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Estimation of a Parameter of Morgenstern Type Bivariate Lindley Distribution by Ranked Set Sampling

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Abstract: In this work, we have discussed the problem of estimation of a parameter of Morgenstern type bivariate Lindley distribution (MTBLDD) using ranked set sampling. We have proposed two estimators, namely an unbiased estimator based on the Stoke's ranked set sample and the best linear unbiased estimator (BLUE) based on the Stoke's ranked set sample. The efficiencies of the BLUE with respect to the unbiased estimator is also evaluated in this work.

Key words: Ranked set sampling; Best linear unbiased estimator; Morgenstern type bivariate Lindley distribution.
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1. Introduction

When we look at the model corresponding to a bivariate data set, one method of constructing bivariate distribution is Morgenstern approach. Morgenstern Family of Distributions (MFD) are constructed by using specified univariate distribution functions $F_U(u)$ and $F_V(v)$ of random variables U and V respectively and a dependence parameter ϕ. A bivariate random vector (U,V) is said to follow a Morgenstern type bivariate distribution if its cumulative distribution function (cdf) $F(u,v)$ can be written in the form (see, [14]),

$$F(u,v) = F_U(u)F_V(v)\{1+\phi[1-F_U(u)][1-F_V(v)]\}, -1 \leq \phi \leq 1, \tag{1.1}$$

where $F_U(u)$ and $F_V(v)$ are the univariate distribution functions of the random variables U and V respectively. The class of all bivariate distributions with cdf $F(u,v)$ holding a representation of the form (1.1) is called in the literature as Morgenstern Family of Distributions (MFD).
The joint probability density function (pdf) corresponding to the cdf defined in (1.1) is given by,

$$f(u,v) = f_U(u)f_V(v)\{1+\phi[1-2F_U(u)][1-2F_V(v)]\}, -1 \leq \phi \leq 1, \tag{1.2}$$

where $f_U(u)$ and $f_V(v)$ are the pdf's corresponding to the marginal random variables U and V respectively.

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Finite mixture distributions occur in a variety of applications which varies from the length distribution of fish to the content of DNA in the nuclei of liver cells. It provides a mathematical based approach in statistical modelling of a wide variety of random phenomena. For flexible method of modelling, finite mixture models have received an increasing attention over the years from practical and theoretical point of view. Application of mixture models spread over astronomy, biology, genetics, medicine, psychiatry, economics, engineering, marketing and other fields in the biological, physical and social sciences. There is a vast literature available on finite mixture models. For example, see, [3], [21] and [12]. Among the various mixture models available in the literature the Lindley distribution (LD) introduced by [9] is most popular one, whose probability density function (pdf) is obtained by mixing the densities exponential (θ) and gamma $(2, \theta)$ with mixing probabilities $\frac{\theta}{1+\theta}$ and $\frac{1}{1+\theta}$ respectively. Thus the pdf of one parameter Lindley distribution is given by

$$h(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x > 0,\ \theta > 0, \tag{1.3}$$

where θ is the scale parameter.
Although the Lindley distribution has drawn little attention in the statistical literature over the great popularity of the well known exponential distribution. [4] studied various properties of Lindley distribution and proved that the advantages of Lindley distribution compared to the exponential distribution using real life data sets. After the work of [4], Lindley distribution has gained momentum in a theoretical perspective as well as in terms of its practical applications. [23] have obtained a generalized Lindley distribution and elucidated its various properties and applications. [5] and [15] have recently proposed two parameter extensions of the Lindley distribution named as the generalized Lindley and power Lindley distributions respectively. A discrete form of Lindley distribution was introduced by [6] by using descretizing phenomenon. [16] introduced gamma Lindley distribution and studied its important properties. Location parameter extension of Lindley distribution is extensively discussed by [13]. [18] proposed an extended version of generalized Lindley distribution and showed that it includes all the existing Lindley models. Another extension of generalized Lindley distribution is elaborately elucidated by [7]. Also in (2017), [10] introduced generalized Stacy-Lindley mixture distribution and thoroughly studied its reliability properties. But in the available literature, we have not seen bivariate Lindley models except the work of [23] and [22]. Hence in this work, we introduce a new bivariate Lindley model using morgenstern approach and discussed its estimation problem using ranked set sampling.
The ranked set sampling (RSS) scheme was first developed by [11] as a process of increasing the precision of the sample mean as an estimator of the population mean. McIntyre's idea of ranking is possible whenever it can be done easily by a judgement method. For a detailed discussion on the theory and applications of ranked set sampling (see, [2]). Basically the procedure involves choosing n sets of units, each of size n, and ordering the units of each of the set by judgement method or by applying some inexpensive method, without making actual measurement on the units. Then the unit ranked as one from the 1^{st} set is actually measured, the unit ranked as two from the 2^{nd} set is measured. The process continuous in this way until the unit ranked as n from the n^{th} set is measured. Then the observations obtained under the afore mentioned criterion is known as ranked set sample.
In some practical problems, variable of primary interest say V, is difficult or expensive to measure, but an auxiliary variable U related with V is easily measurable and can be ordered exactly. In this case, [19] developed another scheme of ranked set sampling, which is as follows: Choose n independent bivariate sets, each of size n. In the first set of size n, the V variate associated with smallest ordered U is measured, in the second set of size n, the V variate associated with the second smallest, U is measured. This process is continued until the V associated with the largest U from the n^{th} set is measured. The measurements on the V variate of the resulting new set of n

units chosen by the above method gives a ranked set sample as suggested by [19]. If $U_{(r)r}$ is the observation measured on the auxiliary variable U from the unit chosen from the r^{th} set, then we write $V_{[r]r}$ to denote the corresponding measurement made on the study variable V on this unit so that $V_{[r]r}, r = 1,2,\cdots,n$ form the ranked set sample. Clearly $V_{[r]r}$ is the concomitant of the r^{th} order statistic (r^{th} induced order statistic) arising from the r^{th} sample.
[20] discussed the estimation of parameters of location-scale family of distributions using RSS. [8] have obtained the best linear unbiased estimators of location and scale parameters of exponential using ranked set sample. [19] has suggested the ranked set sample mean as an estimator for the mean of the study variate V, when an auxiliary variable U is used for ranking the sample units, under the assumption that (U,V) follows a bivariate normal distribution. [1] have improved the estimator of [19] by deriving the Best Linear Unbiased Estimator (BLUE) of the mean of the study variate V, based on ranked set sample obtained on the study variate V.
Hence in this study, our aim is to consider a bivariate version of one parameter Lindley distribution which is a member of well known Morgenstern family introduced by [14], called it as Morgenstern type bivariate Lindley distribution (MTBLDD). Here we proposed two estimators for σ_2 involved in MTBLDD based on the ranked set sample observations obtained as per [19] ranked set sampling procedure. The efficiency comparison of the proposed BLUE with unbiased estimator of σ_2 are also considered in this paper.

2. Morgenstern Type Bivariate Lindley Distribution

A bivariate random variable (U,V) is said to possess a Morgenstern type bivariate Lindley distribution (MTBLDD), if its probability density function (pdf) is given by,

$$f(u,v) = \begin{cases} \frac{1}{2\sigma_1}\left(1+\frac{u}{\sigma_1}\right)e^{-\frac{u}{\sigma_1}}\frac{1}{2\sigma_2}\left(1+\frac{v}{\sigma_2}\right)e^{-\frac{v}{\sigma_2}}\left\{1+\phi\left[2e^{-\frac{u}{\sigma_1}}\left(1+\frac{u}{2\sigma_1}\right)-1\right]\right. \\ \left.\left[2e^{-\frac{v}{\sigma_2}}\left(1+\frac{v}{2\sigma_2}\right)-1\right]\right\}, \quad if \quad u>0, v>0; -1\le\phi\le 1 \\ 0, \qquad otherwise. \end{cases} \tag{2.1}$$

The corresponding cumulative distribution function (cdf) is given by

$$\begin{aligned} F(u,v) = &\left\{1-e^{-\frac{u}{\sigma_1}}\left(1+\frac{u}{2\sigma_1}\right)\right\}\left\{1-e^{-\frac{v}{\sigma_2}}\left(1+\frac{v}{2\sigma_2}\right)\right\} \\ &\times\left\{1+\phi e^{-\frac{u}{\sigma_1}-\frac{v}{\sigma_2}}\left(1+\frac{u}{2\sigma_1}\right)\left(1+\frac{v}{2\sigma_2}\right)\right\}. \end{aligned} \tag{2.2}$$

Clearly the marginal distributions of U and V variables are univariate Lindley distributions with probability density functions (pdf's) are respectively given by

$$f_U(u) = \begin{cases} \frac{1}{2\sigma_1}\left(1+\frac{u}{\sigma_1}\right)e^{-\frac{u}{\sigma_1}}, & if \quad u>0, \sigma_1>0; \\ 0, & otherwise. \end{cases}$$

and

$$f_V(v) = \begin{cases} \frac{1}{2\sigma_2}\left(1+\frac{v}{\sigma_2}\right)e^{-\frac{v}{\sigma_2}}, & if \quad v>0, \sigma_2>0; \\ 0, & otherwise. \end{cases}$$

Clearly, $E(U) = \frac{2}{3}\sigma_1$, $Var(U) = \frac{7}{9}\sigma_1^2$, $E(V) = \frac{2}{3}\sigma_2$, $Var(V) = \frac{7}{9}\sigma_2^2$ and the correlation coefficient ρ between U and V is obtained as $\rho = \frac{121}{448}\phi$.

2.1. An unbiased estimator of σ_2 using Stoke's RSS

Suppose the bivariate random vector (U,V) follows a MTBLDD with pdf given in (2.1). Select a ranked set sample as per [19] ranked set sampling scheme. Let $U_{(r)r}$ be the observation obtained on the auxiliary variate U in the r^{th} unit of the ranked set sample and let $V_{[r]r}$ be the measurement made on the variate related with $U_{(r)r}, r=1,2,\cdots,n$. Clearly $V_{[r]r}$ is the r^{th} induced order statistic of a random sample of size n arising from MTBLDD. Using the results of [17], we obtain the pdf of $V_{[r]r}, r=1,2,\cdots,n$, and is given by

$$f_{[r]r}(v) = \frac{1}{2\sigma_2}\left(1+\frac{v}{\sigma_2}\right)e^{-\frac{v}{\sigma_2}}\left\{1+\phi\frac{(n-2r+1)}{(n+1)}\left(2e^{-\frac{v}{\sigma_2}}\left\{1+\frac{v}{2\sigma_2}\right\}-1\right)\right\}.$$

The mean and variance of $V_{[r]r}$ for $r=1,2,\cdots,n$, is obtained as

$$E[V_{[r]r}] = \sigma_2\left[\frac{3}{2}-\frac{11}{16}\phi\frac{(n-2r+1)}{(n+1)}\right] \tag{2.3}$$

and

$$Var[V_{[r]r}] = \sigma_2^2\left[\frac{7}{4}-\frac{3}{4}\phi\frac{(n-2r+1)}{(n+1)}-\frac{121}{256}\phi^2\frac{(n-2r+1)^2}{(n+1)^2}\right]. \tag{2.4}$$

Since $V_{[r]r}$ and $V_{[s]s}$ for $r\neq s$ are arising from two independent samples, we obtain

$$Cov[V_{[r]r},V_{[s]s}] = 0, r\neq s. \tag{2.5}$$

Next, we derive an unbiased estimator of σ_2 and its variance using the ranked set sample observations $V_{[r]r}$ for $r=1,2,\cdots,n$, on the variable V of primary interest and are given by the following theorem.

THEOREM 1. *Let (U,V) follows a MTBLDD with pdf given by (2.1). Let $V_{[r]r}$, $r=1,2,\cdots,n$ be the ranked set sample observations on a study variate V generated out of ranking made on an auxiliary variate U. Then the ranked set sample mean given by,*

$$\sigma_2^* = \frac{2}{3n}\sum_{r=1}^{n}V_{[r]r} \tag{2.6}$$

is an unbiased estimator of σ_2 and its variance is given by

$$Var[\sigma_2^*] = \frac{4}{9n}\left[\frac{7}{4}-\frac{121}{256n}\phi^2\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right]\sigma_2^2. \tag{2.7}$$

PROOF.

$$\begin{aligned} E[\sigma_2^*] &= \frac{2}{3n}\sum_{r=1}^{n}E[V_{[r]r}] \\ &= \frac{2}{3n}\sum_{r=1}^{n}\left[\frac{3}{2}-\frac{11}{16}\phi\frac{(n-2r+1)}{(n+1)}\right]\sigma_2. \end{aligned} \tag{2.8}$$

Using the result,

$$\sum_{r=1}^{n}(n-2r+1) = 0. \tag{2.9}$$

Applying (2.9) in (2.8) we get,

$$E[\sigma_2^*] = \sigma_2.$$

Therefore, σ_2^* is an unbiased estimator of σ_2. The variance of σ_2^* is given by,

$$Var[\sigma_2^*] = \frac{4}{9n^2}\sum_{r=1}^{n} Var[V_{[r]r}]. \tag{2.10}$$

Applying (2.4) and (2.9) in (2.10), we get

$$Var[\sigma_2^*] = \frac{4}{9n}\left[\frac{7}{4} - \frac{121}{256n}\phi^2\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right]\sigma_2^2.$$

Hence the proof.

Remark 1. For given value of $\phi \in (0,1]$, once the variance of σ_2^* is evaluated, then this variance is equal to the variance of σ_2^* for $-\phi$ because the variance given in (2.7) depends only on ϕ by a term containing ϕ^2 only.

2.2. Best linear unbiased estimator of σ_2 using Stoke's RSS

In this section, we derive a best linear unbiased estimator of σ_2 provided the dependence parameter ϕ is known.
Suppose $V_{[r]r}$ for $r = 1, 2, \cdots, n$, are the ranked set sample observation generated from (2.1) as per Stokes ranked set sampling scheme. Let

$$\zeta_{r,n} = \frac{3}{2} - \frac{11}{16}\phi\frac{(n-2r+1)}{(n+1)} \tag{2.11}$$

and

$$g_{r,r,n} = \frac{7}{4} - \frac{3}{4}\phi\frac{(n-2r+1)}{(n+1)} - \frac{121}{256}\phi^2\frac{(n-2r+1)^2}{(n+1)^2}. \tag{2.12}$$

Using (2.3) and (2.4), we get

$$E[V_{[r]r}] = \sigma_2\zeta_{r,n}, \ 1 \le r \le n \tag{2.13}$$

and

$$Var[V_{[r]r}] = \sigma_2^2 g_{r,r,n}, \ 1 \le r \le n. \tag{2.14}$$

Also, we have from (2.5)

$$Cov[V_{[r]r}, V_{[s]s}] = 0, r, s = 1, 2, \cdots, n \text{ and } r \neq s. \tag{2.15}$$

Let $\mathbf{V}_{[n]} = (V_{[1]1}, V_{[2]2}, \cdots, V_{[n]n})'$ denote the column vector of n ranked set sample observations. Then from (2.13), (2.14) and (2.15), we can write,

$$E[\mathbf{V}_{[n]}] = \sigma_2\zeta \tag{2.16}$$

and the dispersion matrix of $\mathbf{V}_{[n]}$,

$$D[\mathbf{V}_{[n]}] = \sigma_2^2\mathbf{G}, \tag{2.17}$$

where $\zeta = (\zeta_{1,n}, \zeta_{2,n}, \cdots, \zeta_{n,n})'$ and $\mathbf{G} = diag(g_{1,1,n}, g_{2,2,n}, \cdots, g_{n,n,n})$, where $\zeta_{r,n}$ and $g_{r,r,n}$ for $r = 1, 2, \cdots, n$ are respectively given by equations (2.11) and (2.12). If ϕ contained in ζ and $\mathbf{G}$ are known then (2.16) and (2.17) together defines a generalized Gauss-Markov setup and then the BLUE of σ_2 is given by

$$\tilde{\sigma_2} = (\zeta'\mathbf{G}^{-1}\zeta)^{-1}\zeta'\mathbf{G}^{-1}\mathbf{V}_{[n]}$$

and the variance of σ_2 is given by

$$Var(\tilde{\sigma_2}) = \frac{\sigma_2^2}{\zeta'\mathbf{G}^{-1}\zeta}.$$

On simplifying, we get

$$\tilde{\sigma_2} = \frac{\sum_{r=1}^{n} \frac{\zeta_{r,n}}{g_{r,r,n}}}{\sum_{r=1}^{n} \frac{\zeta_{r,n}^2}{g_{r,r,n}}} V_{[r]r} \tag{2.18}$$

and

$$Var(\tilde{\sigma_2}) = \frac{\sigma_2^2}{\sum_{r=1}^{n} \frac{\zeta_{r,n}^2}{g_{r,r,n}}}. \tag{2.19}$$

From (2.18), we have $\tilde{\sigma_2}$ is a linear functions of the ranked set sample observations $V_{[r]r}, r = 1, 2, \cdots, n$ and hence $\tilde{\sigma_2}$ can be written as $\tilde{\sigma_2} = \sum_{r=1}^{n} a_r V_{[r]r}$, where

$$a_r = \frac{\sum_{r=1}^{n} \frac{\zeta_{r,n}}{g_{r,r,n}}}{\sum_{r=1}^{n} \frac{\zeta_{r,n}^2}{g_{r,r,n}}}, r = 1, 2, \cdots, n.$$

We have evaluated the numerical values of means and variances using the expressions (2.11) and (2.12) respectively for $\phi = 0.25(0.25)0.75$ and for $n = 2(1)9$. Using these values we have evaluated the coefficients of $V_{[r]r}$ in the BLUE $\tilde{\sigma_2}$ and its variances for $\phi = 0.25(0.25)0.75$ and for $n = 2(1)9$ and are given in Table1. Also we have computed the ratio $\frac{Var(\tilde{\sigma}_2)}{Var(\sigma_2^*)}$ to measure the efficiency of our estimator $\tilde{\sigma}_2$ relative to σ_2^* for $n = 2(1)10$ and $\phi = 0.25(0.25)0.75$ and are presented in Table1. From the table, it is clear that, BLUE of σ_2 performs well compared to the unbiased estimator of σ_2, namely σ_2^*.

Remark 2. As in the case of variance of an unbiased estimator given in (2.7), for a given value of $\phi \in (0, 1]$, once the variance of $\tilde{\sigma_2}$ is evaluated, then there is no need to again evaluate the variance of $\tilde{\sigma_2}$ when $\phi = -\phi$. To establish this argument we prove the following theorem.

Theorem 2. *Let (U, V) follows a MTBLDD with pdf given by (2.1). For a given $\phi \in (0, 1]$, $Var[\tilde{\sigma}_2(\phi_0)]$ is the variance of the BLUE $\tilde{\sigma}_2$ of σ_2 involved in MTBLDD, then*

$$Var[\tilde{\sigma}_2(-\phi_0)] = Var[\tilde{\sigma}_2(\phi_0)]. \tag{2.20}$$

Proof. The terms $\zeta_{r,n}$ and $g_{r,r,n}$ defined by (2.11) and (2.12) are functions of ϕ, r and n and hence $\zeta_{r,n}$ and $g_{r,r,n}$ can be denoted as $\zeta_{r,n}(\phi)$ and $g_{r,r,n}(\phi)$ respectively. From (2.11) and (2.12), it is clear that

$$\zeta_{r,n}(\phi) = \zeta_{n-r+1,n}(-\phi),\ 1 \le r \le n \tag{2.21}$$

and

$$g_{r,r,n}(\phi) = g_{n-r+1,n-r+1,n}(-\phi),\ 1 \leq r \leq n. \tag{2.22}$$

Applying the results (2.21) and (2.11) in (2.19), we get

$$\begin{aligned} Var[\tilde{\sigma}_2(\phi)] &= \frac{\sigma_2^2}{\displaystyle\sum_{r=1}^{n} \frac{\zeta_{r,n}^2(\phi)}{g_{r,r,n}(\phi)}} \\ &= \frac{\sigma_2^2}{\displaystyle\sum_{r=1}^{n} \frac{\zeta_{n-r+1,n}^2(-\phi)}{g_{n-r+1,n-r+1,n}(-\phi)}} \\ &= Var[\tilde{\sigma}_2(-\phi)]. \end{aligned}$$

Hence the proof.

Remark 3. Sometimes our assumption about the dependence parameter ϕ involved in (2.1) is unknown. One method of overcoming this situation is to consider a moment type estimator of ϕ. For that, we have evaluated the correlation coefficient between the variables U and V of MTBLDD and is given by $\rho = \frac{121}{448}\phi$. Also we take the sample correlation coefficient between $(U_{(r)r}, V_{[r]r}), r = 1, 2, \cdots, n$ be τ. Further, we have evaluated the moment type estimator of ϕ namely $\hat{\phi}$ which is obtained by equating the population correlation coefficient and the sample correlation coefficient, and is given by

$$\hat{\phi} = \begin{cases} -1, & if \quad \tau < \frac{-121}{448} \\ \frac{448}{121}\tau, & if \quad \frac{-121}{448} \leq \tau \leq \frac{121}{448} \\ 1, & if \quad \tau > \frac{121}{448}. \end{cases} \tag{2.23}$$

TABLE 1. Coefficients of $V_{[r]r}$ in the best linear unbiased estimator $\tilde{\sigma}_2$, $V_1 = \frac{Var(\tilde{\sigma}_2)}{\sigma_2^2}$, $V_2 = \frac{Var(\sigma_2^*)}{\sigma_2^2}$ and the efficiency $e = \frac{V_2}{V_1}$

n	ϕ	Coefficients										V_1	V_2	e
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}			
2	0.25	0.33250	0.33411									0.38816	0.38816	1.00000
	0.50	0.33174	0.33470									0.38596	0.38597	1.00003
	0.75	0.33123	0.33500									0.38231	0.38232	1.00003
3	0.25	0.22171	0.22159	0.22327								0.25853	0.25853	1.00000
	0.50	0.22207	0.21970	0.22460								0.25632	0.25634	1.00008
	0.75	0.22377	0.21652	0.22600								0.25261	0.25269	1.00032
4	0.25	0.16639	0.16596	0.16645	0.16775							0.19378	0.19379	1.00005
	0.50	0.16738	0.16435	0.16259	0.16394							0.19179	0.19182	1.00016
	0.75	0.17036	0.16178	0.16311	0.17116							0.18842	0.18854	1.00064
5	0.25	0.13320	0.13275	0.13283	0.13339	0.13439						0.15497	0.15947	1.00000
	0.50	0.13450	0.13167	0.13131	0.13285	0.13602						0.15319	0.15322	1.00020
	0.75	0.13815	0.13011	0.12872	0.13159	0.13797						0.15017	0.15030	1.00087
6	0.25	0.11106	0.11065	0.11058	0.11082	0.11133	0.11211					0.12911	0.12911	1.00000
	0.50	0.11251	0.11004	0.10922	0.10967	0.11120	0.11371					0.12752	0.12755	1.00024
	0.75	0.11648	0.10933	0.10697	0.10762	0.11059	0.11566					0.12481	0.12494	1.00104
7	0.25	0.09524	0.09488	0.09476	0.09483	0.09510	0.09555	0.09617				0.11064	0.11064	1.00000
	0.50	0.09674	0.09462	0.09366	0.09361	0.09432	0.09570	0.09770				0.10921	0.10924	1.00027
	0.75	0.10085	0.09457	0.09190	0.09151	0.09280	0.09552	0.09960				0.10676	0.10689	1.00122
8	0.25	0.08337	0.08306	0.08291	0.08291	0.08305	0.08332	0.08370	0.08420			0.09679	0.09680	1.00010
	0.50	0.08488	0.08305	0.08208	0.08178	0.08205	0.08281	0.08402	0.08566			0.09550	0.09552	1.00021
	0.75	0.08900	0.08350	0.08081	0.07989	0.08028	0.08173	0.08413	0.08749			0.09327	0.09339	1.00129
9	0.25	0.07413	0.07387	0.07372	0.07368	0.07374	0.07389	0.07414	0.07447	0.07489		0.08603	0.08461	1.00000
	0.50	.07562	0.07404	0.07311	0.07270	0.07272	0.07312	0.07385	0.07491	0.07627		0.08484	0.08486	1.00024
	0.75	0.07970	0.07487	0.07226	0.07109	0.07097	0.07168	0.07311	0.07522	0.07801		0.08280	0.08292	1.00145
10	0.25	0.06674	0.06651	0.06637	0.06630	0.06632	0.06641	0.06657	0.06679	0.06708	0.06743	0.07742	0.07742	1.00000
	0.50	0.06819	0.06682	0.06595	0.06549	0.06537	0.06554	0.06598	0.06667	0.06759	0.06874	0.07632	0.07635	1.00039
	0.75	0.07219	0.06792	0.06545	0.06418	0.06376	0.06401	0.06484	0.06618	0.06803	0.07040	0.07444	0.07456	1.00161

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