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On Dynamics of A Higher-Order Rational Difference Equation

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Abstract

In this paper, we study the dynamical behavior of the positive solutions of the difference equation

$$y_{n+1}=\frac{A+By_n}{C+D\prod_{i=1}^{k}y_{n-i}^{q_i}},\ n\in\mathbb{N}_0$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the initial conditions and the parameters A, B are non-negative real numbers, the parameters C, D are positive real numbers, q_i for $i\in\{1,2,...k\}$ are fixed positive integers and $1\le k$.

Keywords: *Difference equation; stability; global behavior.*
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1. Introduction

Recently, difference equations have gained a great importance. Many phenomena are modelled with the help of difference equations. Most of the recent applications of these equations have appeared in many scientific areas such as biology, ecology, physics, economics, etc. Particularly, rational difference equations are important in practical classes of difference equations. So, it is very worthy to examine the behaviour of solutions of rational difference equations and to discuss the stability character of their equilibrium points. In recent years, many researchers have investigated global behaviour of solutions of non-linear difference equations, see the references [1] - [26].

In [1] Hamza et al. investigated the global behaviour and boundedness of the rational second-order difference equation

$$x_{n+1}=\frac{a+bx_n}{A+Bx_{n-1}^{k}},\ n\in\mathbb{N}_0 \tag{1.1}$$

with positive coefficients and non-negative initial conditions. Abo-Zeid [21] discussed the global stability and periodic nature of the positive solutions of the difference equation

$$x_{n+1}=\frac{A+Bx_{n-2k-1}}{C+D\prod_{i=l}^{k}x_{n-2i}},\ n\in\mathbb{N}_0 \tag{1.2}$$

where A, B are non-negative real numbers, $C,D>0$ and l, k are non-negative integers such that $l\le k$. Belhannache et al. [11] investigated the global asymptotic behaviour of the difference equation

$$\frac{A+Bx_{n-2k-1}}{C+D\prod_{i=l}^{k}x_{n-2i}^{m_i}},\ n\in\mathbb{N}_0 \tag{1.3}$$

with non-negative initial conditions, the parameters A, B are non-negative real numbers, C,D are positive real numbers and k, l are fixed non-negative integers such that $l\le k$ and m_i for $i\in\{1,2,...k\}$ are fixed positive integers.

Motivated by all above mentioned works, in this paper we study the dynamical behavior of the positive solutions of the difference equation

$$y_{n+1}=\frac{A+By_n}{C+D\prod_{i=1}^{k}y_{n-i}^{q_i}},\ n\in\mathbb{N}_0 \tag{1.4}$$

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where the initial conditions and the parameters A, B are non-negative real numbers, the parameters C, D are positive real numbers, q_i for $i \in \{1,2,...k\}$ are fixed positive integers and $1 \leq k$.
Note that Eq.(1.4) can be written as

$$x_{n+1} = \frac{\alpha + \beta x_n}{1 + \prod_{i=1}^{k} x_{n-i}^{q_i}}, \; n \in \mathbb{N}_0 \tag{1.5}$$

by the change of variables $y_n = \left(\frac{C}{D}\right)^{1/\gamma} x_n$ with $\alpha = \frac{A}{C}\left(\frac{D}{C}\right)^{1/\gamma}$, $\beta = \frac{B}{C}$ and $\gamma = \sum_{i=1}^{k} q_i$. So, we will consider Eq.(1.5) instead of Eq.(1.4) for the remaining part of the paper.

2. Preliminaries

Now we present some definitions and results which will be useful in our investigation, for more details we refer to [6, 19, 22, 24].

Definition 2.1. *Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \;\; n \in \mathbb{N}_0 \tag{2.1}$$

with $x_{-k}, ..., x_0 \in I$. Let $\overline{x}$ be an equilibrium point of Eq.(2.1).
The linearized equation of Eq.(2.1) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1+} ... + c_{(k+1)} y_{n-k}, \;\; n \in \mathbb{N}_0 \tag{2.2}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, ..., \overline{x}), c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, ..., \overline{x}), ..., c_{(k+1)} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, ..., \overline{x}).$$

The characteristic equation of Eq.(2.2) is

$$\lambda^{(k+1)} - c_1 \lambda^k - c_2 \lambda^{(k-1)} - ... - c_k \lambda - c_{(k+1)} = 0. \tag{2.3}$$

Definition 2.2. *Let $\overline{x}$ be an equilibrium point of Eq.(2.1).*

(i) *The equilibrium point $\overline{x}$ is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta$, we have*

$$|x_n - \overline{x}| < \varepsilon \text{ for all } n \geq -k.$$

Otherwise, the equilibrium point $\overline{x}$ is called unstable.

(ii) *The equilibrium point $\overline{x}$ is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $\sum_{i=-k}^{i=0} |x_i - \overline{x}| < \delta$, we have*

$$\lim_{n \to +\infty} x_n = \overline{x}.$$

(iii) *The equilibrium point $\overline{x}$ is called globally asymptotically stable relative to I^{k+1} if it is locally asymptotically stable, and if for every $x_0, x_{-1}, , x_{-k} \in I$, we have*

$$\lim_{n \to +\infty} x_n = \overline{x}.$$

Theorem 2.3. *Let $\overline{x}$ be an equilibrium point of Eq.(2.1). Then, the following statements are true*

(i) *If all roots of Eq.(2.3) lie inside the open unit disk $|\lambda| < 1$, then $\overline{x}$ is locally asymptotically stable.*
(ii) *If at least one root of Eq.(2.3) has absolute value greater than one, then $\overline{x}$ is unstable.*

3. Main Results

3.1. Case $\alpha > 0$

Lemma 3.1. *The following statements are true*

1. *Assume that $1 \leq \beta$. Then, Eq.(1.5) has a unique equilibrium point in $\left(\sqrt[\gamma]{\frac{\beta-1}{\gamma+1}}, +\infty\right)$.*
2. *Assume that $\beta < 1$. Then, the following statements are true*

   (i) *If $\alpha < \gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, then Eq.(1.5) has a unique equilibrium point in $\left(0, \sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right)$.*

   (ii) *If $\gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}} < \alpha$, then Eq.(1.5) has a unique equilibrium point in $\left(\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}, +\infty\right)$.*

Proof. A point $\bar{x}$ is an equilibrium point of Eq.(1.5) if and only if $\bar{x}$ is a zero of the function

$f(x) = x^{\gamma+1} + (1-\beta)x - \alpha.$

If we consider the above function, we get

$f'(x) = (\gamma+1)x^{\gamma} + 1 - \beta.$

1. If $1 \leq \beta$, then f is increasing on $\left(\sqrt[\gamma]{\frac{\beta-1}{\gamma+1}}, +\infty\right)$. But,

$$f\left(\sqrt[\gamma]{\frac{\beta-1}{\gamma+1}}\right) < 0 \text{ and } \lim_{x\to+\infty} f(x) = +\infty.$$

   Then $f(x)$ has a unique zero in $\left(\sqrt[\gamma]{\frac{\beta-1}{\gamma+1}}, +\infty\right)$.

2. Assume that $\beta < 1$. Then, f is increasing on $(0, +\infty)$ and

$$f\left(\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right) = \gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}} - \alpha.$$

   (i) If $\alpha < \gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, then $f\left(\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right) > 0$. Therefore, $f(x)$ has a unique zero in $\left(0, \sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right)$.

   (ii) If $\gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}} < \alpha$, then $f\left(\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right) < 0$. Therefore, $f(x)$ has a unique zero in $\left(\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}, +\infty\right)$.

□

Theorem 3.2. *If $\beta < 1$ and $\alpha < \gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, then the equilibrium point $\bar{x}$ of Eq.(1.5) is locally asymptotically stable.*

Proof. Let $\bar{x}$ the equilibrium point of Eq.(1.5). The linearized equation associated with Eq.(1.5) about $\bar{x}$ is

$$z_{n+1} - \frac{\beta}{1+\bar{x}^{\gamma}} z_n + \frac{q_1\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}} z_{n-1} + \cdots + \frac{q_k\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}} z_{n-k} = 0,\ n \in \mathbb{N}_0. \tag{3.1}$$

The characteristic equaion associated with this equation is

$$\lambda^{k+1} - \frac{\beta}{1+\bar{x}^{\gamma}}\lambda^{k} + \frac{q_1\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\lambda^{k-1} + \cdots + \frac{q_k\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}} = 0. \tag{3.2}$$

Assume that $\beta < 1$. If $\alpha < \gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, then $\bar{x} \in \left(0, \sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right)$. If we consider the functions

$$h_1(\lambda) = \lambda^{k+1} \text{ and } h_2(\lambda) = -\frac{\beta}{1+\bar{x}^{\gamma}}\lambda^{k} + \frac{q_1\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\lambda^{k-1} + \cdots + \frac{q_k\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}.$$

We get

$$|h_2(\lambda)| \leq \left|-\frac{\beta}{1+\bar{x}^{\gamma}}\right| + \left|\frac{q_1\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\right| + \ldots + \left|\frac{q_k\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\right|,\ \lambda \in \mathbb{C} : |\lambda| = 1.$$

But

$$\left|-\frac{\beta}{1+\bar{x}^{\gamma}}\right| + \left|\frac{q_1\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\right| + \cdots + \left|\frac{q_k\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}}\right| = \frac{\beta+\gamma\bar{x}^{\gamma}}{1+\bar{x}^{\gamma}} < 1 = |h_1(\lambda)|.$$

By Rouché's Theorem all roots of Eq.(3.2) lie in the open unit disk, i.e. $|\lambda| < 1$, and the result follows from Theorem 2.3. □

Lemma 3.3. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.5). Then,*

$$x_n < \sum_{i=0}^{n-1} \alpha\beta^{i} + \beta^{n} x_0,\ \ \forall n \in \mathbb{N}. \tag{3.3}$$

Proof. We prove the theorem by induction for n. For $n = 1$, (3.3) holds. Now, suppose that $n > 1$ and our assumption holds for n. That is;

$$x_n < \sum_{i=0}^{n-1} \alpha\beta^{i} + \beta^{n} x_0. \tag{3.4}$$

Then,

$$x_{n+1} = \frac{\alpha + \beta x_n}{1 + \prod_{i=1}^{k} x_{n-i}^{q_i}} < \alpha + \beta x_n.$$

Using (3.4) we obtain

$$x_{n+1} < \alpha + \beta\left(\sum_{i=0}^{n-1} \alpha\beta^{i} + \beta^{n} x_0\right) = \sum_{i=0}^{n} \alpha\beta^{i} + \beta^{n+1} x_0,$$

hence (3.3) holds and the proof is completed by induction. □

Corollary 3.4. *Assume that $\beta<1$. Then, every solution of Eq.(1.5) is bounded and persists.*

Lemma 3.5. *Suppose that $\beta<1$ and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.5). If $\Lambda=\lim_{n\to+\infty}\sup(x_n)$ and $\lambda=\lim_{n\to+\infty}\inf(x_n)$, then Λ and λ satisfy the following inequalities*

$$\frac{\alpha+\beta\lambda}{1+\Lambda^{\gamma}}\leq\lambda\leq\Lambda\leq\frac{\alpha+\beta\Lambda}{1+\lambda^{\gamma}}. \tag{3.5}$$

Proof. Let $\beta<1$. From Corollary 3.4, the solution $\{x_n\}_{n=-k}^{\infty}$ is bounded. Then, for every $\varepsilon\in(0,\lambda)$, there exists $n_0\in\mathbb{N}_0$ such that

$$\lambda-\varepsilon\leq x_n\leq\Lambda+\varepsilon \quad \text{for every } n\geq n_0,$$

so,

$$\frac{\alpha+\beta(\lambda-\varepsilon)}{1+(\Lambda+\varepsilon)^{\gamma}}\leq x_{n+1}\leq\frac{\alpha+\beta(\Lambda+\varepsilon)}{1+(\lambda-\varepsilon)^{\gamma}} \quad \text{for every } n\geq n_0+k.$$

Therefore,

$$\frac{\alpha+\beta\lambda}{1+\Lambda^{\gamma}}\leq\lambda\leq\Lambda\leq\frac{\alpha+\beta\Lambda}{1+\lambda^{\gamma}}.$$

□

Theorem 3.6. *Assume that $\beta<1$. If $\alpha<\gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, then the positive equilibrium point $\bar{x}\in\left(0,\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right)$ is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$be a solution of Eq.(1.5). From $\beta<1$, the solution $\{x_n\}_{n=-k}^{\infty}$ is bounded. Let $\Lambda=\limsup_{n\to\infty}(x_n)$ and $\lambda=\liminf_{n\to\infty}(x_n)$. Using Lemma 3.5 we have

$$\frac{\alpha+\beta\lambda}{1+\Lambda^{\gamma}}\leq\lambda\leq\Lambda\leq\frac{\alpha+\beta\Lambda}{1+\lambda^{\gamma}}.$$

This implies that

$$(1-\beta)\Lambda^{\gamma}-\alpha\Lambda^{\gamma-1}\leq(1-\beta)\lambda^{\gamma}-\alpha\lambda^{\gamma-1}.$$

Now, consider the function

$$h(x)=(1-\beta)x^{\gamma}-\alpha x^{\gamma-1}.$$

Hence

$$h'(x)=x^{\gamma-2}(\gamma(1-\beta)x-\alpha(\gamma-1)),$$

and the $h(x)$ is increasing on $\left(\frac{\alpha(\gamma-1)}{\gamma(1-\beta)},\infty\right)$. As $\alpha<\gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$, we get $\bar{x}\in\left(\frac{\alpha(\gamma-1)}{\gamma(1-\beta)},\sqrt[\gamma]{\frac{1-\beta}{\gamma-1}}\right)$. In view of inequalities (3.5), we have a contradiction. Therefore, $\lambda=\Lambda=\bar{x}$ and so $\bar{x}$ is a global attractor. By combining this result with the local asymptotic stability of $\bar{x}$ we get the global asymptotic stability of $\bar{x}$ when $\alpha<\gamma\left(\frac{1-\beta}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma}}$. □

3.2. Case $\alpha=0$

When $\alpha=0$, Eq.(1.5) becomes

$$x_{n+1}=\frac{\beta x_n}{1+\prod_{i=1}^{k}x_{n-i}^{q_i}},\ n\in\mathbb{N}_0. \tag{3.6}$$

Clearly, $\bar{x}=0$ is always an equilibrium point for Eq.(3.6) and if $\beta>1$, then Eq.(3.6) has also the positive equilibrium point $\bar{x}=\sqrt[\gamma]{\beta-1}$. We summarize the main results for this particular equation.

Lemma 3.7. *Assume that $\beta<1$. Then, every solution of Eq.(3.6) is bounded.*

Proof. $\{x_n\}_{n=-k}^{\infty}$ be a solution of the Eq.(3.6). It is clear that

$$x_n<\beta^n x_0,\ \ \forall n\in\mathbb{N}. \tag{3.7}$$

If $\beta<1$, then

$$x_n<x_0,\ \ \forall n\in\mathbb{N}. \tag{3.8}$$

□

Theorem 3.8. *The zero equilibrium point of Eq.(3.6) is globally asymptotically stable if $\beta<1$ and it is unstable if $\beta>1$.*

Proof. The linearised equation associated with Eq.(3.6) about the equilibrium point $\bar{x}=0$ is

$z_{n+1}=\beta z_n,\ n\in\mathbb{N}_0.$

Its characteristic equation is

$\lambda-\beta=0,$

then $\lambda=\beta$. Therefore, the equilibrium point $\bar{x}=0$ is locally asymptotically stable if $\beta<1$ and it is unstable if $\beta>1$.
On the other hand (3.7) implies that

$\lim_{n\longrightarrow+\infty} x_n=0,$

then $\bar{x}=0$ is globally asymptotically stable. □

3.3. Numerical Examples

In this section, we give some numerical examples to support our theoretical results related to Eq.(1.5) with some restrictions on the parameters.

Example 3.9. *When $k=2$, Eq.(1.5) becomes*

$$x_{n+1}=\frac{\alpha+\beta x_n}{1+x_{n-1}^{q_1}x_{n-2}^{q_2}},\ n\in\mathbb{N}_0. \tag{3.9}$$

We visualize the solutions of Eq.(3.9) in figures 3.1-3.6 for the initial conditions $x_{-2}=2.4$, $x_{-1}=0.4$, $x_0=1.3$.

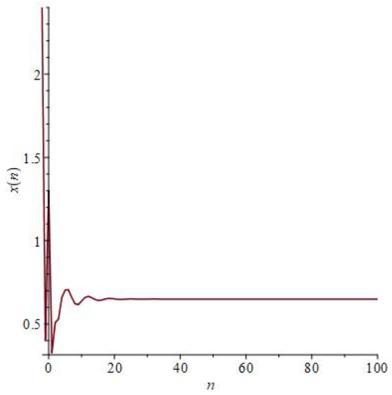

Figure 3.1: $\alpha=0.4$, $\beta=0.5$, $q_1=2$, $q_2=3$.

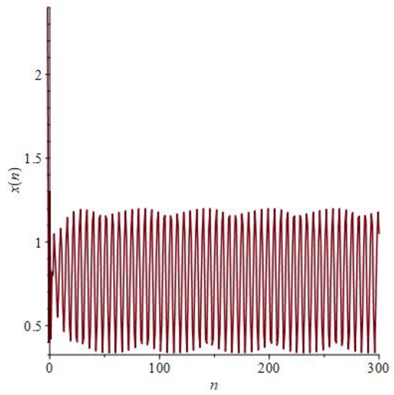

Figure 3.2: $\alpha=0.7$, $\beta=0.5$, $q_1=2$, $q_2=3$.

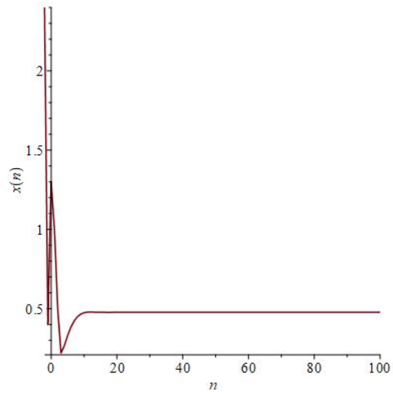

Figure 3.3: $\alpha=0.12$, $\beta=0.8$, $q_1=3$, $q_2=1$.

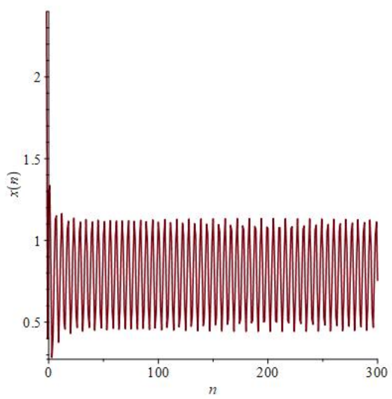

Figure 3.4: $\alpha=0.5$, $\beta=0.8$, $q_1=3$, $q_2=1$.

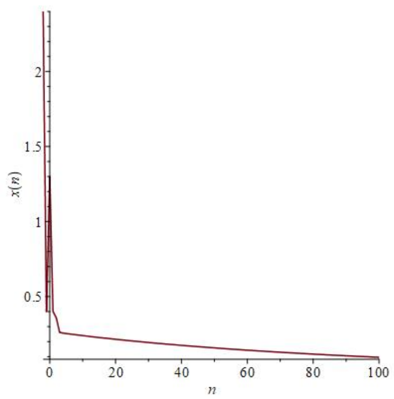

Figure 3.5: $\alpha=0$, $\beta=0.99$, $q_1=2$, $q_2=3$.

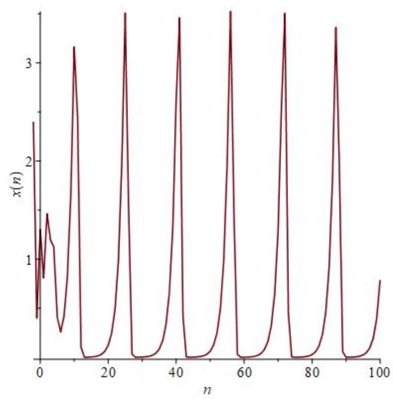

Figure 3.6: $\alpha=0$, $\beta=2$, $q_1=2$, $q_2=3$.

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