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Controllability of Higher Order Fractional Damped Delay Dynamical Systems with Time Varying Multiple Delays in Control

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Abstract

This paper is concerned with the controllability of higher order fractional damped delay dynamical systems with time varying multiple delays in control, which involved Caputo derivatives of any different orders. A necessary and sufficient condition for the controllability of linear fractional damped delay dynamical system is obtained by using the Grammian matrix. Sufficient conditions for controllability of the corresponding non-linear damped delay dynamical system has established by the successive approximation technique. Examples have provided to verify the results.

Keywords: Damped delay dynamical systems, Controllability, Mittag-Leffler Matrix function, Iterative technique.
2010 MSC: 93B0, 34A08, 37N35.

1. Introduction

Fractional calculus has a long history which goes back to Leibniz, who introduced the notion of "$\frac{1}{2}-$ order derivative" in a letter to L'Hospital from 1695. Nowadays, the subject of fractional calculus has proved to be useful in modelling of many real-world problems in various fields of science and engineering. Fractional order models have the tendency to capture non-local relations in space and time, thus forming an improvised model for analyzing complex phenomena. The most important advantage of using fractional differential

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equations is their non-local property, which means that the next state of a system depends not only upon its current state but also determined by the entire historical states [28]. Studying non-local observed facts, the notion of fractional derivatives has been familiar as a useful mathematical tool, which follows the generalized Mittag-Leffler law, power law and exponential law. Many engineers, technologists, mathematicians and other scientists have modelled various types of complex physical and biological phenomena using fractional operators. Magin [24] presented the usefulness of fractional operators in the areas like, bioimaging, biomechanics and bioelectrodes. Fractional derivative formulations are used to represent many practical systems more accurately than integer order ones and gained significance in the fields of bioengineering, signal processing, in electrochemistry, frequency dependent damping behaviour of many visco-elastic materials, filter design, circuit theory, dynamics of interfaces between nanoparticles and substrates, continuum and statistical mechanics, the nonlinear oscillation of earthquakes and robotics [1, 6, 12, 18, 25, 29].

The Mathematical point of view, the fundamentals of fractional calculus and fractional differential and difference equations are given in the monographs [22, 27]. Very recently, the analysis of fractal-fractional malaria transmission model under control strategies using the Liouville Caputo fractional order derivatives with the exponential decay law and power law was studied by Gomez-Aguilar et al. [16]. Aziz Khan et al. [4, 5] examined the dynamical study of fractional order mutualism, parasitism food web module and also stability and numerical simulation of a fractional order plant-nectar pollinator model.

On the other hand, some researchers have generalized integer order controllers to noninteger order controllers. In 1961, fractional order systems in the area of automatic control were investigated by Manabe [26]. In 2009, Chen et al. [11] discussed the fractional calculus as well as fractional order controllers and the discretisation techniques. Review, design, optimization, and stability analysis of fractional-order PID controller established by Ammar Soukkou et al. [32] in 2016. The fractional calculus in dynamic systems and controls are developed by many researchers [9, 14, 15, 20, 36].

Controllability plays a vital role in the development of the modern mathematical control theory and it is used to influence an object's behaviour of a dynamical system to accomplish a desired goal. The study of control systems modeled using fractional differential equations is significant in various problems of an applied nature. Nowadays, the controllability has applied in the fields of industrial and chemical process control, reactor control, control of electric bulk power systems, aerospace engineering and recently in quantum systems theory. Controllability theory of linear and nonlinear dynamical systems in finite-dimensional spaces has been studied by many researchers [2, 7, 8, 13].

A remarkable feature for delay system is that the system's future evolution depends not only on the present control state, but also in a period of control history. Such system occurs in automatic control, biology, economics, medicine and other areas. Mathematical description of these processes can be done with the help of equations with delay, integral and integrodifferential equations. Delay differential equations were initially introduced in the 18th century by Laplace and Condorcet. The principal difficulty in studying delay differential equations lies in their special transcendental character. Delay equations always lead to an infinite spectrum of frequencies. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions. Delay differential equations are often solved using numerical methods, asymptotic solutions, and graphical tools. Several attempts have been made to find analytical solutions for delay differential equations by solving their characteristic equations under different conditions. The differential equations with delay was investigated by Bellman and Cooke [10] and Hale [17]. Wiess [34] studied the controllability of delayed differential systems and Dauer and Gahl [13] established the controllability of nonlinear delay dynamical systems by using fixed point technique.

Yonggang and Xiu'e [35] introduced a fractional oscillator equation in which the restoring force is represented by a term containing fractional derivative and the property of oscillation is retained. In the fractional oscillator model, numerous specific forcing functions and their resonance were analysed by Achar et al. [1]. Tofighi [33] has described and attained the expression of the intrinsic damping force in the fractional oscillator system. Some authors extended the interpretation to the fractional oscillator and reported that fractional oscillations have finite numbers of zeros. Al-rabth et al. [3] used the differential transform method to solve a fractional oscillator system. Recently, some researchers [7, 19, 21] has discussed the controllability of fractional damped dynamical systems.

To the best our knowledge, there are no relevant work has been reported on the control problem of fractional damped delay dynamical systems of higher order. In this paper, we make an attempt to study the controllability of the higher order fractional damped delay dynamical systems with time varying delays in control. Numerical examples are provided to illustrate the theoretical results.

The paper has been developed as follows, in Section 1, the background, motivations and objective of this paper has been discussed. Some preliminary facts, definitions and notations are recalled in the Section 2. In Section 3, necessary and sufficient conditions for controllability results of linear damped delay system has been established. In Section 4, controllability criteria of corresponding nonlinear fractional damped delay dynamical systems with time varying multiple delays in control are provided. In Section 5, numerical examples has given to illustrate the effectiveness and applicability of our results. Finally, some concluding remark has been drawn in Section 6.

2. Preliminaries

In this section, let us recall some notations, basic definitions and preliminary facts [22, 27].

Definition 2.1. *Let f be a real-or complex-valued function of the variable $t > 0$ and let s be a real or complex parameter. The Laplace transform of f is defined as*

$$F(s) = \int_0^\infty e^{-st} f(t)dt, \quad for \quad Re(s) > 0.$$

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha \leq n$, is defined as*

$${}^C D^{\alpha}_{0_+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $f^{(n)}(s) = \frac{d^n f}{ds^n}$ and the function $f(t)$ has absolutely continuous derivative up to order $n-1$. For the brevity, Caputo fractional derivative ${}^C D^{\alpha}_{0_+}$ is taken as ${}^C D^{\alpha}$.

The Laplace transform of Caputo derivative is

$$\mathcal{L}[{}^c D^{\alpha} x(t)](s) = s^{\alpha} \mathcal{L}[x(t)](s) - \sum_{k=0}^{n-1} x^k(0) s^{\alpha-1-k}, \quad n-1 < \alpha \leq n.$$

Definition 2.3. *The Mittag-Leffler functions of various type are defined as*

$$E_{\alpha}(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in C, \quad Re(\alpha) > 0,$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \beta \in C, \quad Re(\alpha) > 0,$$

$$E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (-\lambda)^k}{k! \Gamma(\alpha k + \beta)} t^{\alpha k},$$

where $(\gamma)_n$ is a Pochhamer symbol which is defined as $\gamma(\gamma+1)\cdots(\gamma+n-1)$ and$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$.

Definition 2.4. *The Laplace transforms of various types of Mittag-Leffler functions are defined as*

$$\mathcal{L}[E_{\alpha,1}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-1}}{(s^{\alpha} \pm \lambda)}, \quad Re(\alpha) > 0,$$

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \pm \lambda)}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0,$$

$$\mathcal{L}[t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \pm \lambda)^{\gamma}}, \quad Re(s) > 0, \quad Re(\beta) > 0, \quad |\lambda s^{-\alpha}| < 1.$$

Definition 2.5. *The Mittag-leffler matrix function derivative of order $p (p \in N)$ is defined as*

$$\left(\frac{d}{dt}\right)^{(p)}\left(t^{\alpha-1}E_{\alpha-\beta,\alpha}(At^{\alpha-\beta})\right) = t^{\alpha-p-1}E_{\alpha-\beta,\alpha-p}(At^{\alpha-\beta}),\ (p \in N).$$

3. Linear Damped Delay System

Consider the linear fractional damped delay dynamical systems with time varying multiple delays in control of the form

$$\begin{cases} {}^CD^{\alpha}x(t) - A\,{}^CD^{\beta}x(t) = Bx(t-\tau) + \sum_{i=0}^{M} C_i u(\rho_i(t)), \quad t \in J = [0,T], \\ x(t) = \varphi(t), \quad x'(t) = \varphi'(t), \cdots, x^{(p)}(t) = \varphi^{(p)}(t), \\ u(t) = \psi(t), \ -\tau \le t \le 0, \end{cases} \tag{1}$$

where $p-1 < \alpha < p,\ q-1 < \beta < q,\ q \le p-1,\ x \in \mathbb{R}^n, u \in \mathbb{R}^m,\ \rho_i(t) = t - \tau_i(t)$ and $\tau_i(t) \ge 0, i = 0,1,\dots,M$ are time-dependent delays in control, A, B are $n \times n$ matrices and C_i are $n \times m$ matrices for $i = 0,1,\dots,M$. Assume the following conditions:

(**H1**) The functions $\rho_i(t) : J \to \mathbb{R},\ i = 0,1,\dots,M$, are twice continuously differentiable and strictly increasing in J . Moreover

$$\rho_i(t) \le t, i = 0,1,\dots,M, \text{ for } t \in J.$$

(**H2**) Introduce the time lead functions $r_i(t) : [\rho_i(0), \rho_i(T)] \to [0,T], i = 0,1,\dots,M$, such that $r_i(\rho_i(t)) = t$ for $t \in J$. Further $\rho_0(t) = t$ and for $t = T$. The following inequalities holds

$$\rho_M(T) \le \rho_{M_1}(T) \le \cdots \rho_{m+1}(T) \le 0 = \rho_m(T) < \rho_{m-1}(T) = \cdots = \rho_1(T) = \rho_0(T) = T. \tag{2}$$

Definition 3.1. *The set $y(t) = \{x(t), \psi(t,s)\}$ where $\psi(t,s) = u(s)$ for $s \in [\min \tau_i(t), t)$ is said to be the complete state of the system (1) at time t .*

Definition 3.2. *System (1) is said to be relatively controllable on $[0,T]$ if for every complete state $\varphi(t)$, $\varphi'(t)$, $\cdots$, $\varphi^{(p)}(t)$, $z \in \mathbb{R}^n$, there exists a control $u(t)$ defined on $[0,T]$, such that the solution of system (1) satisfies $x(T) = z$.*

In order to get the solution of system (1), by taking Laplace and inverse Laplace transform of both sides of the equation (1), and using convolution of Laplace transforms, we have the solution of the form [30, 31]

$$\begin{aligned} x(t) =& \sum_{k=0}^{p-1} x^k(0)t^k\Phi_{\alpha-\beta,1+k}(At^{\alpha-\beta}) - \sum_{k=0}^{q-1} x^k(0)t^{\alpha-\beta+k}\Phi_{\alpha-\beta,\alpha-\beta+1+k}(At^{\alpha-\beta}) \\ &+ B\int_{-\tau}^{0}(t-s-\tau)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s-\tau)^{\alpha-\beta})\varphi(s)ds \\ &+ \int_{o}^{t}(t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\sum_{i=0}^{M} C_i u(\rho_i(s))ds. \end{aligned} \tag{3}$$

Using the time lead function $r_i(t)$, the solution is of the form

$$\begin{aligned} x(t) =& \sum_{k=0}^{p-1} x^k(0)t^k\Phi_{\alpha-\beta,1+k}(At^{\alpha-\beta}) - \sum_{k=0}^{q-1} x^k(0)t^{\alpha-\beta+k}\Phi_{\alpha-\beta,\alpha-\beta+1+k}(At^{\alpha-\beta}) \\ &+ B\int_{-\tau}^{0}(t-s-\tau)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s-\tau)^{\alpha-\beta})\varphi(s)ds \\ &+ \sum_{i=0}^{M}\int_{\rho_i(0)}^{\rho_i(t)}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(\rho_i(s))ds. \end{aligned} \tag{4}$$

The solution (4) is expressed as

$$x(t) = x(t;\varphi) + \sum_{i=0}^{M} \int_{\rho_i(0)}^{\rho_i(t)} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) u(\rho_i(s)) ds,$$

where

$$\begin{aligned} x(t;\varphi) =& \sum_{k=0}^{p-1} x^k(0) t^k \Phi_{\alpha-\beta,1+k}(At^{\alpha-\beta}) - \sum_{k=0}^{q-1} x^k(0) t^{\alpha-\beta+k} \Phi_{\alpha-\beta,\alpha-\beta+1+k}(At^{\alpha-\beta}) \\ &+ B \int_{-\tau}^{0} (t - s - \tau)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - s - \tau)^{\alpha-\beta}) \varphi(s) ds, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Phi_{\alpha-\beta,1+k}(At^{\alpha-\beta}) &= \mathcal{L}^{-1}\left[\frac{s^{\alpha-\beta-1-k}}{S^{\alpha-\beta} I - A - Bs^{-\beta} e^{-sh}}\right](t), \\ \Phi_{\alpha-\beta,\alpha-\beta+1+k}(At^{\alpha-\beta}) &= \mathcal{L}^{-1}\left[\frac{s^{\alpha-\beta-(\alpha-\beta+1+k)}}{S^{\alpha-\beta} I - A - Bs^{-\beta} e^{-sh}}\right](t), \\ \Phi_{\alpha-\beta,\alpha}(At^{\alpha-\beta}) &= \mathcal{L}^{-1}\left[\frac{s^{-\beta}}{S^{\alpha-\beta} I - A - Bs^{-\beta} e^{-sh}}\right](t). \end{aligned}$$

Now using the inequality (2) in the above equation, we get

$$\begin{aligned} x(t) =& x(t;\varphi) + \sum_{i=0}^{m} \int_{\rho_i(0)}^{0} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) \psi(s) ds \\ &+ \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) u(s) ds \\ &+ \sum_{i=m+1}^{M} \int_{\rho_i(0)}^{\rho_i(t)} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) \psi(s) ds. \end{aligned}$$

Further simplify we get

$$x(t) = x(t;\varphi) + \sigma(t) + \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) u(s) ds, \tag{6}$$

where

$$\begin{aligned} \sigma(t) =& \sum_{i=0}^{m} \int_{\rho_i(0)}^{0} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) \psi(s) ds \\ &+ \sum_{i=m+1}^{M} \int_{\rho_i(0)}^{\rho_i(t)} (t - r_i(s))^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s) \psi(s) ds. \end{aligned} \tag{7}$$

The controllability Grammian matrix is defined by as follows

$$W = \sum_{i=0}^{m} \int_{0}^{T} (T - r_i(s))^{2\alpha-2} [\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s)][\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta}) C_i \dot{r}_i(s)]^* ds,$$

where the $*$ denotes the matrix transpose.

Theorem 3.1. *The linear fractional damped delay dynamical system* (1) *is controllable on* $[0, T]$ *if and only if the controllability Grammian matrix* W *is positive definite.*

Proof. Since W is positive definite, it is non singular and so its inverse is well defined. Let control function u is defined by

$$u(t) = [(T - r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T - r_i(t))^{\alpha-\beta})C_i\dot{r}_i(t)]^* W^{-1}[z - x(T; \varphi) - \sigma(T)]. \tag{8}$$

Substituting $t = T$ in (6) and inserting (8) we have,

$$\begin{aligned} x(T) =& x(T; \varphi) + \sigma(T) + \sum_{i=0}^{m}\int_0^T \Big[(T - r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)\Big] \\ & \times \Big[(T - r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T - r_i(t))^{\alpha-\beta})C_i\dot{r}_i(t)\Big]^* W^{-1}\Big[z - x(T; \varphi) - \sigma(T)\Big]ds, \\ =& z. \end{aligned}$$

Thus system (1) is controllable on $[0, T]$.

On the other hand, W is not positive definite. Then there exists a non zero vector y such that

$$\begin{aligned} y^*Wy =& y^*\sum_{i=0}^{m}\int_0^T (T - r_i(s))^{2\alpha-2}[\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)][\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)]^* y ds \\ =& 0, \end{aligned}$$

$y^*\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s) = 0$ on $[0, T]$.
Let the initial points $\varphi(t) = \varphi'(t) = \cdots = \varphi^{(p)}(t) = 0$ and the final point $z = y$. By assumption, there exists control input u on $[0, T]$ such that it steers the response from 0 to z at $t = T$.
It follows that

$$z = y = \sum_{i=0}^{m}\int_0^T (T - r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(s)ds,$$

then,

$$y^*y = y^*\sum_{i=0}^{m}\int_0^T (T - r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T - r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(s)ds = 0.$$

This is a contradiction to $y \neq 0$. Thus W is positive definite. □ □

4. Nonlinear Damped Delay Dynamical System

Consider the nonlinear fractional damped delay dynamical systems with time varying multiple delays in control of the form

$$\begin{cases} {}^CD^{\alpha}x(t) - A\,{}^CD^{\beta}x(t) \\ \qquad = Bx(t - \tau) + \sum_{i=0}^{M} C_i u(\rho_i(t)) + f(t, x(t), x(t - \tau), {}^CD^{\alpha}x(t), {}^CD^{\beta}x(t), u(t)) \quad t \in J = [0, T], \\ x(t) = \varphi(t), \quad x'(t) = \varphi'(t), \cdots, x^{(p)}(t) = \varphi^{(p)}(t), \\ u(t) = \psi(t), \ -\tau \le t \le 0, \end{cases} \tag{9}$$

where $p-1<\alpha<p,\ q-1<\beta<q,\ q\leq p-1,\ x\in\mathbb{R}^n, u\in\mathbb{R}^m,\ \rho_i(t)=t-\tau_i(t)$ and $\tau_i(t)\geq 0, i=0,1,\ldots,M$ are time-dependent delays in control, A,B are $n\times n$ matrices and C_i are $n\times m$ matrices for $i=0,1,\ldots,M$, and $f: J\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is continuous function. The solution of (9) using the time lead function $r_i(t)$ is given by

$$\begin{aligned} x(t) =& x(t;\varphi)+\sigma(t)+\sum_{i=0}^{m}\int_0^t (t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(s)ds \\ &+\int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,x(s),x(s-\tau),{}^C D^\alpha x(s),{}^C D^\beta x(s),u(s))ds, \end{aligned} \tag{10}$$

where $x(t;\varphi)$ and $\sigma(t)$ are defined as in (5) and (7).

Consider the space $X=\{x(t)\in C(J:\mathbb{R}^n),\ {}^CD^\alpha x(t)\in C(J:\mathbb{R}^n),\ {}^CD^\beta x(t)\in C(J:\mathbb{R}^n)$ and $u\in L^\infty(J,\mathbb{R}^m)\}$ be a Banach space endowed with the norm $||x||_x=\max_{t\in J}\{|x(t)|,|{}^CD^\alpha x(t)|,|{}^CD^\beta x(t)|,|u(t)|\}$. Further we assume the following hypothesis:

(**H3**) $f: J\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is continuous and there exist positive constants K and L such that

$$\begin{aligned} &||f(t,x(t),x(t-\tau),{}^C D^\alpha x(t),{}^C D^\beta x(t),u(t))||\leq K,\ \text{for } t\in J,\\ &||f(t,x_1,y_1,x_{\alpha1},x_{\beta1},z_1)-f(t,x_2,y_2,x_{\alpha2},x_{\beta2},z_2)||\leq L\Big[||x_1-x_2||+||y_1-y_2||+||x_{\alpha1}-x_{\alpha2}||\\ &\qquad\qquad +||x_{\beta1}-x_{\beta2}||+||z_1-z_2||\Big],\\ &x_1,x_2,y_1,y_2,x_{\alpha1},x_{\alpha2},x_{\beta1},x_{\beta2}\in\mathbb{R}^n, z_1,z_2\in\mathbb{R}^m. \end{aligned}$$

For brevity, let us define

$$\begin{aligned} a_i =& \sup\left\|(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A^*(t-r_i(s))^{\alpha-\beta}\right)\right\|,\quad b_i=\sup\|\dot{r}_i(s)\|,\ i=0,1,2,\ldots,M,\\ c_i =& \sup\left\|\Phi_{\alpha-\beta,\alpha}\left(A(t-r_i(s))^{\alpha-\beta}\right)\right\|,\quad n_1=\sup||x(t;\varphi)||,\quad n_2=\sup||\psi(s)||,\\ n_3 =& \sup\left\|(T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A(T-s)^{\alpha-\beta}\right)\right\|,\ n_4=a_ib_i||C_i^*||||W^{-1}||,\\ n_5 =& \sum_{i=0}^{m}c_ib_i||C_i||N_i+\sum_{i=m+1}^{M}c_ib_i||C_i||M_i,\\ n_6 =& \sum_{i=0}^{m}c_ib_i||C_i||L_i,\ n_7=\sup||\varphi(t)||,\ n_8=\sup||(t-s)^{-1}\Phi_{\alpha-\beta}(A(t-s)^{\alpha-\beta})||,\ n_9=||C_i||,\\ n_{10} =& \sup||(t-s)^{\alpha-\beta-p+q-1}\Phi_{\alpha-\beta,\alpha-\beta-p+q}(A(t-s)^{\alpha-\beta})||,\\ N_i =& \int_{\rho_i(0)}^{0}(T-r_i(s))^{\alpha-1}ds,\ M_i=\int_{\rho_i(0)}^{\rho_i(T)}(T-r_i(s))^{\alpha-1}ds,\ L_i=\int_0^T(T-r_i(s))^{\alpha-1}ds. \end{aligned}$$

Theorem 4.1. *Assume that the function f satisfies the condition **(H3)**. Suppose that the linear system (1) is controllable then the nonlinear system (9) is controllable on J.*

Proof. To prove the controllability results we apply the successive approximation technique . For that, we define

$$\begin{aligned}
x_0(t) =&\varphi(t),\\
x_{n+1}(t) =&x(t;\varphi) + \sum_{i=0}^{m}\int_{\rho_i(0)}^{0}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)\psi(s)ds\\
&+\sum_{i=0}^{m}\int_{0}^{t}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u_n(s)ds\\
&+\sum_{i=m+1}^{M}\int_{\rho_i(0)}^{\rho_i(t)}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)\psi(s)ds\\
&+\int_0^t(t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,x_n(s),x_n(s-\tau),^C D^{\alpha}x_n(s),^C D^{\beta}x_n(s),u_n(s))ds, \quad (11)
\end{aligned}$$

where

$$\begin{aligned}
u_n(t) =&(T-r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A^*(T-r_i(t))^{\alpha-\beta})C_i^*\dot{r}_i(t)W^{-1}\Big[z-x(T;\varphi)\\
&-\sum_{i=0}^{m}\int_{\rho_i(0)}^{0}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)\psi(s)ds\\
&-\sum_{i=m+1}^{M}\int_{\rho_i(0)}^{\rho_i(t)}(t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)\psi(s)ds\\
&-\int_0^t(t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,x_n(s),x_n(s-\tau),^C D^{\alpha}x_n(s),^C D^{\beta}x_n(s),u_n(s))ds\Big], \quad (12)
\end{aligned}$$

and $n = 0, 1, 2, \dots$.
Since $\varphi(0)$ is a given vector, and note that $\{x_n(t)\}$ are the known sequence of functions. Now we have to show that $\{x_n(t)\}$ is a Cauchy sequence in X. Noting that $x_{n+1}(t) = \varphi(t) + \sum_{j=0}^{n}(x_{j+1}(t) - x_j(t))$, and it is necessary to prove that the series $\sum_{j=0}^{n}(x_{j+1}(t) - x_j(t))$ converges uniformly with respect to $t \in J$. It is clear that

$$\begin{aligned}
||u_n(t)|| \leq&||(T-r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A^*(T-r_i(t))^{\alpha-\beta})||||C_i^*||||\dot{r}_i(t)||||W^{-1}||\Big[||z||+||x(T;\varphi)||\\
&+\sum_{i=0}^{m}\int_{\rho_i(0)}^{0}||(T-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-r_i(s))^{\alpha-\beta})||||C_i||||\dot{r}_i(s)||||\psi(s)||ds\\
&+\sum_{i=m+1}^{M}\int_{\rho_i(0)}^{\rho_i(T)}||(T-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-r_i(s))^{\alpha-\beta})||||C_i||||\dot{r}_i(s)||||\psi(s)||ds\\
&+\int_0^T||(T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})||||f(s,x_n(s),x_n(s-\tau),^C D^{\alpha}x_n(s),^C D^{\beta}x_n(s),u_n(s))||ds\Big]\\
\leq&a_ib_i||C_i^*||\,\|W^{-1}\|\Big[||z||+n_1+n_2\sum_{i=0}^{m}c_ib_i||C_i||\int_{\rho_i(0)}^{0}(T-r_i(s))^{\alpha-1}ds\\
&+n_2\sum_{i=m+1}^{M}c_ib_i||C_i||\int_{\rho_i(0)}^{\rho_i(T)}(T-r_i(s))^{\alpha-1}ds+n_3KT\Big]\\
\leq&n_4\Big[||z||+n_1+n_2n_5+n_3KT\Big]=p,
\end{aligned}$$

and
$||u_n(t) - u_{n-1}(t)||$

$$\begin{aligned}
&\leq ||(T - r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A^*(T - r_i(t))^{\alpha-\beta})C_i^*\dot{r}_i(t)W^{-1}||\bigg[\int_0^T ||(T - s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T - s)^{\alpha-\beta})||\\
&\quad \times ||f(s, x_{n-1}(s), x_{n-1}(s - \tau), {}^C D^\alpha x_{n-1}(s), {}^C D^\beta x_{n-1}(s), u_{n-1}(s))\\
&\quad - f(s, x_n(s), x_n(s - \tau), {}^C D^\alpha x_n(s), {}^C D^\beta x_n(s), u_n(s))||ds\bigg]\\
&\leq n_4 n_3 LT\bigg[||x_{n-1}(s) - x_n(s)|| + ||x_{n-1}(s - \tau) - x_n(s - \tau)|| + ||{}^C D^\alpha x_{n-1}(s) - {}^C D^\alpha x_n(s)||\\
&\quad + ||{}^C D^\beta x_{n-1}(s) - {}^C D^\beta x_n(s)|| + ||u_{n-1}(s) - u_n(s)||\bigg].
\end{aligned}$$

Then
$||x_{n+1}(t) - x_n(t)||$

$$\begin{aligned}
&\leq \sum_{i=0}^{m}\int_0^t ||(t - r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta})||||C_i||||\dot{r}_i(s)||||u_n(s) - u_{n-1}(s)||ds\\
&\quad + \int_0^t ||(t - s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t - s)^{\alpha-\beta})||||f(s, x_n(s), x_n(s - \tau), {}^C D^\alpha x_n(s), {}^C D^\beta x_n(s), u_n(s))\\
&\quad - f(s, x_{n-1}(s), x_{n-1}(s - \tau), {}^C D^\alpha x_{n-1}(s), {}^C D^\beta x_{n-1}(s), u_{n-1}(s))||ds\\
&\leq n_6 n_4 n_3 LT\bigg[||x_{n-1}(s) - x_n(s)|| + ||x_{n-1}(s - \tau) - x_n(s - \tau)|| + ||{}^C D^\alpha x_{n-1}(s) - {}^C D^\alpha x_n(s)||\\
&\quad + ||{}^C D^\beta x_{n-1}(s) - {}^C D^\beta x_n(s)|| + ||u_{n-1}(s) - u_n(s)||\bigg]\\
&\quad + n_3 LT\bigg[||x_{n-1}(s) - x_n(s)|| + ||x_{n-1}(s - \tau) - x_n(s - \tau)|| + ||{}^C D^\alpha x_{n-1}(s) - {}^C D^\alpha x_n(s)||\\
&\quad + ||{}^C D^\beta x_{n-1}(s) - {}^C D^\beta x_n(s)|| + ||u_{n-1}(s) - u_n(s)||\bigg]\\
&\leq (n_6 n_4 n_3 LT + n_3 LT)\bigg[||x_{n-1}(s) - x_n(s)|| + ||x_{n-1}(s - \tau) - x_n(s - \tau)|| + ||{}^C D^\alpha x_{n-1}(s) - {}^C D^\alpha x_n(s)||\\
&\quad + ||{}^C D^\beta x_{n-1}(s) - {}^C D^\beta x_n(s)|| + ||u_{n-1}(s) - u_n(s)||\bigg].
\end{aligned}$$

Also,

$$\begin{aligned}
||x_1(t) - x_0(t)|| &\leq n_1 + n_2 n_5 + n_7 + \sum_{i=0}^{m}\int_0^t ||(t - r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t - r_i(s))^{\alpha-\beta})||||C_i||||\dot{r}_i(s)||||u_0(s)||ds\\
&\quad + \int_0^t ||(t - s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t - s)^{\alpha-\beta})||||f(s, x_0(s), x_0(s - \tau), {}^C D^\alpha x_0(s), {}^C D^\beta x_0(s), u_0(s))||ds\\
&\leq n_1 + n_2 n_5 + n_7 + (n_6 p + n_3 K t)\\
&\leq PT, \quad P > 0,
\end{aligned}$$

assuming that $T \geq 0$. The method of induction and using the above inequality we have the estimate

$$||x_{n+1}(t) - x_n(t)|| \leq P(n_6 n_4 n_3 LT + n_3 LT)\frac{T^{n+1}}{(n+1)!}.$$

By choosing sufficiently large value of n, then the right-hand side of the above inequality can be made arbitrarily small. This implies that $\{x_n(t)\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n(t)\}$ converges uniformly to a continuous function $x(t)$ on J. Thus we have

$$\begin{aligned} x(t) =& x(t;\varphi) + \sigma(t) + \sum_{i=0}^{m} \int_0^t (t-r_i(s))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(s)ds \\ &+ \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,x(s),x(s-\tau),{}^C D^{\alpha}x(s),{}^C D^{\beta}x(s),u(s))ds, \end{aligned}$$

where

$$\begin{aligned} u(t) =& (T-r_i(t))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A^*(T-r_i(t))^{\alpha-\beta})C_i^*\dot{r}_i(t)W^{-1}\Big[z - x(T;\varphi) - \sigma(T) \\ &- \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,x(s),x(s-\tau),{}^C D^{\alpha}x(s),{}^C D^{\beta}x(s),u(s))ds\Big], \end{aligned}$$

which follows by taking limit as $n \to \infty$ on both sides of (11) and (12). Then

$$\begin{aligned} x_{n+1}^{(p)}(t) =& x^{(p)}(t;\varphi) + \sigma_1(t) + \sum_{i=0}^{m} \int_0^t (t-r_i(s))^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u_n(s)ds \\ &+ \int_0^t (t-s)^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})f(s,x_n(s),x_n(s-\tau),{}^C D^{\alpha}x_n(s),{}^C D^{\beta}x_n(s),u_n(s))ds, \end{aligned}$$

where

$$\begin{aligned} \sigma_1(t) =& \sum_{i=0}^{m} \int_{\rho_i(0)}^{0} (t-r_i(s))^{\alpha-p-1}\Phi_{\alpha,\alpha-p}(t-r_i(s))C_i\dot{r}_i(s)\psi(s)ds \\ &+ \sum_{i=m+1}^{M} \int_{\rho_i(0)}^{\rho_i(t)} (t-r_i(s))^{\alpha-p-1}\Phi_{\alpha,\alpha-p}(t-r_i(s))C_i\dot{r}_i(s)\psi(s)ds, \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} x_{n+1}^{(p)}(t) =& \lim_{n\to\infty}\Bigg(x^{(p)}(t;\varphi) + \sigma_1(t) + \sum_{i=0}^{m} \int_0^t (t-r_i(s))^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u_n(s)ds \\ &+ \int_0^t (t-s)^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})f(s,x_n(s),x_n(s-\tau),{}^C D^{\alpha}x_n(s),{}^C D^{\beta}x_n(s),u_n(s))ds\Bigg) \\ =& x^{(p)}(t;\varphi) + \sigma_1(t) + \sum_{i=0}^{m} \int_0^t (t-r_i(s))^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-r_i(s))^{\alpha-\beta})C_i\dot{r}_i(s)u(s)ds \\ &+ \int_0^t (t-s)^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})f(s,x(s),x(s-\tau),{}^C D^{\alpha}x(s),{}^C D^{\beta}x(s),u(s))ds \\ =& x^{(p)}(t). \end{aligned}$$

Further, we have

$$||^C D^\alpha x(t) -^C D^\alpha x_{n+1}(t)||$$

$$=\left|\left|\frac{1}{\Gamma(p-\alpha)}\int_0^t (t-s)^{p-\alpha-1}\bigg(\int_0^s (s-\xi)^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(s-\xi)^{\alpha-\beta})\right.\right.$$
$$\times [C_i(u(\xi)-u_n(\xi)) + f(\xi, x(\xi), x(\xi-\tau),^C D^\alpha x(\xi),^C D^\beta x(\xi), u(\xi))$$
$$\left.\left. - f(\xi, x_n(\xi), x_n(\xi-\tau),^C D^\alpha x_n(\xi),^C D^\beta x_n(\xi), u_n(\xi))]d\xi\bigg)ds\right|\right|$$
$$=\left|\left|\int_0^t (t-s)^{-1}\Phi_{\alpha-\beta}(A(t-s)^{\alpha-\beta})[C_i(u(s)-u_n(s)) + f(s, x(s), x(s-\tau),^C D^\alpha x(s),^C D^\beta x(s), u(s))\right.\right.$$
$$\left.\left. - f(s, x_n(s), x_n(s-\tau),^C D^\alpha x_n(s),^C D^\beta x_n(s), u_n(s))]ds\right|\right|$$
$$\leq n_8 n_9 T||u(t)-u_n(t)|| + n_8 LT\bigg[||x(t)-x_n(t)|| + ||x(t-\tau)-x_n(t-\tau)|| + ||^C D^\alpha x(t) -^C D^\alpha x_n(t)||$$
$$+ ||^C D^\beta x(t) -^C D^\beta x_n(t)|| + ||u(t)-u_n(t)||\bigg].$$

Moreover,

$$||^C D^\beta x(t) -^C D^\beta x_{n+1}(t)||$$

$$=\left|\left|\frac{1}{\Gamma(q-\beta)}\int_0^t (t-s)^{q-\beta-1}\bigg(\int_0^s (s-\tau)^{\alpha-p-1}\Phi_{\alpha-\beta,\alpha-p}(A(s-\tau)^{\alpha-\beta})\right.\right.$$
$$\times [C_i(u(\tau)-u_n(\tau)) + f(\tau, x(\tau), x(t-\tau),^C D^\alpha x(\tau),^C D^\beta x(\tau), u(\tau))$$
$$\left.\left. - f(\tau, x_n(\tau), x_n(t-\tau),^C D^\alpha x_n(\tau),^C D^\beta x_n(\tau), u_n(\tau))]d\tau\bigg)ds\right|\right|$$
$$=\left|\left|\int_0^t (t-s)^{\alpha-\beta-p+q-1}\Phi_{\alpha-\beta,\alpha-\beta-p+q}(A(t-s)^{\alpha-\beta})[C_i(u(s)-u_n(s))\right.\right.$$
$$+ f(s, x(s), x(s-\tau),^C D^\alpha x(t),^C D^\beta x(t), u(s))$$
$$\left.\left. - f(s, x_n(s), x_n(s-\tau),^C D^\alpha x_n(t),^C D^\beta x_n(t), u_n(s))]ds\right|\right|$$
$$\leq n_{10} n_9 T||u(t)-u_n(t)|| + n_{10} LT\bigg[||x(t)-x_n(t)|| + ||x(t-\tau)-x_n(t-\tau)|| + ||^C D^\alpha x(t) -^C D^\alpha x_n(t)||$$
$$+ ||^C D^\beta x(t) -^C D^\beta x_n(t)|| + ||u(t)-u_n(t)||\bigg].$$

As $n \to \infty$, ${}^C D^\alpha x_{n+1}(t) \to^C D^\alpha x(t)$ and ${}^C D^\beta x_{n+1}(t) \to^C D^\beta x(t)$. Clearly $x(T) = z$ which means that the control $u(t)$ steers the system from the initial state $\varphi(t), \varphi'(t), \cdots, \varphi^{(p)}(t)$ to z in time T. Hence the system (9) is controllable on J. □

5. Examples for controllability results

In this section, we have provided two examples for our proposed criteria to illustrate the controllability results.

Example 5.1. *Consider the problem of nonlinear fractional damped delay dynamical system with time varying delay in control of the form*

$$\begin{aligned} {}^C D^{\frac{5}{2}} x(t) - {}^C D^{\frac{3}{2}} x(t) =& x(t-2) + u(t) + u(t-1) \\ &+ f \begin{pmatrix} 0 \\ \frac{x_1(t) sint}{x_1^2(t)+x_2^2(t)} + \frac{e^{x_1(t-2)}}{1+{}^C D^{\alpha} x_1^2(t-2) + {}^C D^{\beta} x_2^2(t-2) + u(t)} \end{pmatrix}, \end{aligned} \tag{13}$$

where $\alpha = \frac{5}{2}, \beta = \frac{3}{2},\ \tau = 2,\ \rho_0 = t,\ \rho_1 = t-1,$ *as a consequence,* $\tau_0(t) = 0,\ \tau_1(t) = 1,\ A = 1, B = 1, C_0 = 1, C_1 = 1.$

The solution of the system (13) *can be written as*

$$\begin{aligned} x(t) =& [E_{1,1}(t) - tE_{1,2}(t)]\varphi(t) + [tE_{1,2}(t) - t^2 E_{1,3}(t)]\varphi'(t) + t^2 E_{1,3}(t)\varphi''(t) \\ &+ \int_{-2}^{0} (t-s-2)^{\frac{3}{2}} E_{1,\frac{5}{2}}((t-s-2))\varphi(s)ds \\ &+ \sum_{i=0}^{1} \int_0^t (t - r_i(s))^{\frac{1}{2}} E_{1,\frac{5}{2}}((t-r_i(s))^{\frac{3}{2}}) \dot{r}_i(s) u(s) ds \\ &+ \int_0^t (t-s)^{\frac{3}{2}} E_{1,\frac{5}{2}}((t-s)) f(s, x(s), x(s-2), {}^C D^{\alpha} x(s), {}^C D^{\beta} x(s), u(s)) ds. \end{aligned}$$

The Grammian matrix is defined by

$$W = \sum_{i=0}^{1} \int_0^2 (2 - r_i(s))^3 \left[E_{1,\frac{5}{2}}((2-r_i(s)))\dot{r}_i(s)\right] \left[E_{1,\frac{5}{2}}((2-r_i(s)))\dot{r}_i(s)\right]^* ds,$$

where $r_i(s)$ *is a time lead function which is defined by* $r_0(s) = s$ *and* $r_1(s) = s - 1$. *Then the Grammian matrix can be written as*

$$\begin{aligned} W =& \int_0^2 (2-s)^3 \left[E_{1,\frac{5}{2}}((2-s))\right] \left[E_{1,\frac{5}{2}}((2-s))\right]^* ds \\ &+ \int_0^2 (2-s+1)^3 \left[E_{1,\frac{5}{2}}((2-s+1))\right] \left[E_{1,\frac{5}{2}}((2-s+1))\right]^* ds. \end{aligned}$$

Evaluating it, we get

$$W = 145.4159 > 0,$$

which implies it is positive definite. Therefore, the linear system is controllable. And easy to verify that the nonlinear function f *is bounded and Lipschitz continuous and satisfies the Lipschitz condition with the constant* $L = 1$, *the hypotheses of Theorem* 4.1, *and hence the fractional damped delay dynamical system with multiple delays in control* (13) *is controllable on* $[0, 2]$.

Example 5.1 *describe the conditions when* A, B, C_0 *and* C_1 *are constant. Following Example* 5.2 *demonstrate the conditions when* A, B, C_0 *and* C_1 *are matrices.*

Example 5.2. *Consider the problem of nonlinear fractional damped delay dynamical system with time varying delay in control of the form*

$$\begin{aligned} {}^C D^{\frac{5}{2}} x(t) - \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} {}^C D^{\frac{3}{2}} x(t) =& \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t-2) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t-1) \\ &+ f \begin{pmatrix} 0 \\ sin({}^C D^{\alpha} x_1(t)) + cos({}^C D^{\beta} x_2(t)) + \frac{e^{x_1(t-2)}}{1+x_1^2(t)+x_2^2(t-2)+u(t)+u(t-1)} \end{pmatrix}, \end{aligned} \tag{14}$$

where $\alpha = \frac{5}{2}, \beta = \frac{3}{2},\ \tau = 2,\ \rho_0 = t,\ \rho_1 = t-1,$ *as a consequence,* $\tau_0(t) = 0,\ \tau_1(t) = 1,\ x(t) = \varphi(t),\ x'(t) = \varphi'(t),$

$x''(t) = \varphi''(t) \in \mathbb{R}^3,\quad u(t) = \psi(t),\quad A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

The solution of the system (14) *can be written as*

$$\begin{aligned} x(t) =& [E_{1,1}(At) - tE_{1,2}(At)]\varphi(t) + [tE_{1,2}(At) - t^2E_{1,3}(At)]\varphi'(t) + t^2E_{1,3}(At)\varphi''(t) \\ &+ B\int_{-2}^{0}(t-s-2)^{\frac{3}{2}}E_{1,\frac{5}{2}}(A(t-s-2))\varphi(s)ds \\ &+ \sum_{i=0}^{1} C_i \int_0^t (t-r_i(s))^{\frac{1}{2}}E_{1,\frac{5}{2}}(A(t-r_i(s))^{\frac{3}{2}})\dot{r}_i(s)u(s)ds \\ &+ \int_0^t (t-s)^{\frac{3}{2}}E_{1,\frac{5}{2}}(A(t-s))f(s,x(s),x(s-2),^C D^{\alpha}x(s),^C D^{\beta}x(s),u(s))ds. \end{aligned}$$

The Grammian matrix is defined by

$$W = \sum_{i=0}^{1}\int_0^3 (3-r_i(s))^3\left[E_{1,\frac{5}{2}}(A(3-r_i(s)))C_i\dot{r}_i(s)\right]\left[E_{1,\frac{5}{2}}(A(3-r_i(s)))C_i\dot{r}_i(s)\right]^* ds,$$

where $r_i(s)$ *is a time lead function which is defined by* $r_0(s) = s$ *and* $r_1(s) = s-1$. *Then the Grammian matrix can be written as*

$$\begin{aligned} W =& \int_0^3 (3-s)^3\left[E_{1,\frac{5}{2}}(A(3-s))C_0\right]\left[E_{1,\frac{5}{2}}(A(3-s))C_0\right]^* ds \\ &+ \int_0^3 (3-s+1)^3\left[E_{1,\frac{5}{2}}(A(3-s+1))C_1\right]\left[E_{1,\frac{5}{2}}(A(3-s+1))C_1\right]^* ds. \end{aligned}$$

Evaluating it, we get

$$W = \begin{pmatrix} 502.6070 & -373.1780 \\ -1593.7054 & 1305.1044 \end{pmatrix}.$$

Thus $det(W) = 61218.8890 > 0$, *which implies it is positive definite for any* $T > 0$. *Therefore, the linear system is controllable. And easy to verify that the nonlinear function* f *is bounded and Lipschitz continuous and satisfies the Lipschitz condition with the constant* $L = 1$, *the hypotheses of Theorem* 4.1, *and hence the fractional damped delay dynamical system with multiple delays in control* (14) *is controllable on* $[0, 3]$.

6. Conclusion

In this paper has discussed about the controllability of linear and nonlinear fractional damped delay dynamical systems with time varying multiple delays in control. The necessary and sufficient conditions for controllability of linear system has been established by constructing the Grammian matrix. Consequently, a sufficient conditions for the controllability criteria for nonlinear system has been derived by using successive approximation technique. In addition to that, examples are included to verify the effectiveness of the results.

References

[1] B.N.N. Achar, J.W. Hanneken, T. Clarke, Response characteristics of a fractional oscillator, Physica A Stat. Mech. Appl., 309 (3-4) (2002) 275-288.

[2] K. Aissani, M. Benchohra, J.J. Nieto, Controllability for Impulsive Fractional Evolution Inclusions with State-Dependent Delay, Adv. Theory Nonlinear Anal. Appl., 3 (2019) 18-34.

[3] A. Al-rabtah, V. S. Erturk, S. Momani, Solutions of a fractional oscillator by using differential transform method, Comput. Math. with Appl., 59 (3) (2010) 1356-1362.

[4] A. Khan, T. Abdeljawad, J.F. Gómez-Aguilar, Hasib Khan, Dynamical study of fractional order mutualism parasitism food web Module, Chaos, Solitons and Fractals, 134 (2020) 109685.
[5] Aziz Khan, J.F. Goómez-Aguilar, Thabet Abdeljawad, Hasib Khan, Stability and numerical simulation of a fractional order plant-nectar-pollinator model, Alex. Eng. J., 59 (2020) 49-59.
[6] R.L. Bagley, A. Torvik, A Theoretical basis for the application of fractional calculus to viscoelasticity. Journal of Rheology, 27 (3) (1983) 201-210.
[7] K. Balachandran, V. Govindaraj, M. Rivero, J. J. Trujillo, Controllability of fractional damped dynamical systems, Appl Math Comput., 257 (2015) 66-73.
[8] K. Balachandran, J. Y. Park, J. J. Trujillo, Controllability of nonlinear fractional dynamical systems, Nonlinear Anal Theory Methods Appl., 75 (4) (2012) 1919-1926.
[9] D. Baleanu, G. C. Wu, Y. R. Bai, F. L. Chen, Stability analysis of Caputo-like discrete fractional systems, Commun. Nonlinear Sci.Numer. Simul., 48 (2017) 520-530.
[10] R. Bellman, K. L. Cooke, Differential-Difference equations, Academic Press, New York, NY (1963).
[11] Y.Q. Chen, I.D. Petras, D. Xue, Fractional order control: A tutorial, in Proceedings of the 2009. American Control Conference (ACC'09), June 10-12, 2009, St. Louis, IEEE, Piscataway, NJ, 2009, 1397-1411.
[12] T.S. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, Phys. Lett. A, 342 (1-2) (2005) 148-155.
[13] J. Dauer, R. Gahl, Controllability of nonlinear delay systems, J. Optim. Theory Appl., 21 (1) (1977) 59-70.
[14] A.M.A. El-Sayed, H. M. Nour, A. Elsaid, A. E. Matouk, A. Elsonbaty, Dynamical behaviors, circuit realization, chaos control, and synchronization of a new fractional order hyperchaotic system, Appl. Math. Model., 40 (5) (2016) 3516-3534.
[15] G. Fernández-Anaya, G. Nava-Antonio, J. Jamous-Galante, R. Muñoz-Vega, E. G. Hernández Martínez, Asymptotic stability of distributed order nonlinear dynamical systems, Commun. Nonlinear Sci.Numer. Simul., 48 (2017) 541-549.
[16] J.F. Gomez-Aguilar, T. Cordova-Fraga, T. Abdeljawad, A. Khan, H. Khan, Analysis of fractal-fractional malaria transmission model, Fractals. 28 (8) (2020) 2040041 (25 pages).
[17] J. Hale, Theory of functional differential equations, Springer, New York (1977).
[18] J. He, Nonlinear oscillation with fractional derivative and its applications, International Conferences on Vibrating Engineering, Dalian, China, (1998) 288-291.
[19] B.B. He, H.C. Zhou, C.H. Kou, The controllability of fractional damped dynamical systems with control delay, Commun.Nonlinear Sci.Numer. Simul., 32 (2016) 190-198.
[20] S. Huang, R. Zhang, D. Chen, Stability of nonlinear fractional-order time varying systems, J. Comput. Nonlinear Dyn., 11 (3) (2016) 031007 (9 pages).
[21] Junpeng Liu, Suli Liu, Huilai Li, Controllability result of nonlinear higher order fractional damped dynamical system, J. Nonlinear Sci. Appl., 10 (2017) 325-337.
[22] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam (2006).
[23] R.L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng., 32 (2004) 1-104.
[24] R.L. Magin, Fractional calculus models of complex dynamics in biological tissues, Comput. Math. with Appl., 59 (5) (2010) 1586-1593.
[25] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, A Carpinteri, Francesco Mainardi (Eds.), Fractals and Fractional calculus in Continuum Mechaics, Springer-Verlag, New York, (1997) 291-348.
[26] S. Manabe, The non-integer integral and its application to control systems, Journal of Institute of Electrical Engineers of Japan, 6 (3-4) (1961) 83-87.
[27] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York (1993).
[28] Z. Odibat, S. Momani, The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics, Comput. Math. with Appl., 58 (2009) 2199-2208.
[29] J. Sabatier, O.P. Agarwal, J.A. Tenreiro Machado, Advances in fractional calculus, Theoretical developments and applications in physics and engineering, Springer-Verlag (2007).
[30] M. Sivabalan, K. Sathiyanathan, Relative controllability results for nonlinear higher order fractional delay integrodifferential systems with time varying delay in control, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68 (1) (2018) 889-906.
[31] M. Sivabalan, K. Sathiyanathan, Controllability of higher order fractional damped delay dynamical systems, J. Phys. Conf. Ser., 1139 (2018) 12030.
[32] A. Soukkou, M. C. Belhour, S. Leulmi, Review, design, optimization and stability analysis of fractional-order PID controller, Int. j. intell. syst. appl., 8 (7) (2016) 73-96.
[33] A. Tofighi, The intrinsic damping of the fractional oscillator, Phys. A, 329 (1-2) (2003), 29-34.
[34] L. Wiess, On the controllability of delayed differential systems, SIAM j. control, 5 (4) (1967) 575-587.
[35] K. Yongyong, Z. Xiue, Some comparison of two fractional oscillators, Physica B: Physics of Condensed Matter, 405 (1) (2010) 369-373.
[36] Z. Zhou, W. Gong, Finite element approximation of optimal control problems governed by time fractional diffusion equation, Comput. Math. with Appl., 71 (1) (2016) 301-318.