Available online: February 5, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 699–716 (2020)
DOI: 10.31801/cfsuasmas.635829
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ANALYSIS OF JOINT RELIABILITY IMPORTANCE IN LINEAR m-CONSECUTIVE-k,l-OUT-OF-n:F SYSTEM

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Abstract. Combinatorial techniques have an important role to compute the joint reliability importance (JRI) of some coherent systems. We obtain combinatorial formula for calculation of the JRI of two components in a generalised version of consecutive type systems consisting of n linearly ordered components such that system fails if and only if (iff) there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l<k$). Overlapping runs mean having common elements which is denoted by l. We concentrate on both s-independent & identical components and exchangeable components. Explicit combinatorial formulae are provided for computing the JRI of the above mentioned cases. For both cases, we also compare the results with linear m-consecutive-k-out-of-n:F system (nonoverlapping case when $l=0$). In addition, some numerical and illustrative examples are presented.

Acronyms and Notations

MRI	Marginal Reliability Importance
JRI	Joint Reliability Importance
Lin/m/Con/k/l/$n:F$	Linear m-consecutive-k,l-out-of-n:F
n	number of components
X_i	the state of component i, $i=1,\ldots,n$ ($X_i=1$ if the ith component fails, and $X_i=0$ if the ith component works)
E	the event that the system works
k_ϕ	minimum number of failed components that may cause system failure
z_ϕ	maximum number of failed components such that a system can still work successfully

Received by the editors: October 22, 2019; Accepted: January 20, 2020.

2010 *Mathematics Subject Classification.* Primary 60K10, 60G09 ; Secondary 62P30, 62N05.

Key words and phrases. m-consecutive-k,l-out-of-n:F systems, joint reliability importance, exchangeability, system reliability, combinatorial method.

1. Introduction

The marginal reliability importance (MRI) of a component measures the change in system reliability with respect to the change in component reliability([5, 6, 24, 35, 36, 37]). MRI is very useful in engineering fields such as design and improvement of a system. If $MRI_{E+j}(i)$ ($MRI_{E-j}(i)$) denotes MRI of ith component when jth component is functioning (failed), then $JRI(i,j) = MRI_{E+j}(i) - MRI_{E-j}(i)$ where the JRI is a measure of the interaction of the components in their contribution to system reliability (see [3, 14, 19, 20, 21]). Type and degree of interactions between two components are represented not only by the sign but also by the value of the JRI of two components in a coherent system. If the sign of the JRI of two components is nonnegative (nonpositive), it is called reliability complements (substitutes) ([19]). Moreover, if $JRI > 0$ ($JRI < 0$), then one component becomes more (less) important when the other is functioning which is also considered as synergy (diminishing returns). For $JRI = 0$, then one component's importance is not affected by the functioning of the other component ([3]). In literature, there are many studies on computation and analysis of JRI. Hong, Koo, and Lie [22] obtained a closed-form equation for the JRI of two components in a k-out-of- $n : G$ system, and examined its properties with respect to component reliability, and system parameters k and n. Gao, Cui, and Li [14] deeply analyzed JRI of three components in a k-out-of-$n : G$ system with independent components. Gertsbakh and Shpungin [17] combinatorially computed the JRI of two components. Rani, Jain, and Dewan [34] presented conditional marginal and conditional JRI in series–parallel systems. Eryilmaz [10] presented JRI in linear m-consecutive-k-out-of-$n : F$ systems. Mahmoud and Eryilmaz [28] studied exchangeable dependent components which is generalization of some results in Hong, Koo, and Lie [22] and Gao, Cui, and Li [14]. Zhu, Mahmoud, and Mohamed [38] presented JRI in m-consecutive-k-out-of-$n : F$ system that consists of Markov dependent components. Zhu, Mahmoud, and Mohamed [39] computed the JRI in consecutive-k-within-m-out-of-$n : F$ system with Markov dependent components. Eryilmaz and Mahmoud [8] firstly proposed and studied the m-consecutive-k, l-out-of-$n : F$ system. Zhu et al. [40] derived closed-form formulas for the reliability of the m-consecutive-k, l-out-of-$n : F$ and G systems, and computed JRI of this system when the components are non-homogenous Markov-dependent. One can see an extensive review of reliability importance measures in Kuo and Zhu [24] and Kuo, Way, and Zuo [25].

Eryilmaz, Oruc, and Oger [12] obtained general formula for computing the joint reliability importance of two components for a binary coherent system that consists of exchangeable dependent components. In that study, the joint reliability importance can be easily calculated if the path sets of the system are known. On the other hand, achieving the full list of path sets for the computation of JRI of any coherent system is not an easy task. Hence, only combinatorial formula for a series-parallel system is given in the study of Eryilmaz, Oruc, and Oger [12]. From this point of view, combinatorial techniques have an important role to compute JRI of some coherent systems. In this paper, combinatorial method has been used for computing the JRI of two components in Lin/m/Con/k/l/$n:F$ systems consisting of n linearly ordered components such that the system fails iff there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l < k$). Unlike the study done by Zhu et al. [40], we concentrate on both s-independent & identical components and exchangeable components. For both cases, we also compare the results with linear m-consecutive-k-out-of-n:F system (nonoverlapping case when $l=0$). Eryilmaz [11] mentioned Birnbaum importance of a component when the system consists of exchangeable dependent components which is distinguished from our paper. We give explicit formula for calculation of JRI of two components under these two cases. And finally, some numerical and illustrative examples are presented.

2. Lin/m/Con/k/l/$n:F$ System

The Lin/m/Con/k/l/$n:F$ is a system that consists of n linearly ordered components such that the system fails iff there are at least m l-overlapping runs of k consecutive failed components ($n \geq m(k-l)+l, l < k$). Overlapping runs mean having common elements. For instance, 1111 is a sequence which contains two overlapping runs of length three and three overlapping runs of length two. Now, consider the states of a system with 16 components be 1110011100110100. For $m=5, k=2$ and $l=0$, this system is functioning when $l=1$ it is failed. When $l=0$, the Lin/m/Con/k/l/$n:F$ system becomes the non-overlapping Lin/m/Con/k/$n:F$ system which is introduced by Griffith [18] and Papastavridis [33]. When $l=k-1$, it reduces to the overlapping Lin/m/Con/k/$n:F$ systems. When $m=1$, the Lin/m/Con/k/l/$n:F$ system reduces to the Lin/Con/k/$n:F$ system. Also when $k=1$, the Lin/m/Con/k/l/$n:F$ system becomes m-out-of-$n:F$ system. This advanced system model with addition of the new parameter l creates diversity for real life applications in quality control, statistics and probability. Recently, there are many discussions on Lin/m/Con/k/l/$n:F$ system. For instance, Agarwal and Mohan [1] computed reliability of the system with the help of graphical evaluation and review technique under assumptions of i.i.d. components and $(k-1)$-step Markov dependent components. Some recent contributions on Lin/m/Con/k/l/$n:F$ system are the works of Gera [16], Levitin and Dai [27], Cui, Lin, and Du [7] and Zhu et al. [40].

The reliability of Lin/m/Con/k/l/$n:F$ system is closely related with the run statistics $N_{n,k,l}^{L(1:n)}$, which denotes the total number of $l-$overlapping runs of failures of length k in a linearly ordered sequence of binary trials $X_1, X_2, \ldots, X_n$. The distribution of the random variable $N_{n,k,l}^{L(1:n)}$ has been named the binomial distribution of order k for $l-$overlapping runs of length k, and introduced and studied by Aki and Hirano [2]. The reliability of Lin/m/Con/k/l/$n:F$ system can be expressed as $P\{E\} = P\{N_{n,k,l}^{L(1:n)} < m\}$. Some recent discussions on this topic are Eryilmaz [9], Eryilmaz and Mahmoud [8], Levitin [26], Makri and Psillakis [29, 30] and Makri and Psillakis [31]. For extensive reviews of the runs related literature, we refer to Balakrishnan and Koutras [4], Fu and Lou [13], and Koutras [23].

3. The Reliability of Lin/m/Con/k/l/$n : F$ system

Eryilmaz [9] computed that the reliability of Lin/m/Con/k/l/$n : F$ system consisting of s-independent components with common working probability $P\{X_i = 0\} = p$, and $r_i(n)$ denotes the total number of path sets of this structure including i working components,

$$\sum_{i=n-z_\phi}^{n} r_i(n) p^i (1-p)^{n-i}$$

where

$$z_\phi = n - 1 - \left[\frac{n - m(k-l) - l}{k}\right]$$

$n \geq m(k-l)+l$, and $[x]$ denotes the integer part of x and for the derivation of $r_i(n)$, see, Theorems 2.1 of Makri, Philippou, and Psillakis [32] and Eq. (1) of Eryilmaz and Mahmoud [8]). For simplicity of calculation, throughout this paper, we will denote $N_{i,a,k,l,s,n}$ as $N_{n,k,l}^{L(1:n)}$, hence

$$r_i(n) = \sum_{s=0}^{m-1} \sum_{a} \binom{i+1}{a} N_{n,k,l}^{L(1:n)}.$$

In an explicit way, by using Theorem 1 of Eryilmaz and Mahmoud [8] it can be written as

$$\begin{aligned} r_i(n) &= C(n-i; i+1, 0; k-1; k-1) \\ &+ \sum_{s=1}^{m-1} \sum_{a=1}^{\min(i+1,s)} \binom{i+1}{a}\binom{s-1}{a-1} C(n-i-al-s(k-l); a, i-a+1; k-l-1, k-1) \end{aligned}$$

where the quantities $C(\beta; a, r-a; m_1-1, m_2-1)$ can be calculated via the following formula (see, e.g. Makri, Philippou, and Psillakis [32]):

$$C(\beta; a, r-a; m_1-1, m_2-1) = \sum_{j_1=0}^{\left[\frac{\beta}{m_1}\right]} \sum_{j_2=0}^{\left[\frac{\beta - m_1 j_1}{m_2}\right]} (-1)^{j_1+j_2} \binom{a}{j_1}\binom{r-a}{j_2}\binom{\beta - m_1 j_1 - m_2 j_2 + r - 1}{r-1}$$

4. The JRI of Lin/m/Con/k/l/$n : F$ system

Consider Lin/m/Con/k/l/$n : F$ system consists of n binary components. Let X_i denote the state of ith component ($X_i = 1$ if the ith component fails, and $X_i = 0$ if it works, $i = 1, 2, \ldots, n$.) and E be the event that system functions. Then the JRI of components i and j can be defined as (see Kuo and Zhu [2012])

$$\begin{aligned} JRI(i,j) &= P\{E|X_i = 1, X_j = 1\} - P\{E|X_i = 1, X_j = 0\} - P\{E|X_i = 0, X_j = 1\} \\ &+ P\{E|X_i = 0, X_j = 0\}. \end{aligned} \tag{1}$$

Eryilmaz [10] expressed (1) by using the law of total probability as follows

$$JRI(i,j) = \frac{P\{E\} - P\{E, X_j = 0\} - P\{E, X_i = 0\} + P\{E, X_i = 0, X_j = 0\}}{1 - P\{X_j = 0\} - P\{X_i = 0\} + P\{X_i = 0, X_j = 0\}}$$

$$- \frac{P\{E, X_j = 0\} - P\{E, X_i = 0, X_j = 0\}}{P\{X_j = 0\} - P\{X_i = 0, X_j = 0\}}$$
$$- \frac{P\{E, X_i = 0\} - P\{E, X_i = 0, X_j = 0\}}{P\{X_i = 0\} - P\{X_i = 0, X_j = 0\}} + \frac{P\{E, X_i = 0, X_j = 0\}}{P\{X_i = 0, X_j = 0\}}.$$

So we need to calculate $P\{E, X_i = 0\}$ and $P\{E, X_i = 0, X_j = 0\}$ for the computation of JRI. It can easily be written as

$$\begin{aligned} P\{E, X_i = 0\} & \\ &= P\{N_{n,k,l}^{L(1:n)} < m, X_i = 0\} \\ &= P\{N_{i-1,k,l}^{L(1:i-1)} + N_{n-i,k,l}^{L(i+1:n)} < m, X_i = 0\} \\ &= \sum_{s_1+s_2<m}\sum P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, N_{n-i,k,l}^{L(i+1:n)} = s_2, X_i = 0\} \qquad (2) \end{aligned}$$

and

$$\begin{aligned} P\{E, X_i = 0, X_j = 0\} & \\ &= P\{N_{i-1,k,l}^{L(1:i-1)} + N_{j-i-1,k,l}^{L(i+1:j-1)} + N_{n-j,k,l}^{L(j+1:n)} < m, X_i = 0, X_j = 0\} \\ &= \sum_{s_1+s_2+s_3<m}\sum\sum P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, N_{n-j,k,l}^{L(j+1:n)} = s_3, X_i = 0, X_j = 0\}. \end{aligned} \qquad (3)$$

In the following subsections, we will obtain combinatorial formulas for the JRI of Lin/m/Con/k/l/n : F systems consisting of

i. s-independent and identical components (common working probability $P\{X_i = 0\} = p$),
ii. exchangeable s-dependent components.

4.1. **S-Independent and Identical Components.** Consider a Lin/m/Con/k/l/n : F system when the components are s-independent with same working probability $P\{X_i = 0\} = p$. Assume that in the sequence of the first $i-1$ components there are m_1 working ones and in the sequence of the last $n-i$ components there are m_2 working components, and let $S_{b-a+1}^{(a:b)}$ denote the number of working components among the components $a, a+1, \dots, b$, for $a < b$. That is $S_{b-a+1}^{(a:b)} = \sum_{i=a}^{b}(1-X_i)$, then by conditioning on s_1, s_2 and working components (2) can be rewritten as

$$\begin{aligned} P\{E, X_i = 0\} = \sum_{s_1+s_2<m}\sum \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} P\Big\{ & N_{i-1,k,l}^{L(1:i-1)} = s_1, N_{n-i,k,l}^{L(i+1:n)} = s_2, \\ & S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\Big\}. \end{aligned}$$

For the simplicity of calculation, above equation can be written as a sum of 4 terms (4,5,6 and 7) as follows

$$\begin{aligned} P\{E, X_i = 0\} = \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} P\Big\{ & N_{i-1,k,l}^{L(1:i-1)} = 0, N_{n-i,k,l}^{L(i+1:n)} = 0, \\ & S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\Big\} \end{aligned} \qquad (4)$$

$$+\sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)}\sum_{m_1=0}^{i-1}\sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N_{i-1,k,l}^{L(1:i-1)}=0, N_{n-i,k,l}^{L(i+1:n)}=s_2,\right.$$
$$\left.S_{i-1}^{(1:i-1)}=m_1, S_{n-i}^{(i+1:n)}=m_2, X_i=0\right\} \tag{5}$$

$$+\sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)}\sum_{m_1=low_{m_1}}^{up_{m_1}}\sum_{m_2=0}^{n-i} P\left\{N_{i-1,k,l}^{L(1:i-1)}=s_1, N_{n-i,k,l}^{L(i+1:n)}=0,\right.$$
$$\left.S_{i-1}^{(1:i-1)}=m_1, S_{n-i}^{(i+1:n)}=m_2, X_i=0\right\} \tag{6}$$

$$+\sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)}\sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)}\sum_{m_1=low_{m_1}}^{up_{m_1}}\sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N_{i-1,k,l}^{L(1:i-1)}=s_1,\right.$$
$$\left.N_{n-i,k,l}^{L(i+1:n)}=s_2, S_{i-1}^{(1:i-1)}=m_1, S_{n-i}^{(i+1:n)}=m_2, X_i=0\right\} \tag{7}$$

where

$$up_{m_1} = i-1-k-(s_1-1)(k-l),$$
$$low_{m_1} = \left| \begin{array}{ll} 1+\left[\frac{i-2-s_1(k-l)-l}{k}\right] & \text{if } \frac{i-1-l}{k-l} < m-1 \\ 0 & \text{otherwise} \end{array}\right. ,$$
$$up_{m_2} = n-i-k-(s_2-1)(k-l),$$
$$low_{m_2} = \left| \begin{array}{ll} 1+\left[\frac{n-i-1-s_2(k-l)-l}{k}\right] & \text{if } \frac{n-i-l}{k-l} < m-1 \\ 0 & \text{otherwise} \end{array}\right. .$$

For better understanding of the terms 4,5,6 and 7, an explanation is given at Appendix.

Now, consider the probability $P\{E, X_i=0, X_j=0\}$. From (3)

$$P\{E, X_i=0, X_j=0\}$$
$$=\sum\sum\sum_{s_1+s_2+s_3<m}\sum_{m_1=0}^{i-1}\sum_{m_2=0}^{j-i-1}\sum_{m_3=0}^{n-j} P\left\{N_{i-1,k,l}^{L(1:i-1)}=s_1, S_{i-1}^{(1:i-1)}=m_1, N_{j-i-1,k,l}^{L(i+1:j-1)}=s_2,\right.$$
$$\left.S_{j-i-1}^{(i+1:j-1)}=m_2, N_{n-j,k,l}^{L(j+1:n)}=s_3, S_{n-j}^{(j+1:n)}=m_3, X_i=0, X_j=0\right\}$$

By using the independence of components, we can write,

$$P\left\{N_{i-1,k,l}^{L(1:i-1)}=s_1, N_{j-i-1,k,l}^{L(i+1:j-1)}=s_2, N_{n-j,k,l}^{L(j+1:n)}=s_3,\right.$$
$$\left.S_{i-1}^{(1:i-1)}=m_1, S_{j-i-1}^{(i+1:j-1)}=m_2, S_{n-j}^{(j+1:n)}=m_3, X_i=0, X_j=0\right\}$$
$$=P\left\{N_{i-1,k,l}^{L(1:i-1)}=s_1, S_{i-1}^{(1:i-1)}=m_1\right\} P\left\{N_{j-i-1,k,l}^{L(i+1:j-1)}=s_2, S_{j-i-1}^{(i+1:j-1)}=m_2\right\}$$
$$\times P\left\{N_{n-j,k,l}^{L(j+1:n)}=s_3, S_{n-j}^{(j+1:n)}=m_3\right\} P\left\{X_i=0\right\} P\left\{X_j=0\right\}$$

So, the cardinality of $N_{j-i-1,k,l}^{L(i:j)}$ and $S_{j-i-1}^{(i:j)}$ denotes the number of having s l-overlapping failure runs of length k in a linear binary sequence of length $j-i-1(i<j)$ with M number of working component(s) which is given as

$$\{N_{j-i-1,k,l}^{L(i:j)}=s, S_{j-i-1}^{(i:j)}=M\}=$$

$$\begin{cases} C(j-i-1-M; M+1, 0; k-1, k-1), & \text{for } s=0, \\ \sum_a \binom{M+1}{a}\binom{s-1}{a-1} C(j-i-M-al-s(k-l); a, M+1-a; k-l-1, k-1), & \text{for } s>0. \end{cases} \quad (8)$$

The quantity $C(\beta; a, r-a; m_1-1, m_2-1)$ can be calculated via the following formula (see, e.g. Makri, Philippou, and Psillakis [32]):

$$C(\beta; a, r-a; m_1-1, m_2-1) = \sum_{j_1=0}^{\left[\frac{\beta}{m_1}\right]} \sum_{j_2=0}^{\left[\frac{\beta-m_1 j_1}{m_2}\right]} (-1)^{j_1+j_2} \binom{a}{j_1}\binom{r-a}{j_2}\binom{\beta-m_1 j_1-m_2 j_2+r-1}{r-1}.$$

$P\{E, X_i=0, X_j=0\}$ can be rewritten explicitly which was shown at Appendix.

4.1.1. *Numerical Studies and Illustrations.* In this subsection, we present illustrative computational results for the JRI of components in a Lin/m/Con/k/l/$n : F$ system when the components are s-independent with same working probability $P\{X_i = 0\} = p$. In Figure 1., we compare graph of JRI(2,5) considered as a function of component reliability p for $m = 3, k = 2, n = 20$ and $l = 0, 1$.

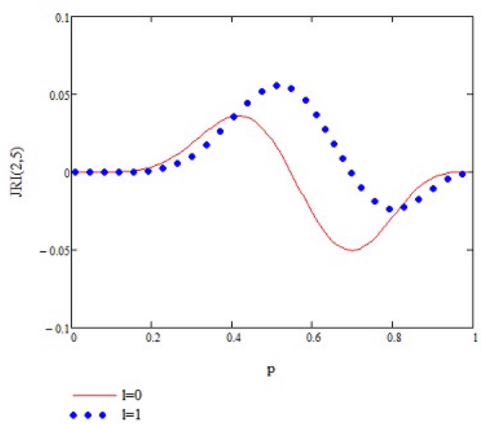

FIGURE 1. JRI(2,5) in a Lin/3/Con/2/l/20:F system as a function of p for $l = 0$ and 1.

It can be easily seen that for Lin/3/Con/2/0/20 : F system, the sign of JRI(2,5) changes around at the point $p = 0.55$. On the other hand, for Lin/3/Con/2/1/20 : F system this point shifts around $p = 0.7$. As a result for the values $i = 2$ and $j = 5$, the graph of Lin/3/Con/2/1/20 : F system can be considered as a graph of Lin/3/Con/2/0/20 : F system shifted to the right. The sign of JRI(2,5) may not change for some values of n, m, k as seen in Figure 2.

In Table 1., we present all pairwise JRI values of the Lin/2/Con/2/1/5 : F system for different values of p.

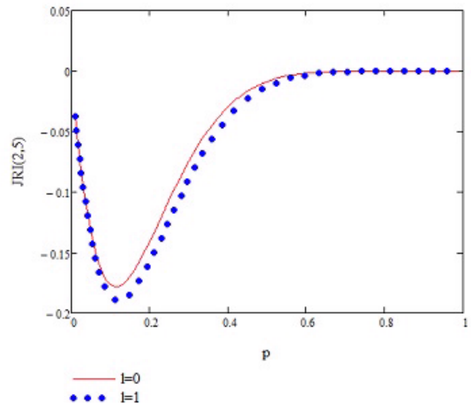

FIGURE 2. JRI(2,5) in a Lin/2/Con/5/l/12:F system as a function of p for $l = 0$ and

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
JRI(1,2)	−0.171	−0.288	−0.357	−0.384	−0.375	−0.336	−0.273	−0.192	−0.099
JRI(1,3)	0.639	0.352	0.133	−0.024	−0.125	−0.176	−0.183	−0.152	−0.089
JRI(1,4)	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001
JRI(1,5)	−0.081	−0.128	−0.147	−0.144	−0.125	−0.096	−0.063	−0.032	−0.009
JRI(2,3)	0.549	−0.192	−0.073	−0.264	−0.375	−0.416	−0.393	−0.312	−0.179
JRI(2,4)	0.639	0.352	0.133	−0.024	−0.125	−0.176	−0.183	−0.152	−0.089

Table 1. All Pairwise JRI Values for the Lin/2/Con/2/1/5 : F System

Note that the Lin/m/Con/k/l/n : F system is symmetric, more precisely $JRI(i, j) = JRI(n-i+1, n-j+1)$, By Table 1, $JRI(1, 2) < JRI(1, 5) < 0$ for all $0 < p < 1$. which means component 2 should be more reliable than component 5 to decrease the diminishing return effect of component 1. while $JRI(1, 4) > 0$ for all $0 < p < 1$. which means that component 1 and 4 have complementary synergy. On the other hand, one can see that components 1 and 3, and 2 and 4 are reliability complements for $p < 0.4$, while they are reliability substitutes for $p \geq 0.4$. The components 2 and 3 are reliability substitutes for $p \geq 0.2$.

In Table 2., we show the sign of JRI between component 1 and others for different values of m, k, l and n with different component reliability p when Lin/m/Con/k/l/n : F system contains s-independent and identical components.

p	n	m	k	l	$JRI(1,2)$	$JRI(1,3)$	$JRI(1,4)$	$JRI(1,5)$	$JRI(1,6)$	$JRI(1,7)$	$JRI(1,j), j>7$
0.85	20	3	2	1	−	−	−	−	−	−	−
0.9	20	3	2	1	−	−	−	−	−	−	−
0.9	30	3	2	1	−	−	−	−	−	−	−
0.9	30	3	3	1	−	−	−	+	−	−	−
0.9	30	3	3	2	−	−	−	−	+	−	−
0.9	30	4	3	1	−	−	−	+	−	+	−
0.9	30	4	3	2	−	−	−	−	−	+	−

Table 2. The sign of $JRI(1,j),\ j>1$ for different Lin/m/Con/$k/l/n : F$ Systems

This table shows the effect of the system values n, m, k, l and p on the JRI. Also we observe that components 1 and 5 are reliability complements in Lin/3/Con/3/1/30 : F system when the components are s-independent with same working probability $p = 0.9$. However, they are reliability substitutes in Lin/3/Con/3/2/30 : F system with same component reliability. For Lin/3/Con/3/l/30 : F systems, when we change the value of l from 1 to 2, the reliability complementary components 1 and 5 turns into reliability substitutive components, while the reliability substitutive components 1 and 6 turns into reliability complementary components. As a result, the sign of JRI may change as the value of n, m, k, l and p change.

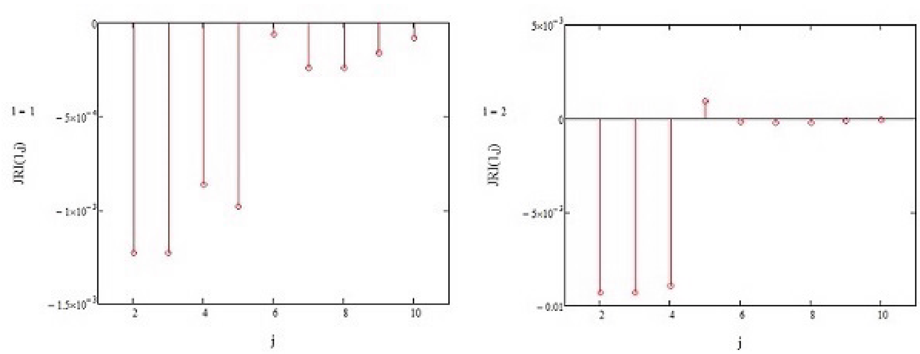

FIGURE 3. JRI$(1, j)$, $j = 2, ..., 10$, of Lin/2/Con/3/l/10:F system with common working probability$p = 0.9$ for $l = 1$ and $l = 2$.

In Figure 3 we present $JRI(1, j)$, $j = 2, \ldots, 10$, of Lin/2/Con/3/l/10 : F system consisting of independent and identical components with working probability $p = 0.9$ for the two cases: when $l = 1$ and $l = 2$. From this figure, we observe that the sign of the $JRI(1, 5)$ for these two cases are different. When $JRI(1, j) < 0$ ($j = 2, \ldots, 10$), increasing in l causes diminishing on the value of $JRI(1, j)$ in Lin/2/Con/3/l/10 : F system, generally.

4.2. **Exchangeable S-Dependent Components.** In this section, JRI formula is obtained for Lin/m/Con/$k/l/n : F$ system consisting of exchangeable components.

A sequence of components $X_1, X_2, \ldots, X_n$ is exchangeable if for each n,

$$P\{X_{\pi_1} = x_1, \ldots, X_{\pi_n} = x_n\} = P\{X_1 = x_1, \ldots, X_n = x_n\}$$

for any permutation $(\pi_1, \ldots, \pi_n)$ of the indices in $(1, \ldots, n)$, i.e. the joint distribution of $X_1, X_2, \ldots, X_n$ is symmetric in $x_1, x_2, \ldots, x_n$. The exchangeability means that the components have identical distribution, but they affect one other within the system. That means, the joint distribution of $X_1, X_2, \ldots, X_n$ is invariant under permutation of its arguments. From George and Bowman (1995), any sequence with a 0 s and $n - a$ 1 s has probability

$$\begin{aligned} g(n, a) &= P\{X_1 = 0, \ldots, X_a = 0, X_{a+1} = 1, \ldots, X_n = 1\} \\ &= \sum_{i=0}^{n-a} (-1)^i \binom{n-a}{i} \lambda_{a+i} \\ &= \sum_{i=0}^{a} (-1)^i \binom{a}{i} \theta_{n-a+i} \end{aligned}$$

where $\lambda_a = P\{X_1 = 0, \ldots, X_a = 0\}$ and $\theta_a = P\{X_1 = 1, \ldots, X_a = 1\}$ with $\lambda_0 = 1, \theta_0 = 1$.

Since (2) can be obtained by the sum of (9), (10), (11) and (12), for exchangeable components $p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$ can be replaced by $g(n, m_1 + m_2 + 1)$ in (2) [see Eryilmaz [10]]. Similarly, $g(n, m_1 + m_2 + m_3 + 2)$ can be substituted in (3).

4.2.1. *Numerical Studies and Illustrations.* In this subsection, we consider Lin/m/Con/k/l/n : F system consisting of exchangeable components and present illustrative computational results for the JRI of components. Suppose p have a Beta distribution with parameters α and β. Hence for exchangeable random variables $X_1, \ldots, X_n$,

$$\begin{aligned} \lambda_a &= P\{X_1 = 0, \ldots, X_a = 0\} \\ &= \int_0^1 p^a \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp \\ &= \frac{\Gamma(a+\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(a+\alpha+\beta)} \end{aligned}$$

when $a \geq 1$.

In Figure 4, we present $JRI(1, j)$ of Lin/m/Con/3/l/10 : F system consisting of exchangeable components with parameters $\alpha = 9$ and $\beta = 1$ for the two cases: when $m = 2$ and $m = 3$. Clearly, when $m = 2$, the sign $JRI(1, j)$, $j = 2, \ldots, 10$, are same for the values $l = 1$ and $l = 2$ but the sign of $JRI(1, 5)$ is different for another case, $m = 3$, when $l = 1$ and $l = 2$ we observe that the sign of the $JRI(1, 5)$ for these two cases are opposite. Similar to s-independent and identical case, increasing in l causes diminishing on the value of JRI between the first and the other components in Lin/m/Con/k/l/n : F system consisting of exchangeable components, for most cases.Since systems in Figure 3 containing s-independent components and systems in Figure 4 containing exchangeable components with parameters $\alpha = 9$ and $\beta = 1$

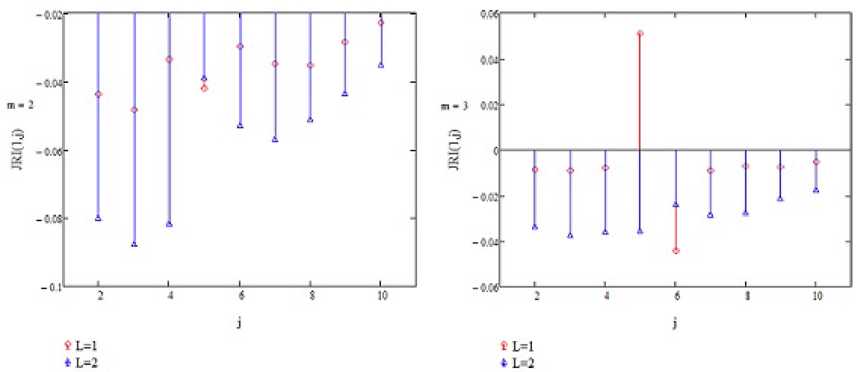

FIGURE 4. JRI$(1,j)$, $j=2,...,10$, of Lin/m/Con/3/l/10:F system with exchangeable components with parameters $\alpha=9$ and $\beta=1$ for the two cases: when $m=2$ ($l=1,2$) and $m=3$ ($l=1,2$).

have the same working component reliability $p = 0.9$, one can easily compare $JRI(1,j)$ of those common systems.
In Table 3, the sign of $JRI(1,j)$, $j=2,\ldots,10$, of various Lin/m/Con/k/l/$n:F$ systems consisting of exchangeable components with parameters $\alpha=9$ and $\beta=1$ are given.

α	β	p	n	m	k	l	$JRI(1,2)$	$JRI(1,3)$	$JRI(1,4)$	$JRI(1,j)$, $j>4$
4	6	0.4	20	3	2	1	+	+	+	+
10	6	0.625	20	3	2	1	−	+	+	+
10	4	0.714	20	3	2	1	−	−	−	−
4	6	0.4	10	3	2	1	−	+	+	+
4	6	0.4	20	2	2	1	+	+	+	+
4	6	0.4	20	2	3	1	−	−	+	+
4	6	0.4	20	2	3	2	−	−	+	+

Table 3. The sign of $JRI(1,j)$, $j>1$ for different Lin/m/Con/k/l/$n:F$ systems consisting of exchangeable components with parameters $\alpha=9$ and $\beta=1$.

From Table 2 and 3 we can see that the dependency may effect the sign of the JRI between the first and the other components. In addition, the sign of JRI between the first and the other components in Lin/m/Con/k/l/$n:F$ systems consisting of exchangeable components highly depend on the values of $n, m.k$, and l. However, from Table 3 one can say that increasing component reliability p will change reliability complement components into reliability substitute components, i.e. increasing p from 0.4 to 0.714 when $n=20, m=3, k=2$, and $l=1$.

5. Conclusions

We have studied on Lin/m/Con/k/l/$n : F$ system which is the generalization of consecutive k-out-of-$n : F$ system. A Lin/m/Con/k/l/$n : F$ system becomes a non-overlapping Lin/m/Con/k/$n : F$ system, an overlapping Lin/m/Con/k/$n : F$ system, a Lin/Con/k/$n : F$ system and a m-out-of-$n : F$ system for $l = 0, l = k-1, m = 1$ and $k = 1$ respectively. We have derived combinatorial formula for the computation of JRI of two components in Lin/m/Con/k/l/$n : F$ system when components are s-independent & identical components and exchangeable. One possible future effort can be carried on the computation of JRI in an arbitrary dependent case or by changing the type of the system from linear form into circular form.

6. Appendix

For better understanding of the terms 4,5,6 and 7, consider a binary sequence in the following form

$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components}}$	$\overset{\text{i}^{th} \text{ working component}}{0}$	$\underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components}}$

For the operation of system, s is the total number of l-overlapping failure runs of length k must be less than m. We can denote this by $s = s_1 + s_2 (s < m)$ where s_1 and s_2 denote the l-overlapping runs of length k in the first sequence $i-1$ components and in the last sequence $n-i$ components, respectively. Hence we have four possible cases for operation of system.

Case 1 (4)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } s_1=0}$	$\overset{\text{i}^{th} \text{ working component}}{0}$	$\underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } s_2=0}$
Case 2 (5)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } s_1=0}$	$\overset{\text{i}^{th} \text{ working component}}{0}$	$\underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } 0<s_2<m}$
Case 3 (6)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } 0<s_1<m}$	$\overset{\text{i}^{th} \text{ working component}}{0}$	$\underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } s_2=0}$
Case 4 (7)	$\underbrace{11\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{first } i-1 \text{ components containing } m_1 \text{ working components with } 0<s_1<m-1}$	$\overset{\text{i}^{th} \text{ working component}}{0}$	$\underbrace{1\ldots10\ldots01\ldots10\ldots01\ldots1}_{\text{last } n-i \text{ components containing } m_2 \text{ working components with } 0<s_2<m-1}$

Now let us consider terms of the sum 4,5,6 and 7 one by one. For term 4, where $s_1 = s_2 = 0$,

$$\sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, N_{n-i,k,l}^{L(i+1:n)} = 0, S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\}$$

$$= \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} C(i-1-m_1; m_1+1, 0; k-1, k-1) \times C(n-i-m_2; m_2+1, 0; k-1, k-1)$$

$$\times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$$

$$= \sum_{m_1=0}^{i-1}\sum_{m_2=0}^{n-i} \sum_{j=0}^{\min(m_1+1, \left[\frac{i-1-m_1}{k}\right])} (-1)^j \binom{m_1+1}{j}\binom{i-1-kj}{m_1}$$

$$\times \sum_{j=0}^{\min(m_2+1,\left[\frac{n-i-m_2}{k}\right])} (-1)^j \binom{m_2+1}{j}\binom{n-i-kj}{m_2} \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1} \quad (9)$$

In term 5, where $s_1 = 0$ and $0 < s_2 < m$,

$$\sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} P\left\{N_{i-1,k,l}^{L(1:i-1)} = 0, N_{n-i,k,l}^{L(i+1:n)} = s_2, S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\right\}$$

$$= \sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} C(i-1-m_1; m_1+1, 0; k-1, k-1)$$

$$\times \sum_{a=1}^{u} \binom{m_2+1}{a}\binom{s_2-1}{a-1} C(n-i-m_2-al-(k-l)s_2; a, m_2+1-a; k-l-1, k-1)$$

$$\times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$$

$$= \sum_{s_2=1}^{\min\left(m-1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{j=0}^{\min(m_1+1,\left[\frac{i-1-m_1}{k}\right])} (-1)^j \binom{m_1+1}{j}\binom{i-1-kj}{m_1}$$

$$\times \sum_{a=1}^{u} \binom{m_2+1}{a}\binom{s_2-1}{a-1} \sum_{j_1=0}^{\min(a,\left[\frac{n-i-al-(k-l)s_2-m_2}{k-l}\right])} \sum_{j_2=0}^{\min(m_2+1-a,\left[\frac{n-i-al-(k-l)(s_2+j_1)-m_2}{k}\right])}$$

$$\times (-1)^{j_1+j_2} \binom{a}{j_1}\binom{m_2+1-a}{j_2}\binom{n-i-al-(k-l)s_2-(k-l)j_1-kj_2}{m_2}$$

$$\times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1} \quad (10)$$

where $u = \left| \begin{array}{ll} \min(m_2+1, s_2) & \text{for } l = 0 \\ \min(m_2+1, s_2, \left[\frac{n-i-s_2(k-l)-m_2}{l}\right]) & \text{otherwise} \end{array} \right.$.

In term 6, where $s_2 = 0$ and $0 < s_1 < m$,

$$\sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} P\left\{N_{i-1,k,l}^{L(1:i-1)} = s_1, N_{n-i,k,l}^{L(i+1:n)} = 0, S_{i-1}^{(1:i-1)} = m_1, S_{n-i}^{(i+1:n)} = m_2, X_i = 0\right\}$$

$$= \sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} \sum_{a=1}^{v} \binom{m_1+1}{a}\binom{s_1-1}{a-1}$$

$$\times C(i-1-m_1-al-s_1(k-l); a, m_1+1-a; k-l-1, k-1)$$

$$\times C(n-i-m_2; m_2+1, 0; k-1, k-1) \times p^{m_1+m_2+1} \times (1-p)^{n-m_1-m_2-1}$$

$$= \sum_{s_1=1}^{\min\left(m-1,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{n-i} \sum_{a=1}^{v} \binom{m_1+1}{a}\binom{s_1-1}{a-1}$$

$$\times \sum_{j_1=0}^{\min(a,\left[\frac{i-1-al-s_1(k-l)-m_1}{k-l}\right])} \sum_{j_2=0}^{\min(m_1+1-a,\left[\frac{i-1-al-(k-l)(s_1+j_1)-m_1}{k}\right])} \Bigg\{(-1)^{j_1+j_2}\binom{a}{j_1}$$

$$\times\binom{m_1+1-a}{j_2}\binom{i-1-al-s_1(k-l)-(k-l)j_1-kj_2}{m_1}\Bigg\}$$

$$\times \sum_{j=0}^{\min(m_2+1,\left[\frac{n-i-m_2}{k}\right])} (-1)^j\binom{m_2+1}{j}\binom{n-i-kj}{m_2}\times p^{m_1+m_2+1}\times(1-p)^{n-m_1-m_2-1} \quad (11)$$

where $v = \left| \begin{array}{ll} \min(m_1+1, s_1) & \text{for } l=0 \\ \min(m_1+1, s_1, \left[\frac{i-1-s_1(k-l)-m_1}{l}\right]) & \text{otherwise} \end{array} \right.$.

In term 7, that is $s_1 \geq 1$, $s_2 \geq 1$, and $s_1+s_2 < m$,

$$\sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}}$$

$$\times P\{N^{L(1:i-1)}_{i-1,k,l}=s_1, N^{L(i+1:n)}_{n-i,k,l}=s_2, S^{(1:i-1)}_{i-1}=m_1, S^{(i+1:n)}_{n-i}=m_2, X_i=0\}$$

$$= \sum_{s_1=1}^{\min\left(m-2,\left[\frac{i-1-l}{k-l}\right]\right)} \sum_{s_2=1}^{\min\left(m-1-s_1,\left[\frac{n-i-l}{k-l}\right]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}}$$

$$\times P\{N^{L(1:i-1)}_{i-1,k,l}=s_1, S^{(1:i-1)}_{i-1}=m_1\}\times P\{N^{L(i+1:n)}_{n-i,k,l}=s_2, S^{(i+1:n)}_{n-i}=m_2\}\times P\{X_i=0\} \quad (12)$$

where

$$\begin{aligned} P\{N^{L(1:i-1)}_{i-1,k,l}=s_1, S^{(1:i-1)}_{i-1}=m_1\} &= \sum_{a=1}^{v}\binom{m_1+1}{a}\binom{s_1-1}{a-1} \\ &\times C(i-1-m_1-al-s_1(k-l); a, m_1+1-a; k-l-1, k-1) \\ &\times p^{m_1}\times(1-p)^{i-1-m_1}, \\ P\{N^{L(i+1:n)}_{n-i,k,l}=s_2, S^{(i+1:n)}_{n-i}=m_2\} &= \sum_{a=1}^{u}\binom{m_2+1}{a}\binom{s_2-1}{a-1} \\ &\times C(n-i-m_2-al-(k-l)s_2; a, m_2+1-a; k-l-1, k-1) \\ &\times p^{m_2}\times(1-p)^{n-i-m_2}. \end{aligned}$$

Explanation for $P\{E, X_i=0, X_j=0\}$:

$P\{E, X_i=0, X_j=0\}$ can be rewritten explicitly as follows

$$P\{E, X_i=0, X_j=0\} =$$

$$\sum_{m_1=0}^{i-1}\sum_{m_2=0}^{j-i-1}\sum_{m_3=0}^{n-j} P\{N^{L(1:i-1)}_{i-1,k,l}=0, S^{(1:i-1)}_{i-1}=m_1\}P\{N^{L(i+1:j-1)}_{j-i-1,k,l}=0, S^{(i+1:j-1)}_{j-i-1}=m_2\}$$

$$\times P\{N^{L(j+1:n)}_{n-j,k,l}=0, S^{(j+1:n)}_{n-j}=m_3\}P\{X_i=0\}P\{X_j=0\}$$

$$+ \sum_{s_1=1}^{\min(m-1,[\frac{i-1-l}{k-l}])} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{j-i-1} \sum_{m_3=0}^{n-j} P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\}$$
$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_2=1}^{\min(m-1,[\frac{j-i-1-l}{k-l}])} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=0}^{n-j} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\}$$
$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_3=1}^{\min(m-1,[\frac{n-j-l}{k-l}])} \sum_{m_1=0}^{i-1} \sum_{m_2=0}^{j-i-1} \sum_{m_3=low_{m_3}}^{up_{m_3}} P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\}$$

$$\times P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\} P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\}$$
$$\times P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-2,[\frac{i-1-l}{k-l}])} \sum_{s_2=1}^{\min\left(m-1-s_1,[\frac{j-i-1-l}{k-l}]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=0}^{n-j}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$
$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = 0, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-2,[\frac{i-1-l}{k-l}])} \sum_{s_3=1}^{\min\left(m-1-s_1,[\frac{n-j-l}{k-l}]\right)} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=0}^{j-i-1} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = 0, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$
$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_2=1}^{\min(m-2,[\frac{j-i-1-l}{k-l}])} \sum_{s_3=1}^{\min\left(m-1-s_2,[\frac{n-j-l}{k-l}]\right)} \sum_{m_1=0}^{i-1} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = 0, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$
$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}$$

$$+ \sum_{s_1=1}^{\min(m-3,[\frac{i-1-l}{k-l}])} \sum_{s_2=1}^{\min(m-2-s_1,[\frac{j-i-1-l}{k-l}])} \sum_{s_3=1}^{\min(m-1-s_1-s_2,[\frac{n-j-l}{k-l}])} \sum_{m_1=low_{m_1}}^{up_{m_1}} \sum_{m_2=low_{m_2}}^{up_{m_2}} \sum_{m_3=low_{m_3}}^{up_{m_3}}$$

$$\times P\{N_{i-1,k,l}^{L(1:i-1)} = s_1, S_{i-1}^{(1:i-1)} = m_1\} P\{N_{j-i-1,k,l}^{L(i+1:j-1)} = s_2, S_{j-i-1}^{(i+1:j-1)} = m_2\}$$
$$\times P\{N_{n-j,k,l}^{L(j+1:n)} = s_3, S_{n-j}^{(j+1:n)} = m_3\} P\{X_i = 0\} P\{X_j = 0\}.$$

where

$$up_{m_1} = i - 1 - k - (s_1 - 1)(k - l),$$
$$low_{m_1} = \left| \begin{array}{ll} 1 + \left[\frac{i-2-s_1(k-l)-l}{k}\right] & \text{if } \frac{i-1-l}{k-l} < m-1 \\ 0 & \text{otherwise} \end{array} \right. ,$$
$$up_{m_2} = j - i - 1 - k - (s_2 - 1)(k - l),$$
$$low_{m_2} = \left| \begin{array}{ll} 1 + \left[\frac{j-i-2-s_2(k-l)-l}{k}\right] & \text{if } \frac{j-i-1-l}{k-l} < m-1 \\ 0 & \text{otherwise} \end{array} \right. ,$$
$$up_{m_3} = n - j - k - (s_3 - 1)(k - l),$$
$$low_{m_3} = \left| \begin{array}{ll} 1 + \left[\frac{n-j-1-s_3(k-l)-l}{k}\right] & \text{if } \frac{n-j-l}{k-l} < m-1 \\ 0 & \text{otherwise} \end{array} \right. .$$

By substituting the equation (8) in (3) one can obtain explicitly.

Acknowledgment. The authors would like to thank two anonymous referees for their helpful comments and suggestions, which were very useful in improving of this paper.

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