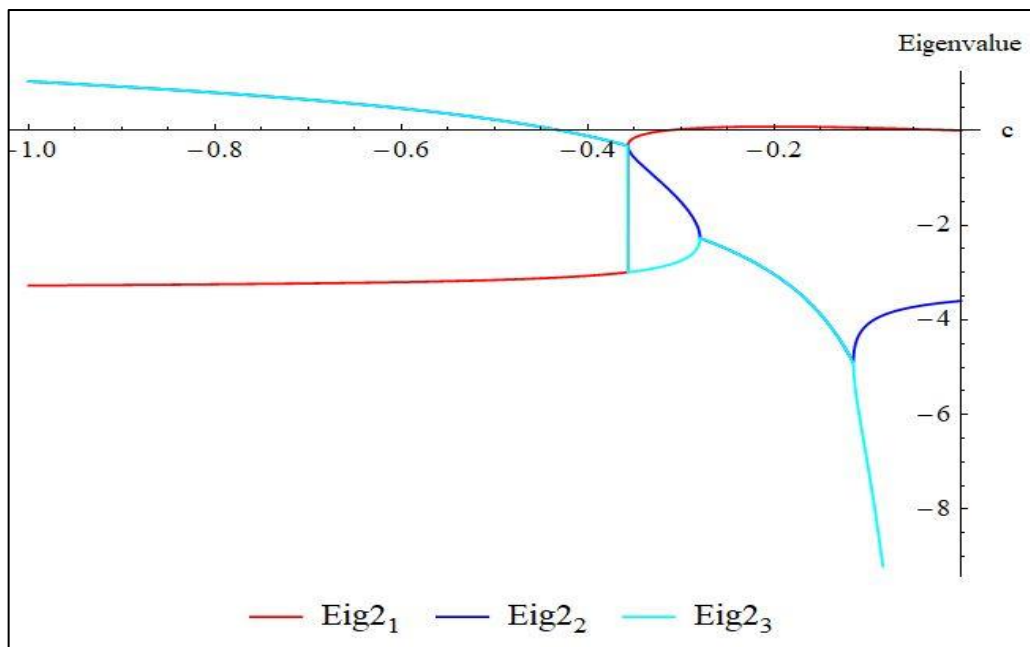
**Figure 3.** *Real parts of the eigenvalues of System (1) at E_2 for $-1 < c < 0$.*

In Figure 4 , 5, and 6, we choose the parameters $a = 3, b = 0.2,$ and $c = 1.61$. The Equilibrium points are $FP_1 = EP_1 =$ (-0.525, 3.62, 0.326), $FP_2 = EP_2 =$ (0,5,0), $FP_3 = EP_3 =$ (0.521, 3.621,-0.326) with the following eigenvectors:

    i. $Eig1_1 = \{-1.187, -0.00066 \pm 0.8647i\}$
    ii. $Eig1_2 = \{1.6976, -1.3076, -0.2\}$
    iii. $Eig1_3 = Eig1_1.$

In order for system (1) to have Hopf bifurcation at E_1, one of the following cases need to be satisfied since a real and a pair of pure imaginary eigenvalues indicate Hopf bifurcation:

    i. $\left(0 \leq a \leq 1 \text{ and } b > \frac{1}{1+a}\right) \text{ or } (a > 1 \text{ and } \frac{1}{1+a} < b < \frac{1}{a-1})$
    ii. $c = \frac{1-ab}{b}.$

When the second case is reorganized as $a = \frac{1-bc}{b}$, E_1 has the following eigenvalues $\{-b, \pm\sqrt{-1 + c^2}\}$. To observe Hopf bifurcation case in a numerical example, we choose $a = 4, b = 1.2,$ and $c = -3.166$. We find the eigenvalues at this equilibrium point $E_1 = \{3.00463, -3.00463, -1.2\}$, at E_2 and E_3 are $\{-3.347, 2.49 \pm 0.474i\}$.
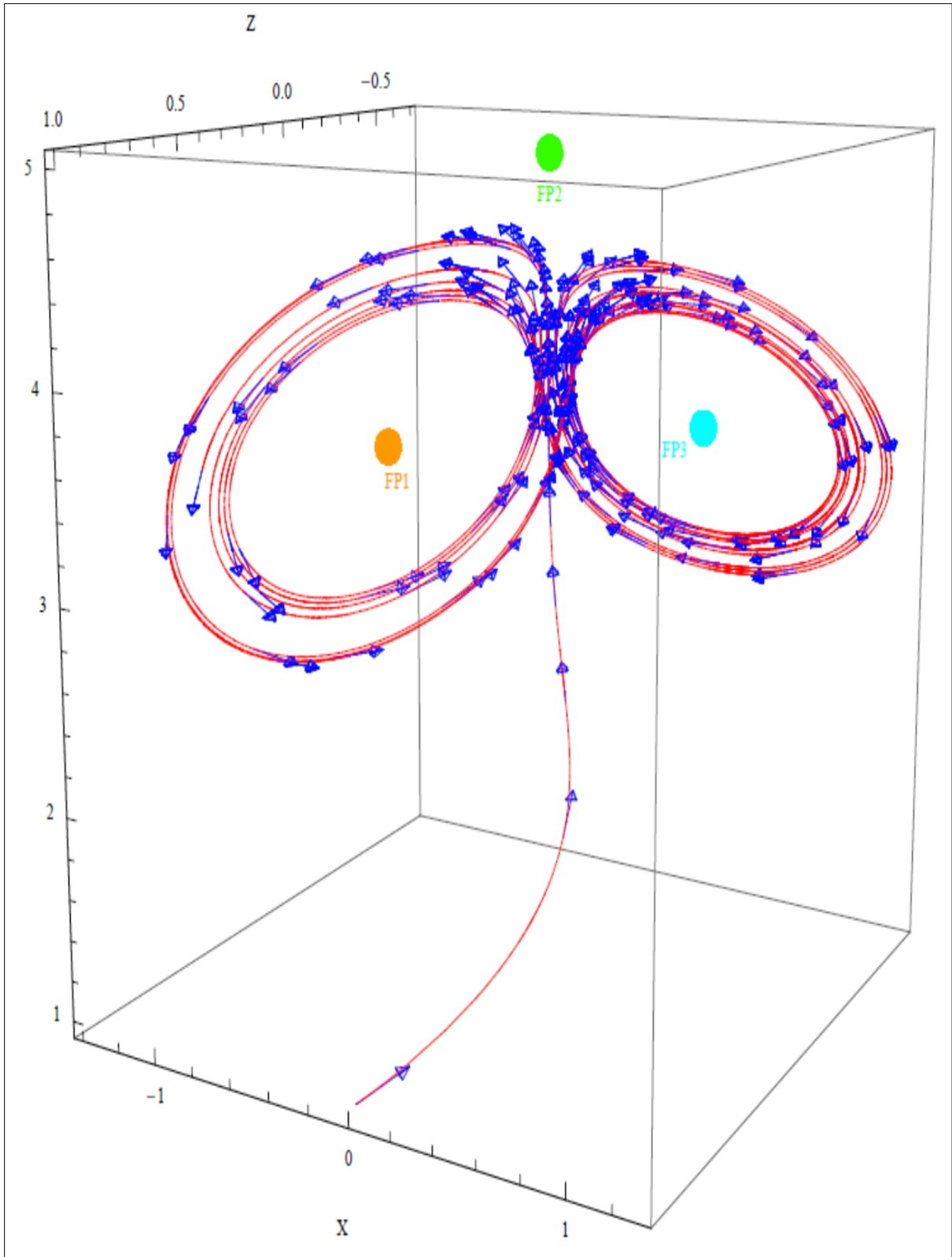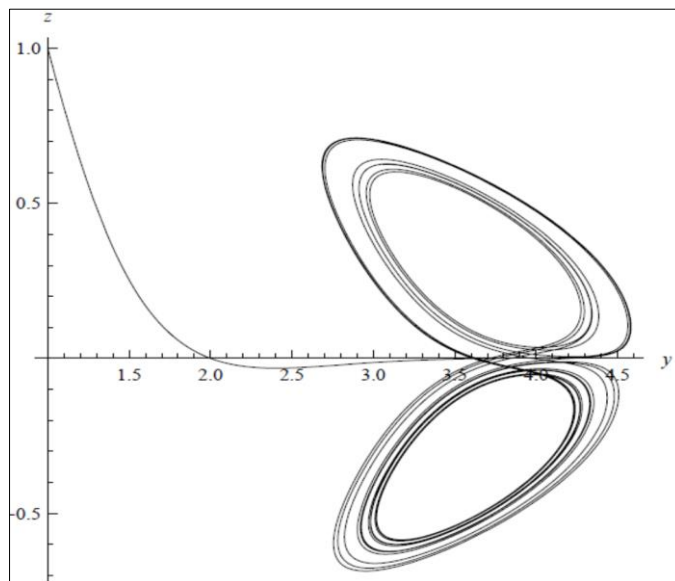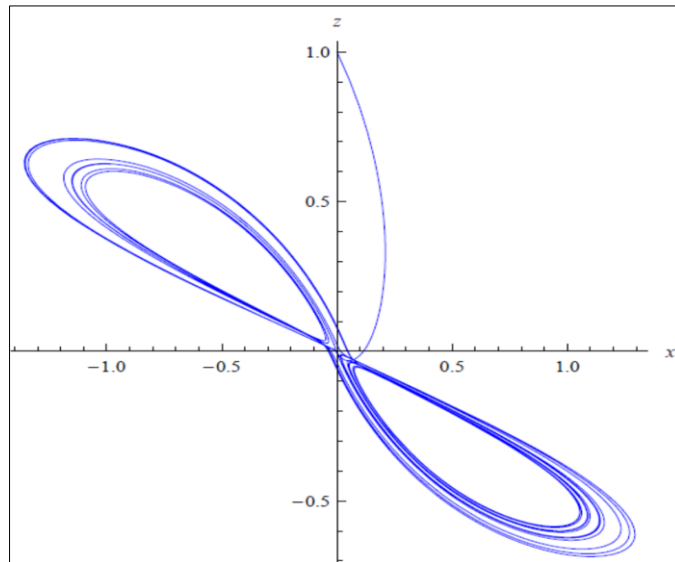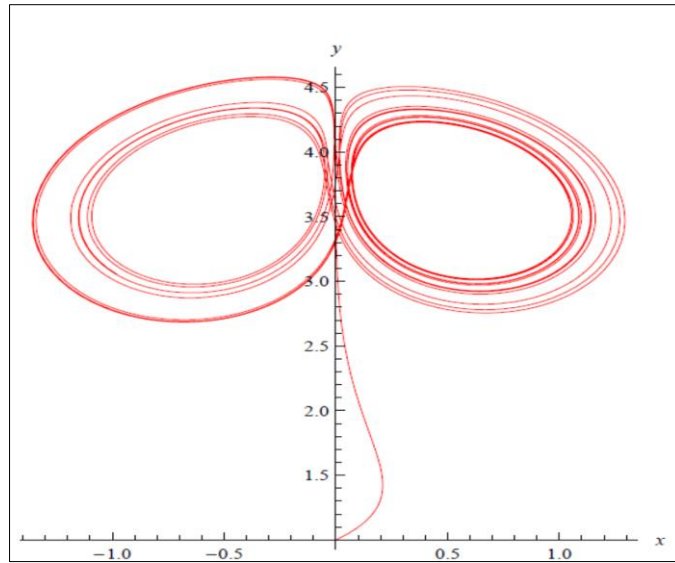
Fig. 4. *3D State Space Trajectory of System (1) for* $a = 3, b = 0.2,$ *and* $c = 1.61$ *where FP defines the equilibrium points.*
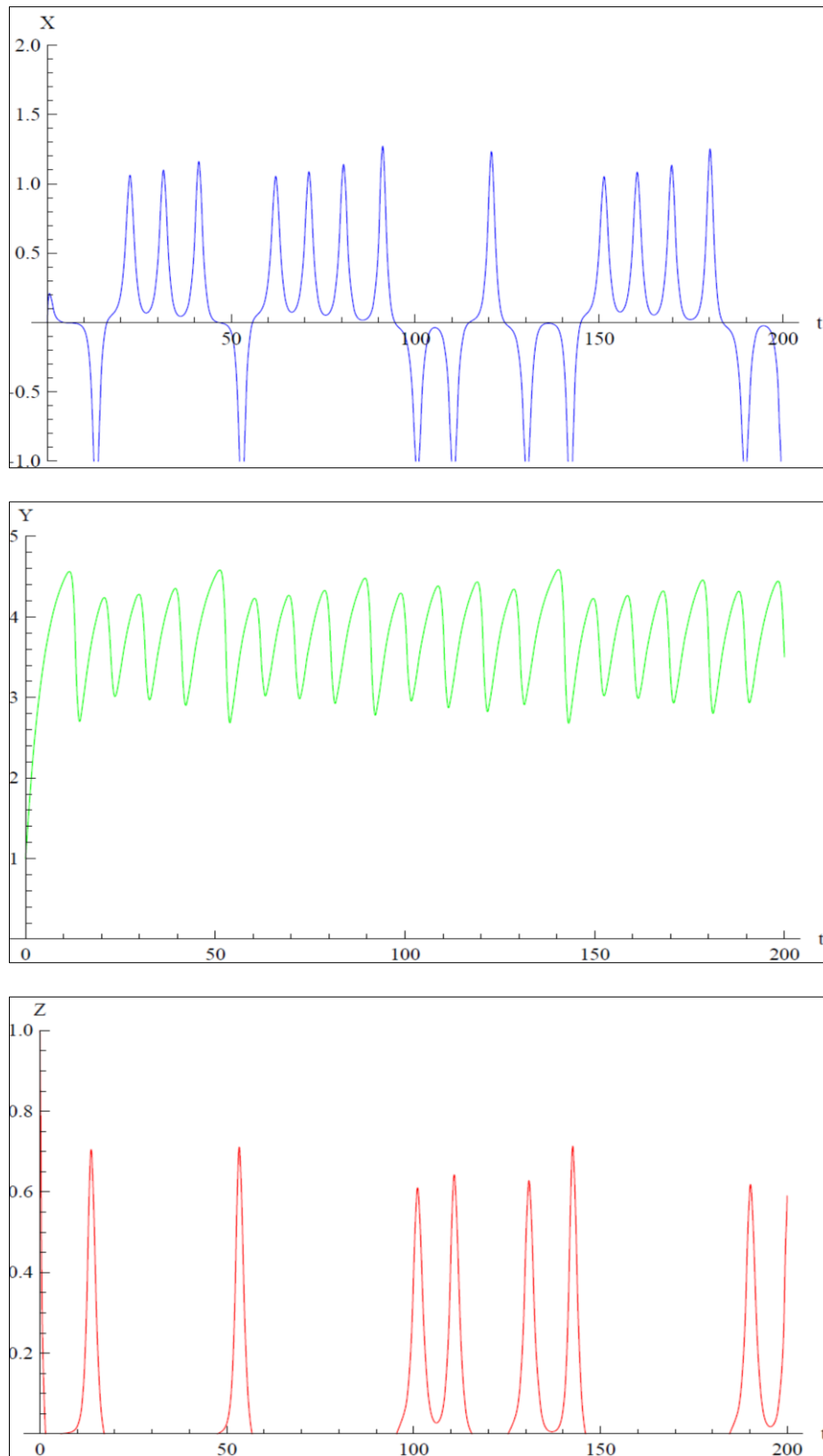
Fig. 5. *Time vs. x, y, and z for a = 3, b = 0.2, and c = 1.61.*

Fig. 6. *2D plot of x, y, and z vs t for $a = 3, b = 0.2$, and $c = 1.61$.*

III. ALGEBRAIC INVARIANT SURFACES

The study of algebraic invariants is a powerful method of computational algebra that can be used to examine the system when methods such as linear stability analysis do not yield results similar to the situation for the equilibrium points $E_{2,3}$ in the nonlinear financial system.

The algebraic invariants are polynomial varieties of the system which remain invariant under given transformations. Hence, it is possible to simplify the system with transformations according to these invariants and to solve the simplified system [7,8]. The simplified system according to the algebraic invariant will then have the same characteristic properties with the original system. So, it will be possible to find the characteristics of the original system.

Let $\dot{x} = P(x, y, z), \dot{y} = Q(x, y, z), \dot{z} = R(x, y, z)$ be a polynomial system of differential equations.

Definition: A polynomial $I(x, y, z)$ is called an algebraic partial integral of system if there exists a polynomial $K(x, y, z)$ such that $XI = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial z}R = KI$ where X is the vector field associated to the given polynomial differential system whose components are P, Q, and R, $K(x, y, z)$ is called a cofactor of $I(x, y, z)$ and the degree of $K(x, y, z)$ is at most $m - 1$ since the degree of system is $m = max(deg(P), deg(Q), deg(R))$. $I(x, y, z)$ is an algebraic invariant surface of the system if and only if $I(x, y, z)$ is an algebraic partial integral of the system. For this reason, mostly in the literature $I(x, y, z)$ is referred as an algebraic invariant surface[7].

System (1) has no algebraic invariants of degree one (Please see the Theorem 3 and Proposition 1 in the publication of M. R. Candido et. Al. [9]). In order to determine the algebraic invariants of degree two, we consider the quadratic form $I = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz$ as an algebraic invariant surface of second degree and $K = s_0 + s_1x + s_2y + s_3z$ as the cofactor corresponding to this invariant.

Solving the equation $\frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial z}R = KI$ according to x, y, z, leads to the following set of polynomials

$I_1 = a_{12} - a_{10}s_0,$

$I_2 = -a\,a_{11} - a_{13} + a_{17} - a_{11}s_0 - a_{10}s_1,$

$I_3 = -a_{12} - 2aa_{14} - a_{18} - a_{14}s_0 - a_{11}s_1,$

$I_4 = -a_{17} - a_{14}s_1,$

$I_5 = -a_{15}s_2,$

$I_6 = 2a_{15} - a_{12}b - a_{12}s_0 - a_{10}s_2,$

$I_7 = a_{11} - aa_{17} - a_{19} - a_{17}b - a_{17}s_0 - a_{12}s_1 - a_{11}s_2,$

$I_8 = -2a_{15}b - a_{15}s_0 - a_{12}s_2,$

$I_9 = 2a_{14} - 2a_{15} - a_{17}s_1 - a_{14}s_2,$

$I_{10} = a_{17} - a_{15}s_1 - a_{17}s_2,$

$$I_{11} = -a_{16}s_3,$$

$$I_{12} = a_{11} + a_{19} - a_{13}c - a_{13}s_0 - a_{10}s_3,$$

$$I_{13} = 2a_{14} - 2a_{16} - aa_{18} - a_{18}c - a_{18}s_0 - a_{13}s_1 - a_{11}s_3,$$

$$I_{14} = a_{17} - a_{19}b - a_{19}c - a_{19}s_0 - a_{13}s_2 - a_{12}s_3,$$

$$I_{15} = a_{18} - 2a_{16}c - a_{16}s_0 - a_{13}s_3,$$

$$I_{16} = -a_{19} - a_{18}s_1 - a_{14}s_3,$$

$$I_{17} = -a_{19}s_2 - a_{15}s_3,$$

$$I_{18} = a_{18} - a_{19}s_1 - a_{18}s_2.$$

By equalizing the right hand sides to zero and eliminating the coefficients of these polynomials, we find a condition for the parameters of system (1) which is $b = c$ and $c(a - c) = 1$. Under this condition system (1) has only one algebraic invariant surface that is $I^* = \frac{1}{c^2} + x^2 - \frac{2y}{c} + y^2 + z^2$ with the corresponding cofactor $K^* = -2c$.

According to this quadratic invariant the following transformation can be used to simplify the system. By solving $I^* = 0$ for x then we obtain $x = \pm \frac{\sqrt{(2cy - c^2(y^2 + z^2) - 1)}}{c}$. When $x \to x + \sqrt{(2cy - c^2(y^2 + z^2) - 1)}$ is transform to system (1) and moved to the origin then we have the following system by computer algebra routine

$$\dot{x} = -x(1 - \frac{1}{c} + c + y + \frac{1}{c(cy - 1)^2 + c^3 z^2})$$

$$\dot{y} = \frac{1}{c^2}(1 - c^3 y + c^2(1 - x^2 + y^2 + z^2) - 2c(y + x\sqrt{2cy - 1 - c^2(y^2 + z^2)})) \tag{6}$$

$$\dot{z} = -(x + cz) + \frac{\sqrt{2cy - 1 - c^2(y^2 + z^2)}}{c}.$$

The Jacobian matrix of the linearized transformed system is $J2 = \begin{pmatrix} J2_{11} & J2_{12} & J2_{13} \\ J2_{21} & J2_{22} & J2_{23} \\ -1 & J2_{32} & J2_{33} \end{pmatrix}$ where

$$J2_{11} = -1 + \frac{1}{c} - c - y - \frac{1}{c(cy - 1)^2 + c^3 z^2},$$

$$J2_{12} = -x(1 + \frac{2 - 2cy}{((cy - 1)^2 + c^2 z^2)^2},$$

$$J2_{13} = \frac{2cxz}{((cy - 1)^2 + c^2 z^2)^2},$$

$$J2_{21} = -\frac{2}{c}\left(cx + \sqrt{-1 + 2cy - c^2(y^2 + z^2)}\right),$$

$$J2_{22} = -\frac{1}{c^2}(c^3 - 2c^2y + 2c\left(1 + \frac{cx(cy-1)}{(cy-1)^2 + c^2 z^2}\right),$$

$$J2_{23=}\frac{1}{(cy-1)^2+c^2 z^2}(2z(1 + c\left(y(cy-2) + cz^2 - x\sqrt{-1 + 2cy - c^2(y^2 + z^2)}\right)),$$

$$J2_{32} = \frac{(1-cy)\sqrt{-1 + 2cy - c^2(y^2 + z^2)}}{(cy-1)^2 + c^2 z^2},$$

$$J2_{33} = \frac{c((cy-1)^2 + c^2z^2 + z\sqrt{-1 + 2cy - c^2(y^2 + z^2)})}{(cy-1)^2 + c^2 z^2}$$

Solving $\dot{x} = \dot{y} = \dot{z} = 0$ yields the term $2cy - 1 - c^2(y^2 + z^2)$ must be negative. Hence, the transformed system is in the complex space and its singular points are all complex. Please see papers that explains the calculation for calculating invariant algebraic surfaces and Hopf Bifurcation arising from that surfaces on two dimension to three dimension (Please see Definition 1, Theorem 1 (Elimination) and Theorem 2 by Kusbeyzi Aybar et. al.[8]) and difficulties arising in complex domains have been extensively investigated for degree one by averaging theory by M. R. Candido et. al. (See Theorem 5[9]).

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