Fundamentals of Contemporary Mathematical Sciences
(2020) **1**(1) 14 – 22

On Biharmonic and Biminimal Curves in 3-dimensional f-Kenmotsu Manifolds

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Received: 07 January 2020 **Accepted:** 20 January 2020

Abstract: In the present paper, we study biharmonicity and biminimality of the curves in 3-dimensional f-Kenmotsu manifolds. We investigate necessary and sufficient conditions for a slant curve in a 3-dimensional f-Kenmotsu manifold to be biharmonic and biminimal, respectively. We give some related characterizations in case such curves are Legendre curves.

Key words: Biharmonic curves, Biminimal curves, f-Kenmotsu manifolds.

1. Introduction

Let $\Psi : (M, g) \to (N, h)$ be a smooth map between (pseudo-)Riemannian manifolds. The energy functional of Ψ is defined by $E(\Psi) = \frac{1}{2}\int_M |d\Psi|^2 v_g$. Critical points of the energy functional are called harmonic maps and the Euler-Lagrange equation for the energy is $\tau(\Psi) := trace\nabla d\Psi = 0$, where ∇ denotes the Levi-Civita connection on M. Biharmonic maps, which can be considered a natural generalization of harmonic maps, are defined as critical points of the bienergy functional given by $E_2(\Psi) = \frac{1}{2}\int_M |\tau(\Psi)|^2 v_g$. The first variation formula for the bienergy is derived by G. Y. Jiang [11, 12] and it is proved that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) := -J(\tau(\Psi)) = -\Delta\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where J is the Jacobi operator, $\Delta = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of pull-back bundle $\Psi^{-1}TN$, ∇^Ψ is the pull-back connection [10] and R^N is the curvature operator on N. One can easily see that harmonic maps are always biharmonic. Biharmonic maps which are not harmonic are called proper biharmonic maps.

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2010 *AMS Mathematics Subject Classification:* 53C25, 53C42, 53C50

An immersion $\Psi : (M, g) \to (N, h)$ between (pseudo-)Riemannian manifolds (or its image) is called biminimal if it is a critical point of the bienergy functional for variations normal to the image $\Psi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that Ψ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\Psi) = E_2(\Psi) + \lambda E(\Psi)$$

for any smooth variation of the map $\Psi_t : (-\varepsilon, \varepsilon) \times M \to N$, $\Psi_0 = \Psi$, such that $V = \frac{d\Psi_t}{dt}|_{t=0}$ is normal to $\Psi(M)$ [13].

In this paper, we study biharmonic and biminimal curves in another important class of almost contact manifolds which can be viewed as the most general case of Kenmotsu geometry defined by a smooth strictly positive function on the given manifold. We obtain necessary and sufficient conditions for biharmonicity and biminimality of a differentiable curve in a 3-dimensional f-Kenmotsu manifold, respectively. Especially, we give some interpretations for slant and Legendre curves.

2. Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called almost contact metric manifold with the almost contact metric structure (φ, ξ, η, g) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a metric (Riemannian) tensor field g satisfying the following conditions [2]:

$$\varphi^2 = -I + \eta \otimes \xi, \tag{1}$$

$$\eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi), \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \Gamma(TM), \tag{3}$$

where I denotes the identity transformation. An almost contact metric manifold is said to be f-Kenmotsu manifold [3] if the Levi-Civita connection ∇ of g satisfies

$$(\nabla_X \varphi) Y = f\ (g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{4}$$

where f is a strictly positive differentiable function on M and $df \wedge \eta = 0$ holds (for $n \geq 2$). If f is equal to a nonzero constant β, then the manifold is called an β-Kenmotsu manifold [4]. As a particular case a 1-Kenmotsu manifold is usually known as a Kenmotsu manifold [5].

In an f-Kenmotsu manifold we have [6]

$$\nabla_X \xi = f\ (X - \eta(X)\xi) \tag{5}$$

for all $X \in \Gamma(TM)$.

In a 3-dimensional f-Kenmotsu manifold we have [7]

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}+2\left(f^2+f'\right)\right)\{g(Y,Z)X-g(X,Z)Y\} \\ &-\left(\frac{r}{2}+3\left(f^2+f'\right)\right)\left\{\begin{array}{c} g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ -\eta(X)\eta(Z)Y+\eta(Y)\eta(Z)X \end{array}\right\}, \end{aligned} \tag{6}$$

$$S(X,Y)=\left(\frac{r}{2}+f^2+f'\right)g(X,Y)-\left(\frac{r}{2}+3\left(f^2+f'\right)\right)\eta(X)\eta(Y), \tag{7}$$

where $X,Y,Z\in\Gamma(TM)$, r is the scalar curvature of M and $f'=\xi(f)$.

Now we recall the notion of Frenet curve. An arbitrary curve $\gamma: I\to M$, $\gamma=\gamma(s)$, parametrized by arclenght s is called an r-Frenet curve ($1\le r\le m=\dim M$) on M if there exist r orthonormal vector fields $E_1=\gamma', E_2,...,E_r$ along γ such that there exist positive differentiable functions $\kappa_1,\kappa_2,...,\kappa_{r-1}$ of s such that

$$\left\{\begin{array}{c} \nabla_{\gamma'}E_1=\kappa_1E_2, \\ \nabla_{\gamma'}E_2=-\kappa_1E_1+\kappa_2E_3, \\ \dots\quad\dots\quad\dots \\ \nabla_{\gamma'}E_r=-\kappa_{r-1}E_{r-1}. \end{array}\right. \tag{8}$$

The function κ_j is called the j-th curvature of γ. The curve γ is known as

(1) a geodesic if $r=1$,

(2) a circle if $r=2$ and κ_1 is a constant,

(3) a helix of order r if $\kappa_1,\kappa_2,...,\kappa_{r-1}$ are constants.

A Frenet curve γ is called non-geodesic if $\kappa_1>0$ on I.

Note that $\gamma: I\to M$ is called a slant curve if the contact angle $\theta: I\to[0,2\pi)$ of γ given by

$$\cos\theta(s)=g(T(s),\xi) \tag{9}$$

is a constant function [8]. In particular, if $\theta\equiv\frac{\pi}{2}$ (or $\frac{3\pi}{2}$) then γ is called a Legendre curve [9].

Remark 2.1 *The integral curves of the Reeb vector field ξ are slant curves with $\theta\equiv 0$. For a Legendre curve in f-Kenmotsu manifolds, we have*

$$N=-\xi,\quad k_1=f\mid_\gamma,\quad k_2=0. \tag{10}$$

In particular, a Legendre curve in a β-Kenmotsu manifold is a circle [1].

We suppose that γ is a non-geodesic curve and in this case γ can not be an integral curve of ξ which means $\theta\neq 0,\pi$. Then we give following result [1] for later use:

Proposition 2.2 *The Frenet curve γ is a slant curve if and only if*

$$\eta(N) = -\frac{f}{k_1}\sin^2\theta. \tag{11}$$

Then a necessary condition for γ to be slant is

$$|\sin\theta| \leq \min\left\{\frac{k_1}{f}, 1\right\}. \tag{12}$$

From the last proposition above for a slant Frenet curve γ, we have [1]

$$\eta(B) = -\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}. \tag{13}$$

Let $\gamma : I \subset \mathbb{R} \to M$ be a differentiable curve parametrized by arclength immersed in a Riemannian manifold (M, g). Then $\tau(\gamma) = \nabla^{\gamma}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) = \nabla_T T$ and the biharmonic equation for γ reduces to $0 = \tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T)T$, that is, γ is called a *biharmonic curve* if it is a solution of this equation (see [14]). On the other hand, the biminimality equation for γ is given by $0 = \tau_{2,\lambda}(\gamma) = [\tau_2(\gamma)]^{\perp} - \lambda[\tau(\gamma)]^{\perp}$, for a value of $\lambda \in \mathbb{R}$, where $[,]^{\perp}$ denotes the normal component of $[,]$, that is, γ is called a *biminimal curve* if it is a solution of this equation. In particular, γ is called free biminimal if it is biminimal for $\lambda = 0$ (see [13]).

3. Biharmonic Curves in 3-dimensional f-Kenmotsu Manifolds

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional f-Kenmotsu manifold. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\gamma : I \to M$ parametrized by arclenght s, where $T = \gamma'(s), N, B$ are, respectively, the tangent, the principal normal, the binormal vector fields. Then for the curve γ the following Frenet equations are given by:

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{14}$$

where κ_1 and κ_2 are the curvature and the torsion of the curve, respectively.

By using the Frenet formulas given in (14), we have

$$\nabla^2_T T = -\kappa_1^2 T + \kappa_1' N + \kappa_1\kappa_2 B \tag{15}$$

and

$$\begin{aligned} \nabla^3_T T &= (-3k_1k_1')T + \left(k_1'' - k_1^3 - k_1k_2^2\right)N \\ &\quad + (2k_1'k_2 + k_1k_2')B. \end{aligned} \tag{16}$$

From (15), (16) and biharmonic equation, we write

$$\begin{aligned}\tau_2(\gamma) &= (-3k_1k_1')T + \left(k_1'' - k_1^3 - k_1k_2^2\right)N \\ &\quad + (2k_1'k_2 + k_1k_2')B - k_1R(T,N)T.\end{aligned} \tag{17}$$

On the other hand, if we use (6), we get

$$R(T,N)T = -\left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)N - \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\begin{array}{c}\eta(T)\eta(N)T \\ -(\eta(T))^2N - \eta(N)\xi\end{array}\right). \tag{18}$$

So one can see that bitension field of γ is as follows:

$$\begin{aligned}\tau_2(\gamma) &= \left(-3k_1k_1' + k_1\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\eta(T)\eta(N)\right)T \\ &\quad + \left(\begin{array}{c} k_1'' - k_1^3 - k_1k_2^2 + k_1\left(\frac{r}{2} + 2\left(f^2 + f'\right)\right) \\ -k_1\left(\frac{r}{2} + 3\left(f^2 + f'\right)(\eta(T))^2\right)\end{array}\right)N \\ &\quad + (2k_1'k_2 + k_1k_2')B - k_1\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\eta(N)\xi.\end{aligned} \tag{19}$$

In this case γ is a biharmonic curve if and only if

$$\left\{\begin{array}{c} k_1k_1' = 0, \\ \left\{\begin{array}{c} k_1'' - k_1^3 - k_1k_2^2 + k_1\left(\frac{r}{2} + 2\left(f^2 + f'\right)\right) \\ -k_1\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left((\eta(T))^2 + (\eta(N))^2\right)\end{array}\right. = 0, \\ 2k_1'k_2 + k_1k_2' - k_1\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\eta(N)\eta(B) = 0.\end{array}\right. \tag{20}$$

Hence we give

Theorem 3.1 *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional f-Kenmotsu manifold and $\gamma : I \to M$ be a Frenet curve parametrized by arclenght s. Then γ is a proper biharmonic curve if and only*

$$\left\{\begin{array}{c} k_1 = const. > 0, \\ \left\{\begin{array}{c} \left(k_1^2 + k_2^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ +\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left((\eta(T))^2 + (\eta(N))^2\right)\end{array}\right. = 0\,, \\ k_2' - \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\eta(N)\eta(B) = 0.\end{array}\right. \tag{21}$$

Now assume that the Frenet curve $\gamma : I \to M$ is a slant curve. In this case, by using (9), (11) and (13) in (21) we get

Theorem 3.2 *A slant Frenet curve γ in a 3-dimensional f-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ is proper biharmonic if and only if*

$$\left\{\begin{array}{c} k_1 = const. > 0, \\ \left\{\begin{array}{c} \left(k_1^2 + k_2^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ +\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right)\end{array}\right. = 0\,, \\ k_2' + \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) = 0.\end{array}\right. \tag{22}$$

In particular case if $\gamma : I \to M$ is a Legendre curve, from (10) and (22) we have

Corollary 3.3 *A Legendre Frenet curve γ in a 3-dimensional f-Kenmotsu manifold is proper biharmonic if and only if it is a Legendre circle with*

$$k_1 = f = const. \tag{23}$$

Now let us assume that $\gamma : I \to M$ is a slant curve in a 3-dimensional f-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ with $\theta > 0$. It is proved in [1] that if the principal normal vector field N of γ is parallel to ξ then $\cos\theta = 0$, i.e. γ is a Legendre curve. So we shall consider non-geodesic slant curves $\gamma : I \to M$ (with $\theta \neq 0, \pi$) such that N is non-parallel to the Reeb vector field ξ.

Case I: If $k_1 = const. > 0$ and $k_2 = 0$, then (22) reduces to

$$\begin{cases} k_1 = const. > 0, \\ \begin{cases} \left(k_1^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ + \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right) \end{cases} = 0, \\ \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) = 0. \end{cases} \tag{24}$$

From the third equation of (24), we get

$$\frac{r}{2} + 3\left(f^2 + f'\right) = 0. \tag{25}$$

By using the last equation in the second equation of (24), we conclude

Theorem 3.4 *Let $\gamma : I \to M$ be a non-geodesic slant curve ($\theta \neq 0, \pi$) with $k_1 = const. > 0$ and $k_2 = 0$ such that N is non-parallel to ξ. Then γ is a proper biharmonic curve if and only if*

$$f' + f^2 + k_1^2 = 0. \tag{26}$$

Case II: If $k_1 = const. > 0$ and $k_2 = const. > 0$, then (22) reduces to

$$\begin{cases} k_1 = const. > 0, \\ \begin{cases} \left(k_1^2 + k_2^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ + \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right) \end{cases} = 0, \\ \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) = 0. \end{cases} \tag{27}$$

From the third equation of (24), we get

$$\frac{r}{2} + 3\left(f^2 + f'\right) = 0. \tag{28}$$

By using the last equation in the second equation of (24), we conclude

Theorem 3.5 *Let $\gamma : I \to M$ be a non-geodesic slant curve ($\theta \neq 0, \pi$) with $k_1 = const. > 0$ and $k_2 = const. > 0$ such that N is non-parallel to ξ. Then γ is a proper biharmonic curve if and only if*

$$f' + f^2 + k_1^2 + k_2^2 = 0. \tag{29}$$

In particular, in a 3-dimensional β-Kenmotsu manifold M, a non-geodesic slant curve with N is non-parallel to ξ and constant curvature k_1 has a constant torsion k_2 (see [1]). So, from (29) we have

Corollary 3.6 *There does not exist a proper biharmonic slant curve with N is non-parallel to ξ and constant curvature k_1 in a 3-dimensional β-Kenmotsu manifold.*

4. Biminimal Curves in 3-dimensional f-Kenmotsu Manifolds

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional f-Kenmotsu manifold. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\gamma : I \to M$ parametrized by arclenght s, where $T = \dot{\gamma}(s), N, B$ are, respectively, the tangent, the principal normal, the binormal vector fields. From the tension field γ and (17) we have

$$\begin{aligned} \tau_{2,\lambda}(\gamma) \quad = \quad & \left(\begin{array}{c} k_1'' - k_1^3 - k_1 k_2^2 + k_1 \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right) \\ -k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)(\eta(T))^2 \right) - \lambda k_1 \end{array} \right) N \\ & + \left(2k_1' k_2 + k_1 k_2'\right) B - k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right) \eta(N) \xi. \end{aligned} \tag{30}$$

Then we obtain that γ is a biminimal curve if and only if

$$\left\{ \begin{array}{l} \left\{ \begin{array}{c} k_1'' - k_1^3 - k_1 k_2^2 + k_1 \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right) \\ -k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)(\eta(T))^2 \right) - \lambda k_1 \\ -k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)(\eta(N))^2 \end{array} \right. = 0, \\ 2k_1' k_2 + k_1 k_2' - k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right) \eta(N) \eta(B) = 0. \end{array} \right. \tag{31}$$

So we have

Theorem 4.1 *A non-geodesic curve $\gamma : I \to M$ parametrized by arclenght in a 3-dimensional f-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ is biminimal if and only if*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{c} k_1'' - k_1^2 - k_2^2 + \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right) \\ -\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left((\eta(T))^2 + (\eta(N))^2 \right) \end{array} \right. = \lambda, \\ 2k_1' k_2 + k_1 k_2' - k_1 \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right) \eta(N) \eta(B) = 0. \end{array} \right. \tag{32}$$

Let $\gamma : I \to M$ be a non-geodesic slant curve ($\theta \neq 0, \pi$) such that N is non-parallel to ξ. Then from (32) we have

Theorem 4.2 *Let* (M,φ,ξ,η,g) *be a 3-dimensional* f*-Kenmotsu manifold and* $\gamma : I \to M$ *be a non-geodesic slant curve (* $\theta \neq 0,\pi$ *) such that* N *is non-parallel to* ξ. *Then* γ *is a biminimal curve if and only*

$$\left\{\begin{array}{l} \left\{\begin{array}{c} \left(k_1'' - k_1^2 - k_2^2 + \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ -\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right) \end{array}\right. = \lambda, \\ \left\{\begin{array}{c} 2k_1'k_2 + k_1k_2' \\ +\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) \end{array}\right. = 0. \end{array}\right. \tag{33}$$

Now, we give the interpretations of (33)

Case I: If $k_1 = const. > 0$ and $k_2 = 0$, then (33) reduces to

$$\left\{\begin{array}{l} \left\{\begin{array}{c} \left(k_1^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ +\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right) \end{array}\right. = \lambda, \\ \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) = 0. \end{array}\right. \tag{34}$$

So we have

Theorem 4.3 *Let* $\gamma : I \to M$ *be a non-geodesic slant curve (* $\theta \neq 0,\pi$ *) with* $k_1 = const. > 0$ *and* $k_2 = 0$ *such that* N *is non-parallel to* ξ. *Then* γ *is a biminimal curve if and only if*

$$f' + f^2 + k_1^2 = \lambda. \tag{35}$$

Case II: If $k_1 = const. > 0$ and $k_2 = const. > 0$, then (33) reduces to

$$\left\{\begin{array}{l} \left\{\begin{array}{c} \left(k_1^2 + k_2^2 - \left(\frac{r}{2} + 2\left(f^2 + f'\right)\right)\right) \\ +\left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\cos^2\theta + \frac{f^2}{k_1^2}\sin^4\theta\right) \end{array}\right. = \lambda, \\ \left(\frac{r}{2} + 3\left(f^2 + f'\right)\right)\left(\frac{f}{k_1}\sin^2\theta\right)\left(\frac{|\sin\theta|}{k_1}\sqrt{k_1^2 - f^2\sin^2\theta}\right) = 0. \end{array}\right. \tag{36}$$

From the second equation of (36), we get

$$\frac{r}{2} + 3\left(f^2 + f'\right) = 0.$$

By using the last equation in the first equation of (36), we conclude

Theorem 4.4 *Let* $\gamma : I \to M$ *be a non-geodesic slant curve (* $\theta \neq 0,\pi$ *) with* $k_1 = const. > 0$ *and* $k_2 = const. > 0$ *such that* N *is non-parallel to* ξ. *Then* γ *is a biminimal curve if and only if*

$$f' + f^2 + k_1^2 + k_2^2 = \lambda. \tag{37}$$

In particular, in a 3-dimensional β-Kenmotsu manifold M, a non-geodesic slant curve with N is non-parallel to ξ and constant curvature k_1 has a constant torsion k_2 (see [1]). So, from (29) we have

Corollary 4.5 *A non-geodesic slant curve ($\theta \neq 0, \pi$) with N is non-parallel to ξ and constant curvature k_1 in a 3-dimensional β-Kenmotsu manifold is a biminimal curve if and only if*

$$k_1^2 + k_2^2 = \lambda - \beta^2. \tag{38}$$

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