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Strongly Far Proximity and Hyperspace Topology

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Abstract: This article introduces strongly far proximity $\underset{\doteq}{\not\delta}$, which is associated with Lodato proximity δ. A main result in this paper is the introduction of a hit-and-miss topology on $\mathrm{CL}(X)$, the hyperspace of nonempty closed subsets of X, based on the strongly far proximity.

Key words: Hit-and-miss topology, Hyperspaces, Proximity, Strongly far.

1. Introduction

This paper introduces the strongly far proximity, which is useful in the study of remote nonempty sets $A \subset \mathrm{int}(E), B$ such that $E \cap B = \emptyset$ and $A \cap B = \emptyset$, *i.e.*, E contains no members in common with B and A resides in the interior of E. Usually, when we talk about proximities, we mean *Efremovič proximities*. Nearness expressions are very useful and also represent a powerful tool because of the relation existing among *Efremovič proximities*, *Weil uniformities* and T_2 compactifications. But sometimes *Efremovič proximities* are too strong. So we want to distinguish between a weaker and a stronger form of proximity. For this reason, we consider at first *Lodato proximity* δ and then, by this, we define a stronger proximity by using the Efremovič property related to proximity.

2. Preliminaries

Recall how a *Lodato proximity* is defined [9–11] (see, also, [12, 14]).

Definition 2.1 *Let X be a nonempty set. A Lodato proximity δ is a relation on $\mathscr{P}(X)$ which satisfies the following properties for all subsets A, B, C of X :*

P0) $A\ \delta\ B \Rightarrow B\ \delta\ A$,

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P1) $A\ \delta\ B \Rightarrow A \neq \emptyset$ *and* $B \neq \emptyset$,

P2) $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$,

P3) $A\ \delta\ (B \cup C) \Leftrightarrow A\ \delta\ B$ *or* $A\ \delta\ C$,

P4) $A\ \delta\ B$ *and* $\{b\}\ \delta\ C$ *for each* $b \in B\ \Rightarrow A\ \delta\ C$.

Further δ *is separated , if*

P5) $\{x\}\ \delta\ \{y\} \Rightarrow x = y$.

When we write $A\ \delta\ B$, we read A *is near to* B and when we write $A \not\delta B$ we read A *is far from* B. A *basic proximity* is one that satisfies $P0) - P3)$. *Lodato proximity* or *LO-proximity* is one of the simplest proximities. We can associate a topology with the space (X, δ) by considering as closed sets the ones that coincide with their own closure, where for a subset A we have

$$\mathrm{cl}A = \{x \in X : x\ \delta\ A\}.$$

This is possible because of the correspondence of Lodato axioms with the well-known Kuratowski closure axioms.

By considering the gap between two sets in a metric space ($d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$ or ∞ if A or B is empty), Efremovič introduced a stronger proximity called *Efremovič proximity* or *EF-proximity.*

Definition 2.2 *An* EF-proximity *[7] is a relation on* $\mathscr{P}(X)$ *which satisfies* $P0)$ *through* $P3)$ *and in addition*

$$A \not\delta B \Rightarrow \exists E \subset X \text{ such that } A \not\delta E \text{ and } (X \setminus E) \not\delta B \text{ EF-property.}$$

A topological space has a compatible EF-proximity if and only if it is a Tychonoff space.

Any proximity δ on X induces a binary relation over the powerset exp X, usually denoted as $\ll_\delta$ and named the *natural strong inclusion associated with* δ, by declaring that A is *strongly included* in B, $A \ll_\delta B$, when A is far from the complement of B, $A \not\delta (X \setminus B)$.

By strong inclusion the *Efremivič property* for δ can be written also as a betweenness property

(EF) If $A \ll_\delta B$, then there exists some C such that $A \ll_\delta C \ll_\delta B$.

A pivotal example of *EF-proximity* is the *Euclidean metric proximity* (denoted by δ_e) in a metric space (X,d) defined by

$$d(A,B) = \inf\{d(a,b) \in \mathbb{R} : a \in A, b \in B\}.$$

$$A\ \delta_e\ B \Leftrightarrow d(A,B) = 0.$$

That is, A and B are either close or far in d, provided A, B are either intersect or asymptotic. In effect, for each natural number n, there is a point a_n in A and a point b_n in B such that $d(a_n, b_n) < \frac{1}{n}$ [2, §2.1, p. 94].

2.1. Hit and Far-Miss Topologies

Let $\mathrm{CL}(X)$ be the hyperspace of all non-empty closed subsets of a space X. *Hit and miss* and *hit and far-miss* topologies on $\mathrm{CL}(X)$ are obtained by the join of two halves. Well-known examples are Vietoris topology [19–22] (see, also, [1, 3–6, 13]) and Fell topology [8]. In this article, we concentrate on an extension of Vietoris based on the strongly far proximity.

Vietoris topology

Let X be an Hausdorff space. The *Vietoris topology* on $\mathrm{CL}(X)$ has as subbase all sets of the form

- $V^- = \{E \in \mathrm{CL}(X) : E \cap V \neq \varnothing\}$, where V is an open subset of X,
- $W^+ = \{C \in \mathrm{CL}(X) : C \subset W\}$, where W is an open subset of X.

The topology ${\tau_V}^-$ generated by the sets of the first form is called **hit part** because, in some sense, the closed sets in this family hit the open sets V. Instead, the topology ${\tau_V}^+$ generated by the sets of the second form is called **miss part**, because the closed sets here miss the closed sets of the form $X \setminus W$.

The Vietoris topology is the join of the two part: $\tau_V = {\tau_V}^- \vee {\tau_V}^+$. It represents the prototype of hit and miss topologies.

The Vietoris topology was modified by Fell. He left the hit part unchanged and in the miss part, ${\tau_F}^+$ instead of taking all open sets W, he took only open subsets with compact complement.

Fell topology: $\tau_F = {\tau_V}^- \vee {\tau_F}^+$

It is possible to consider several generalizations. For example, instead of taking open subsets with compact complement, for the miss part we can look at subsets running in a family of closed sets $\mathscr{B}$. So we define the *hit and miss topology on* $CL(X)$ *associated with* $\mathscr{B}$ as the topology generated by the join of the hit sets A^-, where A runs over all open subsets of X, with the miss sets A^+, where A is once again an open subset of X, but more, whose complement runs in $\mathscr{B}$.

Another kind of generalization concerns the substitution of the inclusion present in the miss part with a strong inclusion associated to a proximity. Namely, when the space X carries a proximity δ, then a proximity variation of the miss part can be displayed by replacing the miss sets with *far-miss sets* $A^{++} := \{ E \in \mathrm{CL}(X) : E \ll_\delta A \}$.

Also in this case we can consider A with the complement running in a family $\mathscr{B}$ of closed subsets of X. Then the *hit and far-miss topology* , $\tau_{\delta,\mathscr{B}}$, *associated with* $\mathscr{B}$ is generated by the

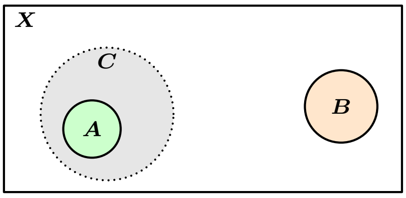

Figure 1: Strongly Far

join of the hit sets A^-, where A is open, with far-miss sets A^{++}, where the complement of A is in $\mathscr{B}$.

Fell topology can be considered as well an example of hit and far-miss topology. In fact, in any EF-proximity, when a compact set is contained in an open set, it is also strongly contained.

3. Main Results

Results for the *strongly far* proximity [16] (see, also, [15, 17, 18]) are given in this section. Let X be a nonempty set and δ be a *Lodato proximity* on $\mathscr{P}(X)$.

Definition 3.1 *We say that A and B are $\delta-$strongly far and we write $\underset{\doublevee}{\not\delta}$ if and only if $A \not\delta B$ and there exists a subset C of X such that $A \not\delta (X \setminus C)$ and $C \not\delta B$, that is the Efremovič property holds on A and B.*

Example 3.2 *In the Figure, let X be a nonempty set endowed with the euclidean metric proximity δ_e, $A, B, C \subset X, A \subset C$. Clearly, $A \underset{\doublevee}{\not\delta_e} B$ (A is strongly far from B), since $A \not\delta_e B$ so that $A \not\delta_e (X \setminus C)$ and $C \not\delta_e B$. Also observe that the Efremovič property holds on A and B.* ■

Remark 3.3 *Observe that $A \not\delta B$ does not imply $A \underset{\doublevee}{\not\delta} B$. In fact, this is the case when the proximity δ is not an* EF-proximity.
Furthermore, $\delta = \underset{\doublevee}{\delta}$ if and only if the proximity δ is an EF-proximity.

Example 3.4 *The Alexandroff proximity is defined as follows: $A \ \delta_A \ B \Leftrightarrow clA \cap clB \neq \emptyset$ or both clA and clB are non-compact. In a T_1 topological space this is a compatible Lodato proximity that is not Efremovič if the space is not locally compact. Suppose that X is a non-locally compact T_4 space. In this case, if we take two far subsets that are relatively compact, i.e. their closures are compact, they are also strongly far, but it doesn't hold for every pair of subsets, being the proximity not Efremovič. So, in general, $A \not\delta_A B$ does not imply $A \underset{\doublevee}{\not\delta_A} B$.* ■

Theorem 3.5 *The relation $\underset{\doublevee}{\delta}$ is a basic proximity.*

Proof Immediate from the properties of δ. □

We can also view the concept of strong nearness in many other ways. For example, let $A \ \hat{\underset{\mathrm{w}}{\not\delta}}\ B$, read A $\hat{\delta}$*-strongly far from* B, defined by

$$A \ \hat{\underset{\mathrm{w}}{\not\delta}}\ B \Leftrightarrow \exists E, C \subset X : \ A \subset \mathrm{int}(\mathrm{cl}E), \ B \subset \mathrm{int}(\mathrm{cl}C) \text{ and } \mathrm{int}(\mathrm{cl}E) \cap \mathrm{int}(\mathrm{cl}C) = \varnothing.$$

This relation is stronger than $\underset{\mathrm{w}}{\not\delta}$.

Theorem 3.6 *The relation* $\underset{\mathrm{w}}{\not\delta}$ *is stronger than* $\hat{\underset{\mathrm{w}}{\not\delta}}$*, that is* $A \ \underset{\mathrm{w}}{\not\delta}\ B \Rightarrow A \ \hat{\underset{\mathrm{w}}{\not\delta}}\ B$.

Proof Suppose $A \ \hat{\underset{\mathrm{w}}{\not\delta}}\ B$. This means that there exists a subset C of X such that $A \not\delta (X \setminus C)$ and $C \not\delta B$. By the Lodato property $P4)$ (see [9]), we obtain that $\mathrm{cl}A \cap \mathrm{cl}(X \setminus C) = \varnothing$ and $\mathrm{cl}C \cap \mathrm{cl}B = \varnothing$. So $\mathrm{cl}A \subset \mathrm{int}(C)$, $\mathrm{cl}B \subset \mathrm{int}(\mathrm{cl}(X \setminus C))$ and $\mathrm{int}(C) \cap \mathrm{int}(\mathrm{cl}(X \setminus C)) = \varnothing$, that gives $A \ \hat{\underset{\mathrm{w}}{\not\delta}}\ B$. □

We now want to consider *hit and far-miss topologies* related to δ and $\underset{\mathrm{w}}{\delta}$ on $\mathrm{CL}(X)$, the hyperspace of nonempty closed subsets of X.
To this purpose, call τ_δ the topology having as subbase the sets of the form:

- $V^- = \{E \in \mathrm{CL}(X) : E \cap V \neq \varnothing\}$, where V is an open subset of X,
- $A^{++} = \{ \ E \in \mathrm{CL}(X) : E \not\delta (X \setminus A) \ \}$, where A is an open subset of X.

and τ_{w} the topology having as subbase the sets of the form:

- $V^- = \{E \in \mathrm{CL}(X) : E \cap V \neq \varnothing\}$, where V is an open subset of X,
- $A_{\mathrm{w}} = \{ \ E \in \mathrm{CL}(X) : E \ \underset{\mathrm{w}}{\not\delta}\ (X \setminus A) \ \}$, where A is an open subset of X.

It is straightforward to prove that these are admissible topologies on $\mathrm{CL}(X)$.
The following results concern comparisons between them. From this point forward, let X be a T_1 topological space.

Proposition 3.7 *Let* $B, C \in \mathrm{CL}(X)$*. If* $A \not\delta B \Rightarrow A \underset{\mathrm{w}}{\not\delta} C$ *for all* $A \in \mathrm{CL}(X)$*, then* $C \subseteq B$*. That is* $(X \setminus B)^{++} \subseteq (X \setminus C)_{\mathrm{w}} \Rightarrow C \subseteq B$.

Proof By contradiction, suppose $C \nsubseteq B$. Then there exists $x \in C and x \notin B$. So $x \not\delta B$ but $x \underset{\mathrm{w}}{\delta} C$, which is absurd. □

Lemma 3.8 *Let* $\delta = \delta_A$*, the Alexandroff proximity on* $X = \mathbb{Q}$*, the space of rational numbers endowed with the topology induced by the natural one on* $\mathbb{R}$*. Let* H *be an open subset of* $\mathbb{Q}$ *and* A *a non-compact closed subset of* $\mathbb{Q}$*. Then* $A \in H_{\mathrm{w}}$ *implies that* $H = \mathbb{Q}$.

Proof We know that $A \in H_{\mathbb{W}}$ means $A \not\delta_A (X \setminus H)$ and $\exists C : A \not\delta_A C$ and $(X \setminus C) \not\delta_A (X \setminus H)$. So, by $A \not\delta_A C$, we have $A \cap \mathrm{cl}C = \varnothing$ and $\mathrm{cl}C$ is compact. Being in $\mathbb{Q}$ this means $\mathrm{int}(\mathrm{cl}C) = \varnothing$, so $\mathrm{int}C = \varnothing$. But we also have $(X \setminus C) \not\delta_A (X \setminus H)$, that is in particular $\mathrm{cl}(X \setminus C) \cap (X \setminus H) = \varnothing$. Knowing that $\mathrm{int}C = \varnothing$, we obtain $\mathrm{cl}(X \setminus C) = \mathbb{Q}$. Therefore $X \setminus H = \varnothing$ and consequently $H = \mathbb{Q}$.

□

Now let τ_δ^{++} be the hypertopology having as subbase the sets of the form A^{++}, where A is an open subset of X, and let $\tau_{\mathbb{W}}^{+}$ the hypertopology having as subbase the sets of the form $A_{\mathbb{W}}$, again with A an open subset of X.

Theorem 3.9 *The hypertopologies τ_δ^{++} and $\tau_{\mathbb{W}}^{+}$ are not comparable.*

Proof First we want to prove that, in general, $\tau_{\mathbb{W}}^{+} \notin \tau_\delta^{++}$. Consider the space of rational numbers $X = \mathbb{Q}$ endowed with the topology induced by the natural one on $\mathbb{R}$ and the *Alexandroff proximity* δ_A (see Example 3.4). Let H be an open subset of X with $X \setminus H$ non-compact and suppose $E \in H_{\mathbb{W}}$, with $E \in \mathrm{CL}(X)$. We ask if there exists a τ_δ^{++}−open set, K^{++}, such that $E \in K^{++} \subseteq H_{\mathbb{W}}$. We have two cases: $X \setminus K$ compact or not. First, suppose $X \setminus K$ compact. In this case, a closed set A belongs to K^{++} if $A \cap (X \setminus K) = \varnothing$, and A can be compact or not. So we choose A non-compact. With this choice $A \overset{\delta_A}{\mathbb{W}} (X \setminus H)$, because for all D, $A \ \delta_A \ (X \setminus D)$ or $D \ \delta_A \ (X \setminus H)$. In fact if $\mathrm{cl}D$ is compact, then $\mathrm{cl}(X \setminus D)$ is not compact. So either both A and $\mathrm{cl}(X \setminus D)$ are non-compact, or both $\mathrm{cl}D$ and $X \setminus H$ are non-compact. So K^{++} is not included in $H_{\mathbb{W}}$.
Instead, suppose $X \setminus K$ non-compact. So, we should have $E \not\delta_A (X \setminus K)$, that means E compact and $E \cap (X \setminus K) = \varnothing$. Then, if we take a non-compact subset $E \in H_{\mathbb{W}}$, we are unable to find K with $X \setminus K$ non-compact such that $E \in K^{++} \subset H_{\mathbb{W}}$.
Conversely, we want to prove that $\tau_\delta^{++} \notin \tau_{\mathbb{W}}^{+}$. Consider again the space of rational numbers $X = \mathbb{Q}$ and the *Alexandroff proximity* δ_A. Take $E^{++} \in \tau_\delta^{++}$ and $A \in E^{++}$, with E open subset of X. We have to identify a $\tau_{\mathbb{W}}^{+}$-open set, $H_{\mathbb{W}}$, such that $A \in H_{\mathbb{W}} \subset E^{++}$. Suppose A be a non-compact closed subset of X. We need H such that $A \in H_{\mathbb{W}}$ and we are in the hypothesis of Lemma 3.8. So we obtain $H = \mathbb{Q}$. Now is it true that $\mathbb{Q}_{\mathbb{W}} \subset E^{++}$? No, it isn't, because a set F that belongs to $\mathbb{Q}_{\mathbb{W}}$ is not forced to belong to E^{++}.

□

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