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Generalized Ricci solitons on K-contact manifolds

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Abstract
The object of the present paper is to study K-contact manifold admitting generalized Ricci solitons. We prove that a K-contact manifold of dimension $(2n+1)$ satisfying the generalized Ricci soliton equation is an Einstein one. Finally, we obtain several remarks.

Keywords: *K-contact manifold; Generalized Ricci soliton; Einstein manifold.*

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1. Introduction

Let M be a $(2n+1)$-dimensional differentiable manifold. Suppose that (ϕ, ξ, η, g) is an almost contact metric structure on M. This means that (ϕ, ξ, η, g) is a quadruple consisting of a $(1,1)$-tensor field ϕ, an associated vector field ξ, a 1-form η and a Riemannian metric g on M satisfying the following relations

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1}$$

where X, Y are smooth vector fields on M. In addition, we have

$$\phi\xi = 0, \ \eta(\phi X) = 0, \ g(X, \xi) = \eta(X), \ g(\phi X, Y) = -g(X, \phi Y). \tag{1.2}$$

An almost contact structure is said to be a contact structure if $g(X, \phi Y) = d\eta(X, Y)$. A contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. A normal contact metric manifold is called a Sasakian manifold. If ξ is a Killing vector field on a contact metric manifold (M, g), then the manifold is called a K-contact metric manifold or simply a K-contact manifold ([1], [15]). An almost contact manifold is Sasakian [1], if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{1.3}$$

where ∇ is the Levi-Civita connection.

A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Kähler structure (J, G) of the base manifold M^{2n+1}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n+1}, J, G) is Kählerian. If (M^{2n+1}, J, G) is only almost Käehler, then (ϕ, ξ, η, g) is only K-contact [1]. In a Sasakian manifold, the Ricci operator Q commutes with ϕ, that is, $Q\phi = \phi Q$. Recently in [11], it has been shown that there exist K-contact manifolds with $Q\phi = \phi Q$ which are not Sasakian. It is seen that K-contact structure is the intermediate between contact and Sasakian structure. K-contact manifolds have

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been studied by several authors ([6], [7], [8], [14], [16], [18]) and many others. Given a smooth function f on M, the gradient of f is defined by

$$g(grad\ f, X) = Xf, \tag{1.4}$$

the Hessian of f is defined by

$$(Hess\ f)(X,Y) = g(\nabla_X grad\ f, Y), \tag{1.5}$$

for all smooth vector fields X, Y. For a smooth vector field X, we have ([12], [13])

$$X^b(Y) = g(X,Y). \tag{1.6}$$

The generalized Ricci soliton equation in a Riemannian manifold (M, g) is defined by [13]

$$\pounds_X g = -2c_1 X^b \odot X^b + 2c_2 S + 2\lambda g, \tag{1.7}$$

where $\pounds_X g$ is the Lie derivative of g along X given by

$$(\pounds_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y), \tag{1.8}$$

for all smooth vector fields X, Y, Z and $c_1, c_2, \lambda \in \mathbb{R}$. For different values of c_1, c_2 and λ, equation (1.7) is a generalization of Killing equation $(c_1 = c_2 = \lambda = 0)$, equation for homotheties $(c_1 = c_2 = 0)$, Ricci soliton $(c_1 = 0, c_2 = -1)$, Vacuum near-horizon geometry equation $(c_1 = 1, c_2 = \frac{1}{2})$ etc. For more details we refer to the reader ([3], [4], [5], [9], [13]).

If $X = grad\ f$, then the generalized Ricci soliton equation is given by

$$Hess\ f = -c_1 df \odot df + c_2 S + \lambda g. \tag{1.9}$$

2. Preliminaries

In an $(2n+1)$-dimensional K-contact manifold, the following relations hold ([1], [17])

$$\nabla_X \xi = -\phi X, \tag{2.1}$$

$$g(R(\xi,X)Y,\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi, \tag{2.3}$$

$$S(X,\xi) = 2n\eta(X), \tag{2.4}$$

$$(\nabla_X \phi)Y = R(\xi,X)Y, \tag{2.5}$$

for any vector fields $X, Y \in \chi(M)$.
A K-contact manifold M of dimension ≥ 3 is said to be Einstein if its Ricci tensor S is of the form $S = ag$, where a is a constant.
In this case we have

$$S(X,Y) = ag(X,Y). \tag{2.6}$$

Substituting $X = Y = \xi$ in (2.6) and then using (2.4) and (1.2), we get

$$a = 2n. \tag{2.7}$$

Thus using (2.7) we obtain from (2.6)

$$S(X,Y) = 2ng(X,Y). \tag{2.8}$$

Again from (2.8) we infer that

$$QX = 2nX. \tag{2.9}$$

3. Generalized Ricci soliton

In this section we characterize K-contact manifolds admitting generalized Ricci soliton. First we prove the following Lemma:

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a K-contact manifold. Then*

$$(\pounds_\xi(\pounds_X g))(Y, \xi) = g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Y g(\nabla_\xi X, \xi),$$

for all smooth vector fields X, Y with Y orthogonal to ξ.

Proof. We have

$$\begin{aligned}(\pounds_\xi(\pounds_X g))(Y, \xi) &= \xi((\pounds_X g)(Y, \xi)) - (\pounds_X g)(\pounds_\xi Y, \xi) - (\pounds_X g)(Y, \pounds_\xi \xi)\\ &= \xi((\pounds_X g)(Y, \xi)) - (\pounds_X g)(\pounds_\xi Y, \xi).\end{aligned} \tag{3.1}$$

Using (1.8) in (3.1) yields

$$\begin{aligned}&(\pounds_\xi(\pounds_X g))(Y, \xi) = \xi g(\nabla_Y X, \xi) + \xi g(\nabla_\xi X, Y) - g(\nabla_{[\xi,Y]} X, \xi)\\ &-g(\nabla_\xi X, [\xi, Y]) = g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_\xi \xi) + g(\nabla_\xi \nabla_\xi X, Y)\\ &+g(\nabla_\xi X, \nabla_\xi Y) - g(\nabla_{[\xi,Y]} X, \xi) - g(\nabla_\xi X, \nabla_\xi Y) + g(\nabla_\xi X, \nabla_Y \xi)\\ &= g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_\xi \xi) + g(\nabla_\xi \nabla_\xi X, Y) - g(\nabla_{[\xi,Y]} X, \xi)\\ &+g(\nabla_\xi X, \nabla_Y \xi).\end{aligned} \tag{3.2}$$

Now by the definition of Riemannian curvature tensor, from (3.2) it follows that

$$(\pounds_\xi(\pounds_X g))(Y, \xi) = g(R(\xi, Y)X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) + Y g(\nabla_\xi X, \xi). \tag{3.3}$$

Using (2.2) in (3.3) and with Y orthogonal to ξ, we infer that

$$(\pounds_\xi(\pounds_X g))(Y, \xi) = g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Y g(\nabla_\xi X, \xi).$$

□

Lemma 3.2. *[12] Let (M, g) be a Riemannian manifold and let f be a smooth function. Then*

$$(\pounds_\xi(df \odot df))(Y, \xi) = Y(\xi(f))\xi(f) + Y(f)\xi(\xi(f)),$$

for every vector field Y.

Lemma 3.3. *Let (M, ϕ, ξ, η, g) be a K-contact manifold which satisfies the generalized Ricci soliton equation. Then*

$$\nabla_\xi grad\ f = (\lambda + 2c_2 n)\xi - c_1 \xi(f) grad\ f.$$

Proof. Using (2.4) we have

$$\lambda \eta(Y) + c_2 S(\xi, Y) = (\lambda + 2c_2 n)\eta(Y). \tag{3.4}$$

Making use of (1.9) and (3.4) implies

$$(Hess\ f)(\xi, Y) = -c_1 \xi(f) g(grad\ f, Y) + (\lambda + 2c_2 n)\eta(Y). \tag{3.5}$$

Hence the Lemma follows from (3.5) and the definition of the Hessian (1.9). □

Now we prove our main theorem as follows:

Theorem 3.1. *Suppose that (M, ϕ, ξ, η, g) is a K-contact manifold of dimension $(2n + 1)$ which satisfies the generalized gradient Ricci soliton equation with $c_1(\lambda + 2c_2 n) \neq -1$,. Then f is a constant function. Furthermore, if $c_2 \neq 0$, then the manifold is an Einstein one.*

Proof. Suppose that Y is orthogonal to ξ. Then from Lemma 3.1 with $X = grad\ f$, we have

$$2(\pounds_\xi(Hess\ f))(Y,\xi) = Y(f) + g(\nabla_\xi \nabla_\xi grad\ f, Y) + Y g(\nabla_\xi grad\ f, \xi). \tag{3.6}$$

Using Lemma 3.3 in (3.6) yields

$$\begin{aligned} &2(\pounds_\xi(Hess\ f))(Y,\xi) = Y(f) + (\lambda + 2c_2 n) g(\nabla_\xi \xi, Y) \\ &-c_1 g(\nabla_\xi(\xi(f) grad\ f), Y) + (\lambda + 2c_2 n) Y - c_1 Y(\xi(f)^2) \\ &= Y(f) - c_1 g(\nabla_\xi(\xi(f) grad\ f), Y) + (\lambda + 2c_2 n) Y - c_1 Y(\xi(f)^2). \end{aligned} \tag{3.7}$$

Again using Lemma 3.3 with Y orthogonal to ξ, from (3.7) it follows that

$$\begin{aligned} &2(\pounds_\xi(Hess\ f))(Y,\xi) = Y(f) - c_1 \xi(\xi(f)) Y(f) + c_1^2 \xi(f)^2 Y(f) \\ &-2c_1 \xi(f) Y(\xi(f)). \end{aligned} \tag{3.8}$$

Since ξ is a Killing vector field, so $\pounds_\xi g = 0$, it implies $\pounds_\xi S = 0$. Using the above fact and taking the Lie derivative to the generalized Ricci soliton equation (1.9) yields

$$2(\pounds_\xi(Hess\ f))(Y,\xi) = -2c_1(\pounds_\xi(df \odot df))(Y,\xi). \tag{3.9}$$

Using (3.8), (3.9) and Lemma 3.2 we infer that

$$Y(f)[1 + c_1 \xi(\xi(f)) + c_1 \xi(f^2)] = 0. \tag{3.10}$$

According to Lemma 3.3 we have

$$\begin{aligned} &c_1 \xi(\xi(f)) = c_1 \xi g(\xi, grad\ f) \\ &= c_1 g(\xi, \nabla_\xi grad\ f) \\ &= c_1(\lambda + 2c_2 n) - c_1^2 \xi(f)^2. \end{aligned} \tag{3.11}$$

Making use of (3.10) and (3.11), we obtain

$$Y(f)[1 + c_1(\lambda + 2c_2 n)] = 0,$$

which implies

$$Yf = 0,$$

provided $1 + c_1(\lambda + 2c_2 n) \neq 0$. Therefore, $grad\ f$ is parallel to ξ. Hence $grad\ f = 0$ as $d = ker\ \eta$ is nowhere integrable, that is, f is a constant function. Thus the manifold is an Einstein one follows from (1.9). □

Remark 3.1. We know that [10] every Sasakian manifold is K-contact, but the converse is not true in general. However, a 3-dimensional K-contact manifold is Sasakian. Thus our main Theorem 3.1 is the generalization of Theorem 1.1 of [12].

Remark 3.2. Since a compact K-contact Einstein manifold is Sasakian [2], therefore a compact K-contact manifold admitting generalized Ricci solitons is Sasakian.

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