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Research Article

Generalization of z-ideals in right duo rings

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Abstract

The aim of this paper is to generalize the notion of z-ideals to arbitrary noncommutative rings. A left (right) ideal I of a ring R is called a left (right) z-ideal if $M_a \subseteq I$, for each $a \in I$, where M_a is the intersection of all maximal ideals containing a. For every two left ideals I and J of a ring R, we call I a left z_J-ideal if $M_a \cap J \subseteq I$, for every $a \in I$, whenever $J \nsubseteq I$ and I is a z_J-ideal, we say that I is a left relative z-ideal. We characterize the structure of them in right duo rings. It is proved that a duo ring R is von Neumann regular ring if and only if every ideal of R is a z-ideal. Also, every one sided ideal of a semisimple right duo ring is a z-ideal. We have shown that if I is a left z_J-ideal of a p-right duo ring, then every minimal prime ideal of I is a left z_J-ideal. Moreover, if every proper ideal of a p-right duo ring R is a left relative z-ideal, then every ideal of R is a z-ideal.

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1. Introduction

Throughout this article all rings are associative with identity. The notion of z-ideals which are both algebraic and topological objects was first introduced in [6] by Kohls. These ideals play a fundamental role in studying the ideal structure of C(X), the ring of real-valued continuous functions on a completely regular Hausdorff space X, see [6]. Although in [6], he defined these ideals topologically, in terms of zero-sets, he showed that they can be characterized algebraically. Gillman and Jerison in [4], have proved it to be a powerful tool in the study of both algebraic properties of function rings and topological properties of Tychonoff spaces.

It was Mason [11], who initiated the study of z-ideals in arbitrary commutative rings with identity. An ideal I of a commutative ring R is called a *z-ideal (z°-ideal)* if for each $a \in I$, the intersection of all maximal ideals (minimal prime ideals) containing a

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is contained in I. A. Rezaei Aliabad and R. Mohamadian in [12], characterized the z-ideals and z°-ideals of formal power series ring on a commutative ring. They showed that if R is a commutative ring, then an ideal I of formal power series ring $R[[x]]$ is a z-ideal if and only if $I = (J, x)$, where J is a z-ideal of R. Also, they characterized a relation between the set of z°-ideals of $R[[x]]$ and the set of z°-ideals of R.

Let I and J be two ideals of a commutative ring R. I is said to be a *z_J-ideal* if $M_a \cap J \subseteq I$, for every $a \in I$, where M_a is the intersection of all maximal ideals containing a. Whenever $J \nsubseteq I$ and I is a z_J-ideal, we say that I is a *relative z-ideal*. This special kind of z-ideals introduced and investigated by F. Azarpanah and A. Taherifar in [2]. They have shown that for any ideal J in $C(X)$, the sum of every two z_J-ideals is a z_J-ideal if and only if X is an F-space, where the F-space is a space for which every finitely generated ideal of $C(X)$ is principal. A space X is called *P-space* if every prime ideal in C(X) is a z-ideal. It is in [2] shown that every principal ideal in $C(X)$ is a relative z-ideal if and only if X is a P-space. Also, they characterized the space X for which the sum of every two relative z-ideals of $C(X)$ is a relative z-ideal. If I is an ideal of a semisimple ring and $Ann(I) \neq 0$, A. R. Aliabad and F. Azarpanah and A. Taherifar in [1], have shown that I is a relative z-ideal and the converse is also true for each finitely generated ideal in $C(X)$.

These ideals are also studied further by others in commutative rings. In the following, we present a generalization of z-ideals to noncommutative rings and investigate the structure of them in right duo rings, which are rings in which every right ideal is a two-sided ideal. In fact, we generalize the results in [1] to right duo rings. This paper is organized as follows:

In the second section, we study some properties of ideals in right duo rings. In the third section, we shall generalize the concept of z-ideal to noncommutative rings and we study their structure in right duo rings. We show that every z-ideal of a right duo ring is semiprime. Mason in [10], showed that if I is a z-ideal of a semisimple commutative ring, then every minimal prime ideal of I is also a z-ideal. In a right duo ring, we consider sufficient conditions that every minimal prime ideal of a z-ideal is also a z-ideal. We will show that every ideal of von Neumann right duo rings is a z-ideal. Also, if every left ideal of a right duo ring R is a z-ideal, then R is a von Neumann ring. Furthermore, every left ideal of a semisimple right duo ring is a z-ideal.

In the fourth section, we generalize left relative z-ideals to noncommutative rings. We define the concept of p-right duo rings to obtain equivalent condition to minimal prime ideals of an ideal, and then study left relative z-ideals of their rings. We will present sufficient conditions in order that if every proper ideal of a ring R is a left relative z-ideal, then every ideal of R is a z-ideal.

Let us close this section by mentioning some symbols. Let R be a ring and I an ideal of R. The set of all prime ideals of R is denoted by $Spec(R)$. Also, $Min(I)$ is the set of all minimal prime ideals containing I, for each ideal I of R, and the Jacobson radical of R is denoted by $rad(R)$.

2. Some properties of structure of right duo rings

Recall that a ring R is called a *right duo ring* if each right ideal of R is a two sided ideal. We can similarly define the notion of a *left duo ring*. A ring R is said to be a *duo ring* if R is a right and left duo ring. Commutative rings and division rings are clearly duo ring. Furthermore, any valuation ring arising from a Krull valuation of a division ring is always duo ring, see [8, Exercise 19.9]. It is easily seen that any finite direct product of a right duo ring is a right duo ring. Proposition 1.1 of [3] says that any homomorphic image of a right duo ring is a right duo ring, and so is any factor ring of it. Gerg Marks in Proposition 5 of [9] shows that any power series ring of a right self injective von Neumann right duo ring is a right duo ring. In particular, the power series ring of a division ring is a right duo ring. Further a von Neumann ring which any idempotent element of it is central, is a right duo ring, by [5, Theorem 2.5]. It follows immediately from [3, Proposition 1.1] that for every ideal I of a right (left) duo ring R, $\dfrac{R}{I}$ is also a right (left) duo ring.

It is well known that in every right duo ring R, $RxR = xR$, for all $x \in R$, and so $Rx \subseteq RxR = xR$, see [3]. The following results will be needed in this paper.

Lemma 2.1 ([3])**.** *Let R be a right duo ring and $x \in R$. Then*
(1) *$RxR = xR$.*
(2) *$Rx \subseteq xR$.*

We know that a ring R is called a *Dedekind-finite ring* if whenever $x, y \in R$ and $xy = 1$, then $yx = 1$. Now, we assume that R is a right duo ring and $ab = 1$, for some $a, b \in R$. Then there exists an element $r \in R$ such that $ab = br$, by Lemma 2.1. Hence, we have $a = a.1 = a(ab) = a(br) = (ab)r = 1.r = r$, and so $1 = ba$. Therefore, every right duo ring is Dedekind-finite, see [8, Theorem 3.2].

It is well known that if P is a prime ideal of a right duo ring and $xy \in P$, then $x \in P$ or $y \in P$, because $xy \in P$ implies that $xRy \subseteq xyR \subseteq P$, by Lemma 2.1. Since P is a prime ideal, we have $x \in P$ or $y \in P$. Therefore, we have the following Lemma:

Lemma 2.2. *Let R be a right duo ring and P be a proper ideal of R. Then the following statements are equivalent:*
(1) *P is a prime ideal.*
(2) *For every $x, y \in R$, if $xy \in P$ then $x \in P$ or $y \in P$.*

Therefore, if P is a prime ideal of R and $x^n \in P$, for some $x \in R$ and $n \in \mathbb{N}$, then $x \in P$.

Let R be a ring and I be an ideal of R. We denote by $\sqrt{I}$ the subset
$$\{r \in R \mid \exists n \in \mathbb{N}\ ,\ r^n \in I\}$$
of R. It is easily seen from Lemma 2.1 that if P is a prime ideal of a right duo ring R, then $\sqrt{P} = P$.

Lemma 2.3. *Let R be a right duo ring and I and J be ideals of R. Then*
$$\sqrt{I} + \sqrt{J} \subseteq \sqrt{I + J}.$$

Proof. Let $a \in \sqrt{I}$ and $b \in \sqrt{J}$. Then there exist $n, m \in \mathbb{N}$ such that $a^m \in I$ and $b^n \in J$. Now, we claim that $(a + b)^{m+n} \in I + J$. In fact, $(a + b)^{m+n}$ is the sum of 2^{m+n} elements of the form $f = c_1c_2 \cdots c_{m+n}$ where each $c_i = a$ or b. If at least m of these c_is are a, then there exists $a' \in R$ such that $f = a^m a'$, by Lemma 2.1, and so $f \in I$, because $a^m \in I$. If the number of the $c_i = a$ is smaller than m, then at least n of them are b, and hence there exists $b' \in R$ such that $f = b^n b'$, by Lemma 2.1. Thus $f \in J$, because $b^n \in J$. Therefore $(a + b)^{n+m} \in I + J$. □

Proposition 2.4. *Let R be a right duo ring and I be an ideal of R. Then $\sqrt{I}$ is an ideal of R.*

Proof. Clearly, $0 \in \sqrt{I}$. If $a, b \in \sqrt{I}$, then $a + b \in \sqrt{I}$, by Lemma 2.3. Now, assume that $a \in \sqrt{I}$ and $r \in R$. Hence there exists $n \in \mathbb{N}$ such that $a^n \in I$, and so there exists an element $r' \in R$ such that $(ra)^n = a^n r' \in I$, by Lemma 2.1. Therefore $ra \in \sqrt{I}$ and similarly we show that $ar \in \sqrt{I}$. □

Let R be a ring and $a \in R$. The intersection of all maximal ideals of R containing a will be denoted by M_a. We set $M_a = R$ when a is a unit.

Lemma 2.5. *Let R be a right duo ring and $a, b \in R$. Then $M_{ab} = M_a \cap M_b$. In particular, if $a \in R$ we conclude that $M_{a^n} = M_a$, for every $n \in \mathbb{N}$.*

Proof. Clearly, for every $x \in M_a$ we have $M_x \subseteq M_a$. Thus $M_{ab} \subseteq M_a$ and $M_{ab} \subseteq M_b$, and so $M_{ab} \subseteq M_a \cap M_b$, for each $a, b \in R$. Conversely, let $x \in M_a \cap M_b$. We show that every maximal ideal containing ab is also containing x. Assume that N is a maximal ideal of R such that $ab \in N$. Then $a \in N$ or $b \in N$, by Lemma 2.2. If $a \in N$, then $x \in M_a \cap M_b \subseteq M_a \subseteq N$. If $b \in N$, then $x \in M_b \subseteq N$. Therefore $x \in N$. Thus $M_a \cap M_b \subseteq M_{ab}$, and consequently $M_{ab} = M_a \cap M_b$. □

3. Generalization of z-ideals in a right duo ring

The z-ideals are studied further in commutative rings. These ideals are useful concept in studying the ideal structure of the ring $C(X)$ of continuous real-valued functions on a topological space X. In the following, we shall present a generalization of z-ideals to noncommutative rings.

Definition 3.1. A left (right) ideal I of a ring R is called a *left (right) z-ideal* if $M_a \subseteq I$, for all $a \in I$.

In the following we show that every one sided z-ideal is an ideal.

Proposition 3.2. *Let R be a ring and I be a left (right) z-ideal of R. Then I is an ideal of R.*

Proof. Let I be a left z-ideal of R and $a \in I$. If N is a maximal ideal of R containing a, then $ar \in N$ for every $r \in R$. Thus $M_{ar} \subseteq M_a$. On the other hand, since I is a left z-ideal, we have $M_a \subseteq I$. Therefore, $ar \in M_{ar} \subseteq M_a \subseteq I$, and so $ar \in I$. Hence I is an ideal. □

Here in after a left (right) z-ideal of a ring is called a z-ideal, by Proposition 3.2.

Example 3.3. Every intersection of maximal ideals of a ring R is a z-ideal. In fact, every intersection of z-ideals is a z-ideal.

Lemma 3.4. *Let R be a ring and I be a left (right) ideal of R. Then the following statements are equivalent:*

(1) *I is a z-ideal.*
(2) *For every $a \in R$ and $b \in I$, if $M_a \subseteq M_b$ then $a \in I$.*

Proof. $1 \Rightarrow 2$. Let $a \in R$ and $b \in I$. Since I is a z-ideal and $b \in I$, we have $M_b \subseteq I$. Hence, if $M_a \subseteq M_b$, then $a \in M_a \subseteq M_b \subseteq I$, and so $a \in I$.
$2 \Rightarrow 1$. Let I be a left ideal and $a \in I$. For each $x \in M_a$, we have $M_x \subseteq M_a$. By hypothesis $x \in I$, and so $M_a \subseteq I$. Therefore I is a z-ideal. □

Let R be a ring and I be a left (right) ideal of R. The intersection of all z-ideals containing I will be denoted by I_z. For each element $a \in I_z$ and for every z-ideal J of R containing I, we have $a \in J$. Then $M_a \subseteq J$, and so $M_a \subseteq I_z$. Therefore, we have the following Lemma:

Lemma 3.5. *For every left (right) ideal I of a ring R, the intersection of all z-ideals containing I, which is denoted by I_z, is a z-ideal. In particular, I_z is the smallest z-ideal containing I.*

Lemma 3.6. *Let R be a ring. Then the following statements hold.*

(1) *For every left ideals I and J of R, if $I \subseteq J$, then $I_z \subseteq J_z$.*
(2) *If $\{I_\lambda\}_{\lambda\in\Lambda}$ is any family of left ideals of R, then*

$$\Big(\bigcap_{\lambda\in\Lambda} I_\lambda\Big)_z \subseteq \bigcap_{\lambda\in\Lambda} (I_\lambda)_z.$$

Proof. 1. Since every z-ideal containing J contains I, we see $I_z \subseteq J_z$.
2. For every $\mu \in \Lambda$, we have $\bigcap_{\lambda\in\Lambda} I_\lambda \subseteq I_\mu$. Hence our claim is true, by part (1). □

It is immediate that for every z-ideal I, we have $I_z = I$. In the next two Propositions, we study the structure of z-ideals in right duo rings.

Proposition 3.7. *Let R be a right duo ring and I be an ideal of R. Then $I \subseteq \sqrt{I} \subseteq I_z$.*

Proof. Clearly, $I \subseteq \sqrt{I}$. Now, we assume that $x \in \sqrt{I}$ and J is a z-ideal containing I. Thus there is a positive integer n such that $x^n \in I \subseteq J$. Hence $x \in M_x = M_{x^n} \subseteq J$, by Lemma 2.5. Therefore $x \in I_z$, and so $\sqrt{I} \subseteq I_z$. □

Recall that a proper ideal I of a ring R is said to be a semiprime ideal if for every ideal J of R, $J^2 \subseteq I$ implies that $J \subseteq I$. As an immediate consequence of Proposition 3.7 and [7, Theorem 10.11], we get the following result

Corollary 3.8. *Let R be a right duo ring and I be a z-ideal of R. Then $\sqrt{I} = I$. In particular, I is a semiprime ideal of R.*

Proposition 3.9. *Let R be a right duo ring and I be an ideal of R. Then the following statements hold.*

(1) $(\sqrt{I})_z = I_z$.
(2) *If I is a z-ideal, then* $(\sqrt{I})_z = I$.
(3) $\sqrt{I_z} = (\sqrt{I})_z$.

Proof. 1. Every z-ideal containing $\sqrt{I}$ also contains I. Therefore $I_z \subseteq (\sqrt{I})_z$. Conversely, Proposition 3.7 gives $\sqrt{I} \subseteq I_z$. This means that I_z is a z-ideal containing $\sqrt{I}$. Thus $(\sqrt{I})_z \subseteq I_z$, and consequently $(\sqrt{I})_z = I_z$.
2. Since I is a z-ideal, we have $I_z = I$. The proof is completed by (1).
3. We know that I_z is a z-ideal of R. Corollary 3.8 yields $\sqrt{I_z} = I_z$. Therefore $\sqrt{I_z} = (\sqrt{I})_z$, by (1). □

The following Proposition is a generalization of [11, Proposition 3.1] to noncommutative case.

Proposition 3.10. *Let R be a right duo ring. Then the following statements are equivalent:*

(1) *For any z-ideals I and J, $I + J$ is a z-ideal.*
(2) *For any ideals I and J, $(I + J)_z = I_z + J_z$.*
(3) *The sum of any nonempty family of z-ideals is a z-ideal.*
(4) *For every nonempty family $\{I_\alpha\}_{\alpha \in A}$ of ideals,*

$$(\sum_{\alpha \in A} I_\alpha)_z = \sum_{\alpha \in A} (I_\alpha)_z.$$

Proof. 1 ⇒ 2. Since I_z and J_z are z-ideals, $I_z + J_z$ is a z-ideal containing $I + J$, by hypothesis. Hence $(I + J)_z \subseteq I_z + J_z$. It follows from Lemma 3.6 that $I_z + J_z \subseteq (I + J)_z$. Therefore $(I + J)_z = I_z + J_z$.
2 ⇒ 3. Let $\{I_\alpha\}_{\alpha \in A}$ be a family of z-ideals and $a \in \sum_{\alpha \in A} I_\alpha$. Then there exists a finite subset F of A such that $a \in \sum_{\alpha \in F} I_\alpha$. Since I_α is a z-ideal, we have $(I_\alpha)_z = I_\alpha$, for every $\alpha \in F$. A simple induction argument shows that

$$(\sum_{\alpha \in F} I_\alpha)_z = \sum_{\alpha \in F} (I_\alpha)_z = \sum_{\alpha \in F} I_\alpha.$$

Consequently, $\sum_{\alpha \in F} I_\alpha$ is a z-ideal, and so

$$M_a \subseteq \sum_{\alpha \in F} I_\alpha \subseteq \sum_{\alpha \in A} I_\alpha.$$

Therefore $\sum_{\alpha\in A} I_\alpha$ is a z-ideal.
$3 \Rightarrow 4$. Let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals. Since $I_\beta \subseteq \sum_{\alpha\in A} I_\alpha$, for all $\beta \in A$, we have $(I_\beta)_z \subseteq (\sum_{\alpha\in A} I_\alpha)_z$, for all $\beta \in A$, by Lemma 3.6. Therefore

$$\sum_{\alpha\in A} (I_\alpha)_z \subseteq (\sum_{\alpha\in A} I_\alpha)_z.$$

Since $(I_\alpha)_z$ is a z-ideal containing I_α, for all $\alpha \in A$, we may conclude from assumption that $\sum_{\alpha\in A} (I_\alpha)_z$ is a z-ideal containing $\sum_{\alpha\in A} I_\alpha$. Hence

$$(\sum_{\alpha\in A} I_\alpha)_z \subseteq \sum_{\alpha\in A} (I_\alpha)_z.$$

Therefore

$$(\sum_{\alpha\in A} I_\alpha)_z = \sum_{\alpha\in A} (I_\alpha)_z.$$

$4 \Rightarrow 1$. If I and J are z-ideals, then $I_z = I$ and $J_z = J$. By hypothesis, we have $(I + J)_z = I_z + J_z$. Therefore $(I + J)_z = I_z + J_z = I + J$, and so $I + J$ is a z-ideal. □

Lemma 3.11. *Let R be a right duo ring and P be a prime ideal of R. Let $n \in \mathbb{N}$, $I_1, ..., I_{n-1}$ be ideals and I_n be a left ideal of R. Then the following statements are equivalent:*

(1) *$I_j \subseteq P$, for some $1 \le j \le n$.*
(2) *$\bigcap_{i=1}^{n} I_i \subseteq P$.*
(3) *$I_1 I_2 \cdots I_n \subseteq P$.*

Proof. $1 \Rightarrow 2$. $\bigcap_{i=1}^{n} I_i \subseteq I_j \subseteq P$.
$2 \Rightarrow 3$. Since I_n is a left ideal of R, we have $I_1 I_2 \cdots I_n \subseteq I_n$. On the other hand, I_i is an ideal, for every $1 \le i \le n-1$, and hence $I_1 I_2 \cdots I_n \subseteq I_i$, for all $1 \le i \le n$. Thus $I_1 I_2 \cdots I_n \subseteq \bigcap_{i=1}^{n} I_i \subseteq P$.
$3 \Rightarrow 1$. Suppose that $I_i \nsubseteq P$ and $x_i \in I_i \setminus P$, for every $1 \le i \le n$. Thus

$$x_1 x_2 \cdots x_n \in I_1 I_2 \cdots I_n \subseteq P$$

which yields $x_j \in P$, for some $1 \le j \le n$, by Lemma 2.2. This contradicts the choice of x_j. □

Proposition 3.12. *Let R be a right duo ring and I an ideal of R. If I is a finite intersection of maximal ideals of R, then any minimal prime ideal of I is a z-ideal.*

Proof. Since I is a finite intersection of maximal ideals, Lemma 3.11 implies that any minimal prime ideal of I is a maximal ideal. Hence each minimal prime ideal of I is a z-ideal. □

Recall from [7, Definition 10.3] that a nonempty set S of a ring R is said to be *m-system* if for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$.

Theorem 3.13. *Let R be a right duo ring, I a z-ideal of R and Q be a minimal prime ideal of I. If for every $a, b \in R$ with $a \notin Q$, there exists $r \in R \setminus Q$ such that $ab = br$, then Q is a z-ideal.*

Proof. Suppose Q is not z-ideal. Then there exist elements $a \in R \setminus Q$ and $b \in Q$ such that $M_a \subseteq M_b$, by Lemma 3.4. We now assume that

$$S = (R \setminus Q) \cup \{\ b^n c \ \mid \ n \in \mathbb{N},\ c \in R \setminus Q\ \}.$$

We first prove that $r_1r_2 \in S$, for every $r_1, r_2 \in S$. Consider $r_1, r_2 \in S$.
- If $r_1, r_2 \in R \smallsetminus Q$, then $r_1r_2 \notin Q$, by Lemma 2.2, and so $r_1r_2 \in S$.
- If there are $c_1, c_2 \in R \smallsetminus Q$ such that $r_1 = b^n c_1$ and $r_2 = b^m c_2$, for some positive integers n and m, then $c_1 b^m = b^m r$, for some $r \in R \smallsetminus Q$, by hypothesis. Therefore

$$r_1r_2 = b^n c_1 b^m c_2 = b^{(n+m)} r c_2 \in S$$

because $r, c_2 \in R \smallsetminus Q$, and so $rc_2 \notin Q$, by Lemma 2.2.
- If $r_1 \in R \smallsetminus Q$ and $r_2 = b^n c$, for some $n \in \mathbb{N}$ and $c \in R \smallsetminus Q$, then there exists an element $r \in R \smallsetminus Q$ such that $r_1 b^n = b^n r$, by hypothesis. This yields

$$r_1r_2 = r_1 b^n c = b^n rc.$$

Since $rc \notin Q$, we have $r_1r_2 \in S$. Also, we see that $r_2r_1 = b^n c r_1 \in S$. Hence for every $r_1, r_2 \in S$ we have $r_1r_2 \in S$. Therefore S is an m-system of R.

Now, we show that $I \cap S = \emptyset$. If $x \in I \cap S$, then $x \in I \subseteq Q$ and $x \in S$. Hence $x = b^n c$, for some $n \in \mathbb{N}$ and $c \in R \smallsetminus Q$. From Lemma 2.5 we see that

$$ac \in M_{ac} = M_a \cap M_c \subseteq M_b \cap M_c = M_{b^n} \cap M_c = M_x \subseteq I$$

because $M_a \subseteq M_b$ and I is a z-ideal. This yields $ac \in Q$. Hence $a \in Q$ or $c \in Q$, by Lemma 2.2. This contradicts the choice of a and c. Therefore $I \cap S = \emptyset$. By Zorn's Lemma, there exists an ideal $I \subseteq P$ which is maximal with respect to being disjoint from S. From [7, Proposition 10.5] it follows that P is a prime ideal. Since $S \cap P = \emptyset$ and $b \in S$, we have $I \subseteq P \subsetneqq Q$. However, this contradicts our assumption that Q is a minimal prime ideal of I. Therefore Q is a z-ideal. □

We know that the Jacobson radical of a ring R, which denoted by $rad(R)$, is the intersection of all maximal right (or left) ideals of R. Now, if R is a right (or left) duo ring, then every maximal right (or left) ideal is a maximal ideal. Therefore, if R is a right (or left) duo ring, then we can say that $rad(R)$ is the intersection of all maximal ideals of R.

Example 3.14. Let D be a division ring and $\mathbb{C}$ be the field of complex numbers. Let $R = D \times \mathbb{C}[x]$. We know that R is a duo ring. If $f \in rad(\mathbb{C}[x])$, then $1 - xf$ is a unit of $\mathbb{C}[x]$, by [7, Lemma 4.1], which yields $f = 0$. Hence $rad(\mathbb{C}[x]) = 0$. This implies that $rad(R) = rad(D) \times rad(\mathbb{C}[x]) = 0$, and so $I = \{0\}$ is a z-ideal of R. If P is a prime ideal of R, then $P = 0 \times \mathbb{C}[x]$ or $P = D \times Q$, where Q is a prime ideal of $\mathbb{C}[x]$. obviously, $D \times 0$ and $0 \times \mathbb{C}$ are minimal prime ideals of I which $0 \times \mathbb{C}$ is maximal, and so is a z-ideal. Consider $(a, f), (c, g) \in R$ such that $(a, f) \notin D \times 0$. It is clear that $ac = cr$, for some $r \in D$. Thus

$$(a, f)(c, g) = (ac, fg) = (cr, gf) = (c, g)(r, f).$$

Since $(a, f) \notin D \times 0$, we have $f \neq 0$, and so $(r, f) \notin D \times 0$. Therefore $D \times 0$ is a z-ideal, by Theorem 3.13.

Proposition 3.15. *Let R be a right duo ring and I be a left ideal of R. Then $(I^n)_z = I_z$, for every $n \in \mathbb{N}$.*

Proof. Clearly, $(I^n)_z \subseteq I_z$. For every $x \in I$, we have $x^n \in I^n \subseteq (I^n)_z$, and so $M_{x^n} \subseteq (I^n)_z$. From Lemma 2.5, we see that $x \in M_x = M_{x^n} \subseteq (I^n)_z$. Hence $I \subseteq (I^n)_z$. By Lemma 3.5, $(I^n)_z$ is a z-ideal, and so $I_z \subseteq (I^n)_z$. Therefore $(I^n)_z = I_z$. □

Recall that a ring R is said to be a *von Neumann regular* ring if for any $a \in R$, there exists an element $r \in R$ such that $a = ara$. Furthermore, for any ideal I of a von Neumann regular ring R, it is clear that $\dfrac{R}{I}$ is also a von Neumann regular ring. Therefore, we have

Proposition 3.16. *Let R be a right (or left) duo ring. If R is a von Neumann regular ring, then every ideal of R is a z-ideal.*

Proof. Let I be a proper ideal of R. Since $\frac{R}{I}$ is a von Neumann regular ring, we have $rad(\frac{R}{I}) = 0$, by [7, Corollary 4.24]. On the other hand, $\frac{R}{I}$ is also a right (or left) duo ring. Thus $rad(\frac{R}{I})$ is the intersection of all maximal ideals of $\frac{R}{I}$. Hence I is the intersection of all maximal ideal of R containing I, and so I is a z-ideal. □

Proposition 3.17. *Let R be a right duo ring. If every left ideal of R is a z-ideal, then R is a von Neumann regular ring.*

Proof. Let $a \in R$ and $I = Ra$. By hypothesis, I is a z-ideal, and so $I_z = I$. Hence $(I^2)_z = I_z = I$, by Proposition 3.15. On the other hand, from Lemma 2.1, we see that $I^2 = RaRa = aRa$. Since I^2 is a left ideal, I^2 is a z-ideal, and so $(I^2)_z = I^2$. Hence $I^2 = (I^2)_z = I_z = I$. Then we may conclude from $a \in I = I^2 = aRa$ that there exists an element $r \in R$ such that $a = ara$. Therefore R is a von Neumann regular ring. □

The following result, which is a generalization of [10, Theorem 1.2] to noncommutative case, follows immediately from Proposition 3.16 and Proposition 3.17.

Corollary 3.18. *Let R be a duo ring. Then R is a von Neumann regular ring if and only if every ideal of R is a z-ideal.*

Recall that if I and J are two left ideals of a ring R, then the subset $\{ x \in R \mid xI \subseteq J\}$ is denoted by $(J :_l I)$. It is easily seen that $(J :_l I)$ is an ideal of R. In particular, for each left ideal I, the subset $(0 :_l I)$, which will be denote by $Ann_l(I)$, is also an ideal of R. We call it the left annihilator of I.

Proposition 3.19. *Let I and J be two left ideals of a right duo ring R. If J is a z-ideal, then $(J :_l I)$ is a z-ideal of R.*

Proof. By Lemma 3.4, it is sufficient to show that for every $a \in R$ and $b \in (J :_l I)$, if $M_a \subseteq M_b$ then $a \in (J :_l I)$. Now, we assume that $a \in R$, $b \in (J :_l I)$ and $M_a \subseteq M_b$. Thus for every $x \in I$, we have $bx \in J$. Moreover

$$M_{ax} = M_a \cap M_x \subseteq M_b \cap M_x = M_{bx}$$

by Lemma 2.5. Since $bx \in J$ and J is a z-ideal, $M_{bx} \subseteq J$, and so $ax \in M_{ax} \subseteq M_{bx} \subseteq J$. Therefore $a \in (J :_l I)$. □

Lemma 3.20. *If e is an idempotent element of a right duo ring R, then*

$$Re = Ann_l(R(1 - e)).$$

Proof. For every $r \in R$, we have $reR(1 - e) \subseteq re(1 - e)R = 0$, by Lemma 2.1. Hence $Re \subseteq Ann_l(R(1-e))$. We now assume that $r \in Ann_l(R(1-e))$. Thus $r-re = r(1-e) = 0$, and so $r = re \in Re$. □

From [7, Theorem 2.5], it follows that every right ideal of a ring R is a direct summand of R if and only if every left ideal of R is a direct summand of R. A ring satisfying these equivalent conditions is called a *semisimple ring*.

Proposition 3.21. *Let R be a semisimple right duo ring. Then every one sided ideal of R is an ideal.*

Proof. Since R is a right duo ring, every right ideal of R is an ideal. By [7, Theorem 4.25], semisimple rings are exactly the left Noetherian von Neumann regular rings. Let I be a left ideal of R. Since R is left Noetherian, every left ideal of R is finitely generated, and so $I = Re$, for an idempotent element e of R, by using the characterization (3) of [7, Theorem 4.23]. Hence $I = Ann_l(R(1 - e))$, by Lemma 3.20, and consequently I is an ideal of R. □

As observed in the proof of Proposition 3.21, every ideal of a semisimple right duo ring is an annihilator of a left ideal. On the other hand, we know from [7, Theorem 4.25] that $rad(R) = 0$, for every semisimple ring R, and hence the zero ideal of R is a z-ideal. Now, we may by using Proposition 3.19 conclude that the following result.

Corollary 3.22. *Every ideal of a semisimple right duo ring is a z-ideal.*

4. Relative z-ideals in a right duo ring

The main goal of this section is to introduce left relative z-ideals. We define the concept of p-right duo rings to obtain equivalent condition to minimal prime ideals of an ideal, and then study left relative z-ideals of their rings. Finally, we prove that if every proper ideal of a p-right duo ring R is a left relative z-ideal, then every ideal of R is a z-ideal.

Definition 4.1. Let J be a left ideal of a ring R. A left ideal I of R is said to be a *left z_J-ideal* if $M_a \cap J \subseteq I$, for every $a \in I$. Whenever, for a left ideal I, there exists a left ideal J such that $J \nsubseteq I$ and I is a left z_J-ideal, we say that I is a *left relative z-ideal* and J is called a *z-factor of I*.

Recall that a ring R is said to be a *reduced ring* if R has no nonzero nilpotent element. In the following, we introduce a class of left relative z-ideals in a right duo ring. Before giving it, let us state the following Lemma which follows immediately from Lemma 2.2.

Lemma 4.2. *For each right duo ring R, if $rad(R) = 0$, then R is a reduced ring.*

Proposition 4.3. *Let R be a right duo ring with $rad(R) = 0$. If I is a left ideal of R such that $Ann_l(I) \neq 0$, then I is a left relative z-ideal.*

Proof. First, we show that $M_a \cap Ann_l(I) = 0$, for every $a \in I$. Suppose that $x \in M_a \cap Ann_l(I)$. Then $M_x \subseteq M_a$ and $xa \in xI = 0$, for every $a \in I$. From Lemma 2.5, it thus follows that

$$x \in M_x = M_x \cap M_a = M_{xa} = M_0 = rad(R) = 0.$$

Hence $M_a \cap Ann_l(I) = 0$. We now put $J = Ann_l(I)$, and show that $J \nsubseteq I$. If $J \subseteq I$, then $J^2 \subseteq JI = Ann_l(I)I = 0$. Thus $J^2 = 0$, and so $J = 0$, because R is a reduced ring, by Lemma 4.2. But this contradicts the assumption that $J = Ann_l(I) \neq 0$. Therefore $J \nsubseteq I$, and so I is a left relative z-ideal. □

Definition 4.4. A right duo ring R is called a *p-right duo ring* if for every prime ideal P of R and every elements $a, b \in R$, which $a \notin P$, there exists $r \in R \setminus P$ such that $ab = br$.

In the following, we give some examples of p-right duo rings.

Proposition 4.5. *Let R be a prime right duo ring. If R has a unique nonzero prime ideal, then R is a p-right duo ring.*

Proof. Let P be the unique nonzero prime ideal of R and $a, b \in R$ such that $a \notin P$. If $b = 0$, then $ab = b1$. Now, we assume that $b \neq 0$. Since R is a right duo ring, $ab = br$, for some $r \in R$. On the other hand, P is the unique nonzero prime ideal of R and $a \notin P$. Thus a is a unit element of R which yields $b = a^{-1}br = br'r$, for some $r' \in R$. It follows $b(1 - r'r) = 0$. Since R is a prime right duo ring and $b \neq 0$, we have $r'r = 1$. Therefore $r \notin P$. □

Example 4.6. Let D be a division ring and $R = D \times \mathbb{Z}$. We show that R is a p-right duo ring. It is easily seen that R is a right duo ring. If P is a prime ideal of R, then $P = D \times 0$, or $P = 0 \times \mathbb{Z}$ or $P = D \times p\mathbb{Z}$, for some prime number p. We assume that $(a, b), (c, d) \in R$ and $(a, b) \notin P$. It is clear that $ac = cr$, for some $r \in D$. Thus

$$(a, b)(c, d) = (ac, bd) = (cr, db) = (c, d)(r, b).$$

We consider the following three cases:
1. If $P = D \times 0$, then $b \neq 0$, because $(a, b) \notin P$, and so $(r, b) \notin P$.
2. If $P = 0 \times \mathbb{Z}$, then $a \neq 0$, because $(a, b) \notin P$. Now, if $c \neq 0$, then $r \neq 0$, and so $(r, b) \notin P$, and if $c = 0$, we have

$$(a, b)(c, d) = (0, bd) = (0, db) = (c, d)(1, b)$$

which $(1, b) \notin P$.
3. If $P = D \times p\mathbb{Z}$, for some prime number p, then $p \nmid b$, because $(a, b) \notin P$, and so $(r, b) \notin P$.

Proposition 4.7. *Let R be a p-right duo ring with $rad(R) = 0$ and P be a prime ideal of R. Let Γ be the set of all z-ideals of R contained in P. Then Γ (partially ordered by inclusion) has a maximal element. Furthermore, every maximal element of Γ is a prime z-ideal of R.*

Proof. Since $rad(R) = 0$, the zero ideal of R is a z-ideal, and so $\Gamma \neq \emptyset$. If P is a z-ideal, then clearly P is the only maximal element of Γ. We now assume that P is not z-ideal. If Σ is a chain in Γ, then it is quite obvious that $\bigcup_{I_\alpha \in \Sigma} I_\alpha$ is a z-ideal contained in P. Therefore Γ has a maximal element J, by Zorn's Lemma. Hence $J \subsetneqq P$, because P is not z-ideal. Suppose that Q is a minimal prime ideal of J such that $J \subseteq Q \subseteq P$. Theorem 3.13 implies that Q is a z-ideal, because R is a p-right duo ring, and so $Q \in \Gamma$. Since J is a maximal element in Γ, we have $J = Q$. Therefore J is a prime z-ideal. □

Proposition 4.8. *Let R be a ring and I be a proper ideal of R. Let*

$$\Gamma = \{\ S \subseteq R \ \mid \ S \ \textit{is an} \ m-\textit{system and} \ S \cap I = \emptyset\ \}.$$

If P is a prime ideal of R, then $P \in Min(I)$ if and only if $R \smallsetminus P$ is a maximal element of Γ.

Proof. We know from [7, Corollary 10.4] that an ideal P of R is prime if and only if $R \smallsetminus P$ is an m-system. Therefore, if $T = R \smallsetminus P$ is a maximal element of Γ, then P is a prime ideal of R. Also, $T \cap I = \emptyset$ implies that $I \subseteq P$. Now, we assume that there exists a prime ideal Q of R such that $I \subseteq Q \subseteq P$. It follows that $(R \smallsetminus Q) \cap I = \emptyset$ and $R \smallsetminus Q$ is an m-system, by [7, Corollary 10.4]. Thus $R \smallsetminus Q \in \Gamma$. Since $T \subseteq R \smallsetminus Q$ and T is a maximal element of Γ, we have $R \smallsetminus Q = T$, and so $P = Q$. Therefore $P \in Min(I)$.

Conversely, if $P \in Min(I)$, then $T = R \smallsetminus P$ is an m-system, by [7, Corollary 10.4]. Furthermore, $T \cap I = \emptyset$. Thus $T \in \Gamma$. Suppose that there exists $S \in \Gamma$ such that $T \subseteq S$. Hence S is an m-system and $S \cap I = \emptyset$. By Zorn's Lemma, there exists an ideal $I \subseteq Q$ which is maximal with respect to being disjoint from S. From [7, Proposition 10.5], it follows that Q is a prime ideal. Since $S \cap Q = \emptyset$ and $T \subseteq S$, we have $Q \cap T = \emptyset$. Hence $I \subseteq Q \subseteq P$, and consequently $Q = P$, because $P \in Min(I)$. It follows from $P \cap S = \emptyset$ that $S \subseteq R \smallsetminus P = T$, and so $T = S$. Therefore, T is a maximal element of Γ. □

It is well known that, if I is an ideal of a commutative ring R, then $P \in Min(I)$ if and only if for each $a \in P$, there exist $c \in R \smallsetminus P$ and $n \in \mathbb{N}$ such that $(ac)^n \in I$. We need generalization of this conclusion for the noncommutative rings. In the following Lemma, we will generalize it to right duo rings.

Proposition 4.9. *Let R be a p-right duo ring and I be a proper ideal of R. If P is a nonzero prime ideal of R containing I and $T = R \smallsetminus P$, then the following statements are equivalent:*

(1) *$P \in Min(I)$.*
(2) *For every $x \in P$, there exist $y, z \in T$ and $n \in \mathbb{N}$ such that $yx^n z \in I$.*

Proof. $1 \Rightarrow 2$. Let $P \in Min(I)$ and $0 \neq x \in P$. If

$$\Gamma = \{\ S \subseteq R \ \mid \ S \ \textit{is an} \ m-\textit{system and} \ S \cap I = \emptyset\ \},$$

then T is a maximal element of Γ, by Proposition 4.8. Now, we assume that

$$T' = \{\ yx^nz\ \mid\ y,z \in T\ ,\ n \in \mathbb{N} \cup \{0\}\ \}.$$

Let $y_1x^mz_1,\ y_2x^nz_2 \in T'$. From Lemma 2.2, it is clear that $z_1y_2 \in T$, and hence there is an element $r \in T$ such that $z_1y_2x^n = x^nr$, because R is a p-right duo ring. Thus

$$y_1x^mz_1y_2x^nz_2 = y_1x^{m+n}rz_2 \in T'.$$

Therefore T' is an m-system. Obviously, $x \in T' \setminus T$, consequently $T \subsetneq T'$. Hence $T' \notin \Gamma$, by the maximality of T. However, this yields $T' \cap I \neq \emptyset$. Therefore, there exist $y,z \in T$ and $n \in \mathbb{N}$ such that $yx^nz \in I$.
$2 \Rightarrow 1$. Let Q be a prime ideal of R such that $I \subseteq Q \subseteq P$. For each $x \in P$, there exist $n \in \mathbb{N}$ and $y,z \in T$ such that $yx^nz \in I \subseteq Q$, by hypothesis. Since Q is a prime ideal and $y,z \notin Q$, we have $x \in Q$, by Lemma 2.2. Therefore $Q = P$, and so $P \in Min(I)$. □

In the following Proposition, which is an analogue of [1, Lemma 2.1], we give conditions that, whenever J is a left ideal, then every minimal prime ideal of a left z_J-ideal, is also a left z_J-ideal.

Proposition 4.10. *Let R be a p-right duo ring, I be an ideal and J be a left ideal of R. If I is a left z_J-ideal, then every minimal prime ideal of I is a left z_J-ideal.*

Proof. Let $P \in Min(I)$. For every element $a \in P$, there exist elements $b,c \in R \setminus P$ and $n \in \mathbb{N}$ such that $ba^nc \in I$, by Proposition 4.9. Since I is a left z_J-ideal, we have $M_{ba^nc} \cap J \subseteq I$. Now, it follows from Lemma 2.5 that

$$M_b \cap M_a \cap M_c \cap J = M_b \cap M_{a^n} \cap M_c \cap J = M_{ba^nc} \cap J \subseteq I \subseteq P.$$

Obviously, $M_b, M_c \nsubseteq P$, because $b,c \notin P$. It follows that $M_a \cap J \subseteq P$, by Lemma 3.11, and so P is a left z_J-ideal of R. □

Proposition 4.11. *Let R be a right duo ring and J be a left ideal of R. If P is a prime ideal of R for which $J \nsubseteq P$, then P is a left z_J-ideal if and only if P is a z-ideal.*

Proof. It is clear that if P is a z-ideal, then P is also a left z_J-ideal. Conversely, if P is a left z_J-ideal, then $M_a \cap J \subseteq P$, for each $a \in P$. Since $J \nsubseteq P$, Lemma 3.11 yields $M_a \subseteq P$, for each $a \in P$. Hence P is a z-ideal of R. □

Let R be a p-right duo ring and I be an ideal of R. From Lemma 2.2, it is clear that $\sqrt{I} \subseteq P$, for each $P \in Min(I)$. On the other hand, if $P \in Min(I)$ and $x \in R \setminus P$, we conclude from Proposition 4.9 that $x^n \notin I$, for every $n \in \mathbb{N}$, and so $x \notin \sqrt{I}$. Therefore, we have

$$\sqrt{I} = \bigcap_{P \in Min(I)} P.$$

The following Lemma corresponds to [1, Lemma 2.2].

Lemma 4.12. *Let R be a p-right duo ring, I be an ideal and J be a left ideal of R. If I is a left z_J-ideal, then $I_z \cap J \subseteq I$.*

Proof. As we have seen in the preceding paragraph

$$\sqrt{I} = \bigcap_{P \in Min(I)} P.$$

From Proposition 3.9, it follows that

$$I_z \cap J = (\sqrt{I})_z \cap J = (\ \bigcap_{P \in Min(I)} P\ )_z \cap J.$$

Moreover, Lemma 3.6 yields

$$(\bigcap_{P\in Min(I)} P)_z \subseteq \bigcap_{P\in Min(I)} P_z.$$

Thus

$$I_z \cap J \subseteq (\bigcap_{P\in Min(I)} P_z) \cap J. \tag{4.1}$$

Since I is a left z_J-ideal, from Proposition 4.10, we see that P is a left z_J-ideal, for every $P \in Min(I)$. However, we conclude from Proposition 4.11 that $J \subseteq P$ or P is a z-ideal, for every $P \in Min(I)$.

We now assume that $P \in Min(I)$. If P is a z-ideal, then $P_z = P$, and so $P_z \cap J = P \cap J$. If $J \subseteq P$, then we also have $P_z \cap J = J = P \cap J$. Therefore

$$(\bigcap_{P\in Min(R)} P_z) \cap J = (\bigcap_{P\in Min(R)} P) \cap J = \sqrt{I} \cap J$$

and from (4.1) we get

$$I_z \cap J \subseteq \sqrt{I} \cap J.$$

Let us finally prove that $\sqrt{I} \cap J \subseteq I$. If $x \in \sqrt{I} \cap J$, then there is a positive integer n such that $x^n \in I$. Since I is a left z_J-ideal, we have $M_{x^n} \cap J \subseteq I$. Hence, from Lemma 2.5, it follows that

$$x \in M_x \cap J = M_{x^n} \cap J \subseteq I.$$

Thus $\sqrt{I} \cap J \subseteq I$, and consequently $I_z \cap J \subseteq \sqrt{I} \cap J \subseteq I$. □

Lemma 4.13. *Let R be a p-right duo ring and J be a left ideal of R. If I is an ideal of R, then I is a left z_J-ideal if and only if I is a left z_{I+J}-ideal.*

Proof. If I is a left z_{I+J}-ideal, then clearly I is a left z_J-ideal. Conversely, let I be a left z_J-ideal. Since $I \subseteq I_z$, from Lemma 4.12 and modular law follow that

$$I_z \cap (I + J) = I + (I_z \cap J) \subseteq I.$$

For every $a \in I$, we have $M_a \subseteq I_z$, because $I \subseteq I_z$ and I_z is a z-ideal. Hence

$$M_a \cap (I + J) \subseteq I_z \cap (I + J) \subseteq I.$$

Therefore, I is a left z_{I+J}- ideal. □

Lemma 4.14. *Let R be a right duo ring. If I and J are two left ideals of R such that at least one of them is ideal, then $I \cap J$ is a left z_J-ideal if and only if I is a left z_J-ideal.*

Proof. We first assume that J is an ideal. If I is a left z_J-ideal, then for every $a \in I \cap J$, we have $M_a \cap J \subseteq I$, and so $M_a \cap J \subseteq I \cap J$. Hence $I \cap J$ is a left z_J-ideal.

Conversely, let $I \cap J$ be a left z_J-ideal and $a \in I$. We must show that $M_a \cap J \subseteq I$. We now assume that $x \in M_a \cap J$. Thus $xa \in I \cap J$, because J is an ideal. Since $I \cap J$ is a left z_J-ideal, $M_{xa} \cap J \subseteq I \cap J$. From Lemma 2.5, we see that

$$x \in M_x \cap M_a \cap J = M_{xa} \cap J \subseteq I \cap J \subseteq I.$$

Therefore $M_a \cap J \subseteq I$, and so I is a left z_J-ideal.

Now, if I is an ideal, then we can prove this Lemma by a similar argument. □

The following result is an analogue of [1, Proposition 2.5].

Proposition 4.15. *Let R be a p-right duo ring and M be a maximal ideal of R. If I is an ideal of R, then I is a z-ideal if and only if $I \cap M$ is a z-ideal.*

Proof. If I is a z-ideal, then clearly $I \cap M$ is a z-ideal. We now assume that $I \cap M$ is a z-ideal of R. If $I \subseteq M$, then $I = I \cap M$, and so I is a z-ideal. If $I \nsubseteq M$, then $M_a \subseteq I \cap M$, for every $a \in I \cap M$, and so $M_a \cap M \subseteq I \cap M$. Thus $I \cap M$ is a left z_M-ideal. It follows from Lemma 4.14 that I is a left z_M-ideal. Now, Lemma 4.13 implies that I is a left z_R-ideal, because $I + M = R$. However, I is a z-ideal of R. □

The following Proposition is an analogue of [1, Proposition 2.6] and [2, Proposition 2.2].

Proposition 4.16. *Let R be a p-right duo ring and J be an ideal of R with $J \nsubseteq rad(R)$. If J is not a z-ideal, then there exists an ideal I of R such that $I \subsetneq J$ and I is a left z_J-ideal which is not a z-ideal.*

Proof. Since $J \nsubseteq rad(R)$, there is a maximal ideal M of R such that $J \nsubseteq M$. Thus $I = J \cap M$ is an ideal of R and $I \subsetneq J$. Obviously, for every $a \in I$, $M_a \cap J \subseteq M \cap J = I$, and so I is a left z_J-ideal of R. From Proposition 4.15, it follows that J is a z-ideal if and only if I is a z-ideal. Therefore, the desired conclusion trivially holds. □

Lemma 4.17. *Let R be a p-right duo ring and I be an ideal of R. If I is a left relative z-ideal of R, then the set*

$$\Gamma = \{\ J \ \mid \ J \text{ is a } z\text{−factor of } I\ \}$$

has a maximal member with respect to inclusion. Furthermore, every maximal element of Γ properly contains I.

Proof. Obviously, $\Gamma \neq \emptyset$. If Σ is a non-empty totally ordered subset of Γ, then clearly $L = \bigcup_{J \in \Sigma} J$ is a left ideal which $L \nsubseteq I$. We will show that I is a z_L-ideal. For every $a \in I$, we have

$$M_a \cap L = M_a \cap (\bigcup_{J \in \Sigma} J) = \bigcup_{J \in \Sigma} (M_a \cap J) \subseteq I,$$

because J is a z-factor of I, for all $J \in \Sigma$, and so $M_a \cap L \subseteq I$. Hence I is a left relative z_L-ideal, and consequently I is an upper bound for Σ in Γ. From Zorn's Lemma, we see that Γ has a maximal element.

Now, we show that every maximal element of Γ properly contains I. If J is a maximal element of Γ, then I is a left z_J-ideal and $J \nsubseteq I$. Hence I is a left z_{I+J}-ideal, by Lemma 4.13. Since $I + J \nsubseteq I$, $I + J$ is a z-factor of I, and so $I + J \in \Gamma$. Therefore, by the maximality of J, we deduce that $J = I + J$, and consequently $I \subsetneq J$. □

Lemma 4.18. *Let R be a right duo ring and I, J and L be left ideals of R such that $I \subseteq J$. If I is a left z_J-ideal and J is a left z_L-ideal, then I is a left z_L-ideal.*

Proof. Since I is a left z_J-ideal, we have $M_a \cap J \subseteq I$, for every $a \in I$. Moreover, $M_a \cap L \subseteq J$, for every $a \in I$, because $I \subseteq J$ and J is a left z_L-ideal. Thus $M_a \cap L \subseteq M_a \cap J \subseteq I$, for every $a \in I$. Therefore I is a left z_L-ideal. □

Theorem 4.19. *Let R be a duo ring such that every proper ideal of R is a left relative z-ideal. If R is a p-right duo ring, then every ideal of R is a z-ideal.*

Proof. It is clear that R is a z-ideal. Let I be a proper ideal of R. Then I is a left relative z-ideal, by hypothesis. It follows from Lemma 4.17 that there exists a maximal z-factor J of I such that $I \subsetneq J$. We claim that $J = R$. If $J \neq R$, then J is also a left relative z-ideal, and hence we can assume that L is a z-factor of J such that $J \subsetneq L$, by Lemma 4.17. It follows that I is a left z_J-ideal and J is a left z_L-ideal. From Lemma 4.18, we may conclude that I is a left z_L-ideal. Since $I \subsetneq J \subsetneq L$, L is a z-factor of I, which contradicts the maximality of J. Therefore, $J = R$, and so I is a z-ideal. □

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References

[1] A.R. Aliabad, F. Azarpanah and A. Taherifar, *Relative z-ideals in commutative rings*, Comm. Algebra, **41**, 325–341, 2013.

[2] F. Azarpanah and A. Taherifar, *Relative z-ideals in $C(X)$*, Topology Appl. **156**, 1711–1717, 2009

[3] R.C. Courter, *Finite dimensional right duo algebras are duo*, Proceedings of the Amer. Math. Soc. **84** (2), 157–161, 1982.

[4] L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Mathematics, New York, Van Nostrand, 1960.

[5] N.K. Kim and Y. Lee, *On a ring property unifying reversible and right duo rings*, J. Korean Math. Soc. **50** (5), 1083-1103, 2013.

[6] C.W. Kohls, *Ideals in rings of continuous functions*, Fund. Math. **45**, 28–50, 1957.

[7] T.Y. Lam, *A first course in noncommutative ring*, Graduate Texts in Mathematics **131**, Springer-Verlag, New York, 1991.

[8] T.Y. Lam and A.S. Dugas, *Quasi-duo rings and stable range descent*, J. Pure Appl. Algebra **195**, 243–259, 2005.

[9] G. Marks, *Duo rings and ore extensions*, J. Algebra, **280**, 463–471, 2004.

[10] G. Mason, *z-ideals and prime ideals*, J. Algebra, **26**, 280–297, 1973.

[11] G. Mason, *Prime z-ideals of $C(X)$ and related rings*, Canad. Math. Bull. **23** (4), 437–443, 1980.

[12] A. Rezaei Aliabad and R. Mohamadian, *On z-ideals and z°-ideals of Power Series Rings*, J. Math. Ext. **7** (2), 93–108, 2013.