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Some Special Ruled Surfaces in Hyperbolic $3-$Space

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Abstract

In this paper, normal and binormal surfaces, a special class of ruled surfaces in hyperbolic $3-$space, are discussed. These special surfaces are defined in hyperbolic $3-$space and the types of these surfaces are introduced. The properties of these surfaces are given under the condition of being constant angle surfaces in hyperbolic $3-$space.

Keywords: *Ruled Surface, Normal Surface, Binormal Surface*
2010 Mathematics Subject Classification: *53A99*

1. Introduction

These surfaces, which we consider in hyperbolic $3-$space, are special surfaces built on the curve. In this study, these surfaces are provided with the condition of being a constant angle surface. Surfaces making a constant angle between tangent planes with a constant vector field of the ambient space is called a constant angle surface. Constant angle surfaces have been investigated by many researchers in many different spaces. Constant angle surfaces have been studied in three dimensional Euclidean space E^3 by Munteanu and Nistor and the class of constant angle surfaces in E^3have been obtained [1] and in E^n have been studied by Scala and Hernandez [2],[3]. Germelli and Scala applied constant angle surfaces to liquid layers and liquid crystal theory [4].
S^2 and H^2 are spherical and hyperbolic planes respectively, constant angle surfaces are studied in multiplication spaces such as $S^2\times R, H^2\times R$ and Nil_3 [5],[6],[7].
Lopez and Munteanu studied and classified such surfaces in Minkowski space E_1^3. In addition in these studies, they delivered the required and sufficient condition that an extensile tangential surface be a constant angle surface [8].
Constant angle surfaces have also been studied in hyperbolic $3-$space and de Sitter $3-$space [9],[10],[11],[12],[13]. The constant angle conditions of a surface in hyperbolic and de Sitter spaces were determined and the invariants of these surfaces were investigated.
Constant angle tangent surfaces are given as typical examples of constant angle surfaces in H^3 and S_1^3 [9]. Also, a ruled surface is formed by moving a line along a curve in hyperbolic $3-$space [14].
Such constant angle surfaces built on the curves have been studied by Nistor, in three dimensional Euclidean space [15]. Again in Minkowski space E_1^3, Karakus studied under constant angle normal, binormal, rectifiying developable, Darboux developable, and conic surfaces [16].
In this study, constant angle normal and constant angle binormal surfaces, which are a special class of ruled surfaces in hyperbolic $3-$space, have been studied. Due to the condition of being a constant angle surface, the varieties of these surfaces have emerged considering the causal character of the constant vector field of the ambiant space and the causal character of the specified constant angle.
Let us now consider the differential geometry of curves and surfaces in this space before investigating these types of surfaces in hyperbolic $3-$space.

2. Preliminaries

Let R_1^4 be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x,y\rangle = -x_1y_1+x_2y_2+x_3y_3+x_4y_4 \ .$$

Then R_1^4 is called Minkowskian or Lorentzian 4-space. Lorentz norm of vector $x\in R_1^4$ is defined as

$$\|x\| = |\langle x,x\rangle|^{\frac{1}{2}} \ ,$$

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where $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4), z = (z_1, z_2, z_3, z_4) \in R_1^4$ and the canonical basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}_1^4$, then the lorentzian cross product $x \wedge y \wedge z$ is defined by the symbolic determinant

$$x \wedge y \wedge z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

. On can easly see that

$$\langle x \wedge y \wedge z, w \rangle = \det(x, y, z, w).$$

Differential geometry of curves and surfaces in the hyperbolic space H^3 has also been studied [17],[18],[19].
Since H^3 is a Riemannian manifold and regular curve γ reparametrized by arclength, we may assume that $\gamma(s)$ is a unit speed curve, that is, there is a tangent vector

$$t(s) = \gamma'(s)$$

with $\|t(s)\| = 1$. If $\left\langle t'(s), t'(s) \right\rangle \neq -1$, then there is a unit vector

$$n(s) = \frac{t'(s) - \gamma(s)}{\|t'(s) - \gamma(s)\|}$$

and also

$$b(s) = \gamma(s) \wedge t(s) \wedge n(s).$$

Then we have a pseudo orthonormal frame $\{\gamma(s), t(s), n(s), b(s)\}$ of $\mathbb{R}_1^4$ along γ.
Since $\left\langle t'(s), t'(s) \right\rangle \neq -1$, we have also the following Frenet-Serre type formulae is obtained

$$\begin{cases} \gamma'(s) & = t(s) \\ t'(s) & = K_h(s) n(s) + \gamma(s) \\ n'(s) & = -K_h(s) t(s) + \tau_h(s) b(s) \\ b'(s) & = -\tau_h(s) n(s) \end{cases} \tag{2.1}$$

where

$$K_h(s) = \|t'(s) - \gamma(s)\|$$

and

$$\tau_h(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(K_h(s))^2}.$$

Since $\left\langle t'(s), t'(s) \right\rangle \neq -1$,. it is easily seen that $K_h(s) \neq 0$.
Let $x : M \longrightarrow \mathbb{R}_1^4$ be an immersion of a surface M into $\mathbb{R}_1^4$. We say that x is timelike (*resp.* spacelike, lightlike) if the induced metric on M via x is Lorentzian (*resp. Riemannian, degenerated*). If $\langle x, x \rangle = -1$, $x_0 > 1$, then x is an immersion of hyperbolic space H^3.
Due to the diversity of the causal character of a vector field in Minkowski space R_1^4, there are multiple angle concepts between the arbitrary two vectors.
Let $Sp\{x, y\}$ be the subspace spanned by the vectors x and y. Let U be unit spacelike vector field on H^3, and $W = Sp\{\xi_p, U_p\}$ be the subspace spanned by U_p and ξ_p.
If U is a unit spacelike vector field on H^3, then the subspace W can be seen spacelike, timelike or lightlike.
If W is a timelike subspace the arclength of the hyperbolic line segment QR is called the measure of the angle between ξ_p and U_p, where Q and R are the endpoints of vectors ξ_p and U_p, respectively. In this case, there is a unique positive real number $\theta(\xi_p, U_p)$ such that

$$|\langle \xi_p, U_p \rangle| = \cosh \theta(\xi_p, U_p). \tag{2.2}$$

The real number $\theta(\xi_p, U_p)$ is called the timelike angle between spacelike vectors U_p and ξ_p in [20].
If W is a spacelike subspace the arclenght of segment QR for each $p \in M$ is called the measure of the angle between ξ_p and U_p . In this case, there is a unique real number $\theta(\xi_p, U_p) \in (0, \pi)$ such that

$$\langle \xi_p, U_p \rangle = \cos \theta(\xi_p, U_p). \tag{2.3}$$

The real number $\theta(\xi_p, U_p)$ is called spacelike angle between spacelike vectors U_p and ξ_p [20].

3. Constant Angle Normal Ruled Surfaces

Constant angle surfaces similar to those in Lorentz space of helicoid surfaces, which are well known in the Euclidean and Lorentz spaces are very important [22], [23]. In this section, spacelike normal surfaces are constructed on a curve in hyperbolic $3-$space will be investigated and these surfaces will be examined under the condition of being a constant angle surface. The curve used to construct this surface will be a spacelike curve in the hyperbolic $3-$space.
Now let's explore the surface types that will be created in this way.

Definition 3.1. *Let $\alpha : I \to H^3 \subset R^4_1$ is a unit spacelike curve given by arclenght, $x : M \to H^3 \subset R^4_1$ is a spacelike immersion. The M surface produced by α curve is called **normal ruled surface** in hyperbolic space H^3 given by*

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,n(s)\,,\ \ (s,t) \in I \times R, \tag{3.1}$$

here, $n(s)$ is unit normal vector of regular curve $\alpha(s)$.

If we get derivative of $x(s,t)$ normal surface given by (3.1) equation according to s and t, we obtain

$$\begin{cases} x_s(s,t) = & (\cosh t)\,\alpha'(s) + (\sinh t)\,n'(s) \\ x_t(s,t) = & (\sinh t)\,\alpha(s) + (\cosh t)\,n(s)\,. \end{cases}$$

By using Frenet-Serret formulas (2.1) as $\{\alpha(s), t(s), n(s), b(s)\}$ is a orthonormal structure along with α curve of R^4_1, we get the system of equations

$$\begin{cases} x(s,t) = & (\sinh t)\,\alpha(s) + (\cosh t)\,n(s) \\ x_s(s,t) = & (\cosh t - \kappa_h(s)\sinh t)\,t(s) + (\tau_h(s)\sinh t)\,b(s) \\ x_t(s,t) = & (\sinh t)\,\alpha(s) + (\cosh t)\,n(s)\,. \end{cases}$$

In that case, we get

$$\begin{cases} E & = \langle x_s, x_s\rangle = (\cosh t - \kappa_h(s)\sinh t)^2 + (\tau_h(s)\sinh t)^2 \\ F & = \langle x_s, x_t\rangle = 0 \\ G & = \langle x_t, x_t\rangle = \cosh^2 t - \sinh^2 t = 1 \end{cases}$$

and

$$\langle \xi, \xi\rangle = F^2 - EG = -E < 0$$

where ξ is the unit normal vector field of surface M in R^4_1, and obviously surface M is a spacelike surface.
Let's find the unit normal defined as

$$\xi = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|}$$

of the spacelike normal ruled surface M is given by equation (3.1) in hyperbolic $3-$space. If we consider definition of Lorentz cross product in Minkowski space R^4_1, then we get

$$x \wedge x_s \wedge x_t = (\cosh t - \kappa_h(s)\sinh t)\,b(s) - (\tau_h(s)\sinh t)\,t(s)$$

and the length of this vector is

$$\|x \wedge x_s \wedge x_t\| = \sqrt{(\cosh t - \kappa_h(s)\sinh t)^2 + (\tau_h(s)\sinh t)^2}$$

and the unit normal vector field of surface M given with (3.1) parametrization is obtained as

$$\xi = \frac{(\cosh t - \kappa_h(s)\sinh t)\,b(s) - (\tau_h(s)\sinh t)\,t(s)}{\sqrt{\Delta}} \tag{3.2}$$

$$\Delta = (\cosh t - \kappa_h(s)\sinh t)^2 + (\tau_h(s)\sinh t)^2\,.$$

Let us now examine the case where the surface M given by equation (3.1) is a constant angle surface. In order to the normal ruled surface M to be a constant angle surface, the normal ξ of the surface and the constant direction of the hyperbolic $3-$space must make a constant angle. We can select this constant direction in the hyperbolic $3-$space to be spacelike. Furthermore, considering the causal character of the angle between the unit normal vector field of the surface and the constant direction of the hyperbolic $3-$space, the normal surfaces produced by a spacelike curve is divided into two parts as follows.

3.1. Normal Ruled Surfaces with Timelike And Spacelike Angles

Definition 3.2. *Let $x : M \to H^3 \subset R^4_1$ is a spacelike immersion and ξ is unit spacelike normal vector of normal surface M. If there is a ξ_{d_1} spacelike direction as $\theta\left(\xi,\xi_{d_1}\right)$ timelike angle on M surface is constant, then M surface is called a* ***constant timelike angled normal ruled surface*** *on hyperbolic $3-$space H^3.*

Theorem 3.3. *The normal spacelike surfaces of constant timelike angles in hyperbolic space H^3 are Lorentz plane parts.*

Proof. $x : M \to H^3 \subset R^4_1$ is a spacelike immersion and M surface is a constant timelike angled normal surface provided with parametrization

$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,n(s),\ \ (s,t) \in I \times R.$

In this case, according to the definition of a constant angle surface, the normal vector field ξ of the surface of M and the constant direction ξ_{d_1} will be constant timelike angle, that is

$\left\langle \xi,\xi_{d_1}\right\rangle = \cosh\theta.$

From the equation (3.2), we get

$$\left\langle \xi,\xi_{d_1}\right\rangle = \frac{(\cosh t - \kappa_h(s)\sinh t)}{\sqrt{\Delta}}\left\langle b(s),\xi_{d_1}\right\rangle - \frac{(\tau_h(s)\sinh t)}{\sqrt{\Delta}}\left\langle t(s),\xi_{d_1}\right\rangle.$$

If both sides of this statement are squared and necessary arrangements are made, we get

$$\begin{aligned} 0 = &\left(\cosh^2\theta - \left\langle b(s),\xi_{d_1}\right\rangle^2\right)\cosh^2 t + \\ &+\left(-2\kappa_h(s)\cosh^2\theta + 2\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle^2 + 2\tau_h(s)\left\langle b(s),\xi_{d_1}\right\rangle\left\langle t(s),\xi_{d_1}\right\rangle\right)\cosh t\sinh t + \\ &+\left[\left(\kappa_h^2(s)+\tau_h^2(s)\right)\cosh^2\theta - \left(\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle + \tau_h(s)\left\langle t(s),\xi_{d_1}\right\rangle\right)^2\right]\sinh^2 t. \end{aligned}$$

Multiply both sides of this expression by $\dfrac{1}{\cosh^2 t}$, we obtain the second order polynomial equation according to w as

$$\begin{aligned} 0 = &\left[\left(\kappa_h^2(s)+\tau_h^2(s)\right)\cosh^2\theta - \left(\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle + \tau_h(s)\left\langle t(s),\xi_{d_1}\right\rangle\right)^2\right]w^2 + \\ &+\left(-2\kappa_h(s)\cosh^2\theta + 2\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle^2 + 2\tau_h(s)\left\langle b(s),\xi_{d_1}\right\rangle\left\langle t(s),\xi_{d_1}\right\rangle\right)w \\ &+\left(\cosh^2\theta - \left\langle b(s),\xi_{d_1}\right\rangle^2\right). \end{aligned}$$

where

$\tanh t = w.$

Obviously,

$w \neq 0.$

Thus, the coefficients of such a quadratic equation must be zero. In that case, we have

$$\begin{cases} \left(\kappa_h^2(s)+\tau_h^2(s)\right)\cosh^2\theta - \left(\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle + \tau_h(s)\left\langle t(s),\xi_{d_1}\right\rangle\right)^2 = 0 \\ -2\kappa_h(s)\cosh^2\theta + 2\kappa_h(s)\left\langle b(s),\xi_{d_1}\right\rangle^2 + 2\tau_h(s)\left\langle b(s),\xi_{d_1}\right\rangle\left\langle t(s),\xi_{d_1}\right\rangle = 0 \\ \left(\cosh^2\theta - \left\langle b(s),\xi_{d_1}\right\rangle^2\right) = 0. \end{cases} \tag{3.3}$$

The third equation of (3.3) system, we conclude that

$$\left\langle b(s),\xi_{d_1}\right\rangle = \pm\cosh\theta. \tag{3.4}$$

If this expression is substituted in the second equation of (3.3) system, we have

$$\tau_h(s)\left\langle b(s),\xi_{d_1}\right\rangle\left\langle t(s),\xi_{d_1}\right\rangle = 0. \tag{3.5}$$

The following conditions apply here.
If $\tau_h(s) = 0$, both the first and second equations of the (3.3) system writing expression are provided. Furthermore, since the hyperbolic torsion of the curve α is zero, curve α is a hyperbolic planar curve and M surface is a Lorentz plane part.
If $\tau_h(s) \neq 0$ and $\left\langle b(s),\xi_{d_1}\right\rangle = 0$, in this case we have

$\cosh\theta = 0.$

Because $\cosh\theta$ cannot be zero by definition of a constant angle surface.

If $\tau_h(s) \neq 0$, $\langle b(s), \xi_{d_1} \rangle \neq 0$ and $\langle t(s), \xi_{d_1} \rangle = 0$, in this case we get

$$\begin{cases} \langle b(s), \xi_{d_1} \rangle = \cosh\theta \\ \langle t(s), \xi_{d_1} \rangle = 0. \end{cases}$$

If the derivative of the above system is taken according to s and Frenet-Serret formulas are used, we have

$$\begin{cases} \langle n(s), \xi_{d_1} \rangle = 0 \\ \langle \alpha(s), \xi_{d_1} \rangle = 0. \end{cases}$$

So the direction vector ξ_{d_1} is orthogonal to all three of $\alpha(s), t(s), n(s)$, and $\{\alpha(s), t(s), n(s), \xi_{d_1}\}$ is orthonormal basis. This is a contradiction.
Considering the above three situations, we get

$$\tau_h(s) = 0$$

and the constant timelike angle normal ruled surface produced by the hyperbolic planar curve $\alpha(s)$ is the Lorentz plane part in the hyperbolic $3-$space.

□

Remark 3.4. *Since stereographic projection is the conformal map, using stereographic projection, constant angle surface in Minkowskian model of hyperbolic space H^3 is visulized in Poincare ball model of hyperbolic space H^3 [21].*

By using that idea, we can give the following examples.

Example 3.5. *Let $\alpha : I \to H^3 \subset \mathbb{R}^4_1$ be a regular spacelike curve given by arc-length*

$$\alpha(s) = \left(\sqrt{3}, \cos s, \sin s, 1\right).$$

Tangent vector, normal vector, binormal vector, hyperbolic curvature and hyperbolic torsion of the spacelike curve $\alpha(s)$ are as follows.

$$t(s) = (0, \cos s, -\sin s, 0),$$

$$n(s) = \left(-\sqrt{\frac{3}{2}}, -\sqrt{2}\sin s, -\sqrt{2}\cos s, -\sqrt{\frac{1}{2}}\right),$$

$$b(s) = \left(\frac{1}{\sqrt{2}}, 0, 0, \sqrt{\frac{3}{2}}\right),$$

$$\kappa_h(s) = \left\| t'(s) - \alpha(s) \right\| = \sqrt{2},$$

$$\tau_h(s) = -\frac{\det\left(\alpha(s), \alpha'(s), \alpha''(s), \alpha'''(s)\right)}{(\kappa_h(s))^2} = 0.$$

The normal ruled surface M generated by α as the surface parametrized by

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,n(s)\ ,\ (s,t) \in I \times \mathbb{R}\,.$$

Similarly, the normal ruled surface M generated by α is Lorentz plane parts and theorem 1 and theorem 2 are provided. The pictures of the Stereografik projection of normal surface appear in figure-3.1

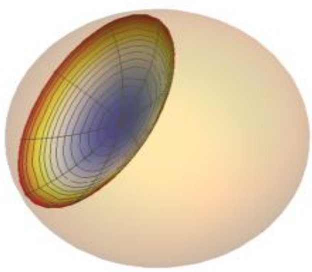

Figure 3.1

Example 3.6. *Let $\alpha : I \to H^3 \subset \mathbb{R}^4_1$ be a regular spacelike curve given by arc-length*

$\alpha(s) = (\cosh s, \sinh s, 1, 1)$.

The normal ruled surface M generated by α as the surface parametrized by

$x(s,t) = (\cosh t)\, \alpha(s) + (\sinh t)\, n(s)\ ,\ (s,t) \in I \times \mathbb{R}$.

Similarly, the normal ruled surface M generated by α is Lorentz plane parts. The pictures of the Stereografik projection of normal surface appear in figure-3.2.

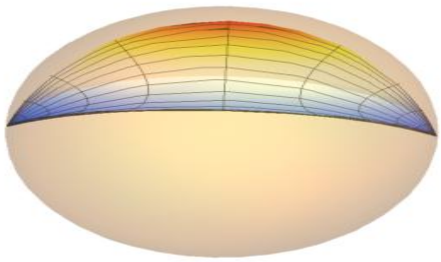

Figure 3.2

We can find the constant spacelike direction ξ_{d_1} of the spacelike normal ruled surface M in hyperbolic $3-$space H^3.

Lemma 3.7. *$x : M \to H^3 \subset R^4_1$ is a spacelike immersion and M is a constant timelike angled normal surface in hyperbolic space H^3. In this case, constant spacelike direction of M surface is as*

$$\begin{aligned} \xi_{d_1} = \ & \frac{(\cosh t - \kappa_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}} t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}} b(s) \\ & + (\cosh\theta)\,\xi \end{aligned}$$

where φ is the angle between the constant spacelike direction ξ_{d_1} and the timelike vector x.

Proof. Let's assume that ξ_d is the constant spacelike direction of the spacelike surface M in Minkowski space R^4_1. The angle between the unit spacelike normal vector field ξ and the constant spacelike direction ξ_d is represented by θ, that is

$\langle \xi, \xi_d \rangle = \cosh\theta$.

We can write

$\xi_d = \xi_d^T + \xi_d^N$

where $\xi_d \in R^4_1$. So, if we take the inner product of both sides of equation

$\xi_d = \xi_d^T + \lambda_1 \xi + \lambda_2 x$

with ξ and x, then we get

$\lambda_1 = \cosh\theta$ and $\lambda_2 = -\langle \xi_d, x \rangle$.

Since x is a timelike vector and ξ_d is a spacelike vector, if we denote the timelike angle between these vectors by φ, then we can write

$\lambda_2 = -\sinh\varphi$.

So, we get

$\xi_d = \xi_d^T + (\cosh\theta)\,\xi - (\sinh\varphi)\, x$.

Taking the inner product of both sides of the last statement with ξ_d, we get

$\|\xi_d\|^2 = \cosh^2\varphi - \cosh^2\theta, \cosh^2\varphi > \cosh^2\theta$.

Also, if we choose without losing generality

$$e_1 = \frac{\xi_d^T}{\|\xi_d^T\|},$$

$$\xi_d = \sqrt{\cosh^2\varphi - \cosh^2\theta}\,e_1 + (\cosh\theta)\,\xi - (\sinh\varphi)\,x$$

as we get. Thus, we find the constant spacelike direction ξ_d in R_1^4. If we take

$$e_1 = \frac{x_s}{\|x_s\|}$$

and choose the part of the ξ_d direction remaining in the hyperbolic space H^3, then the constant spacelike direction ξ_{d_1} of spacelike normal ruled surface M is obtained as

$$\begin{aligned}\xi_{d_1} = \quad & \frac{(\cosh t - \kappa_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}}t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}}b(s) \\ & + (\cosh\theta)\,\xi\end{aligned}$$

where

$$\Delta = \|x_s\|^2 = (\cosh t - \kappa_h(s)\sinh t)^2 + (\tau_h(s)\sinh t)^2.$$

□

Example 3.8. *Let $\alpha : I \to H^3 \subset \mathbb{R}_1^4$ be a regular spacelike curve given by arc-length*

$$\alpha(s) = \left(\sqrt{3}, \cos s, \sin s, 1\right).$$

The normal ruled surface M generated by α as the surface parametrized by

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,n(s)\ ,\ (s,t) \in I \times \mathbb{R}.$$

According to lemma-1, constant spacelike direction of M surface is as

$$\begin{aligned}\xi_{d_1} = \quad & \frac{(\cosh t - \kappa_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}}t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cosh^2\theta\right|}}{\sqrt{\Delta}}b(s) \\ & + (\cosh\theta)\,\xi\end{aligned}$$

where φ is the angle between the constant spacelike direction ξ_{d_1} and the timelike vector x. The pictures of the Stereografik projection of constant spacelike direction appear in figure-3.3 and figure-3.4

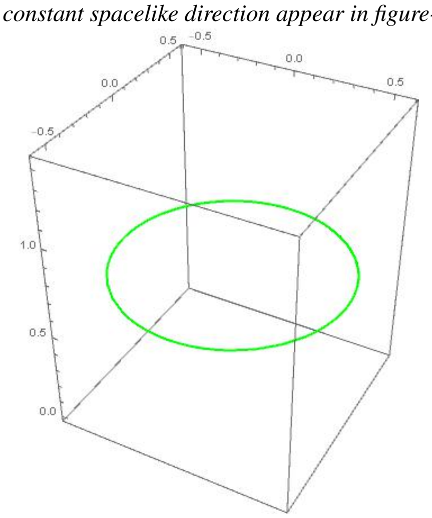

Figure 3.3

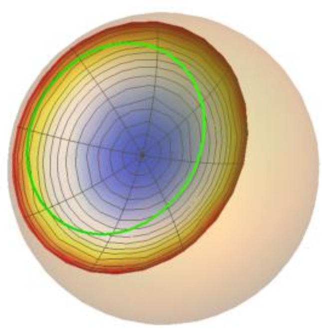

Figure 3.4

Definition 3.9. *Let $x : M \to H^3 \subset R_1^4$ is a spacelike immersion and ξ is unit spacelike normal vector of normal surface M. If there is a ξ_{d_2} spacelike direction as $\theta\left(\xi, \xi_{d_2}\right)$ spacelike angle on M surface is constant, then M surface is called a* ***constant spacelike angled normal ruled surface*** *on hyperbolic $3-$space H^3.*

Theorem 3.10. *The normal spacelike surfaces of constant spacelike angles in hyperbolic space H^3 are Lorentz plane parts.*

Proof. The proof of the theorem can be proved in a similar way to the previous proofs above given. □

Now, we want to find the constant spacelike direction ξ_{d_2} of the spacelike normal ruled surface M in hyperbolic $3-$space H^3. Then, we can give the following lemma.

Lemma 3.11. *$x : M \to H^3 \subset R_1^4$ is a spacelike immersion and M is a constant spacelike angled normal surface in hyperbolic space H^3. In this case, constant spacelike direction of M surface given by*

$$\begin{aligned}\xi_{d_2} = \; & \frac{(\cosh t - \kappa_h(s)\sinh t)\sqrt{\sin^2\theta + \sinh^2\varphi}}{\sqrt{\Delta}} t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\sin^2\theta + \sinh^2\varphi}}{\sqrt{\Delta}} b(s) \\ & + (\cos\theta)\,\xi.\end{aligned}$$

where φ is the angle between the constant spacelike direction ξ_{d_2} and the timelike vector x.

4. Constant Angle Binormal Ruled Surfaces

In this section, spacelike binormal surfaces constructed on a curve in hyperbolic $3-$space will be investigated and these surfaces will be examined under the condition of being a constant angle surface. The curve used to construct this surface will be a spacelike curve in the hyperbolic $3-$space.

Now let's explore the surface types that will be constructed in this way.

Definition 4.1. *Let $\alpha : I \to H^3 \subset R_1^4$ is a unit spacelike curve given by arclenght, $x : M \to H^3 \subset R_1^4$ is a spacelike immersion. The M surface produced by α curve is called* ***binormal ruled surface*** *in hyperbolic space H^3 by given*

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,b(s), \;\; (s,t) \in I \times R \tag{4.1}$$

where, $b(s)$ is unit binormal vector of regular curve $\alpha(s)$.

If we get derivative of $x(s,t)$ normal surface given by (4.1) equation according to s and t, we obtain

$$\begin{cases} x_s(s,t) = & (\cosh t)\,\alpha'(s) + (\sinh t)\,b'(s) \\ x_t(s,t) = & (\sinh t)\,\alpha(s) + (\cosh t)\,b(s). \end{cases}$$

By using Frenet-Serret formulas (2.1) as $\{\alpha(s), t(s), n(s), b(s)\}$ is a orthonormal structure along with α curve of R_1^4, we have

$$\begin{cases} x(s,t) = & (\cosh t)\,\alpha(s) + (\sinh t)\,b(s) \\ x_s(s,t) = & (\cosh t)\,t(s) + (\tau_h(s)\sinh t)\,n(s) \\ x_t(s,t) = & (\sinh t)\,\alpha(s) + (\cosh t)\,b(s). \end{cases}$$

In that case, we get

$$\begin{cases} E & = \langle x_s, x_s\rangle = (\cosh t)^2 + (\tau_h(s)\sinh t)^2 \\ F & = \langle x_s, x_t\rangle = 0 \\ G & = \langle x_t, x_t\rangle = \cosh^2 t - \sinh^2 t = 1 \end{cases}$$

and

$$\langle \xi, \xi \rangle = F^2 - EG = -E < 0$$

where ξ is the unit normal vector field of surface M in R_1^4, and obviously surface M is a spacelike surface.

Let's find the unit normal defined as

$$\xi = \frac{x \wedge x_s \wedge x_t}{\|x \wedge x_s \wedge x_t\|}$$

of the spacelike binormal surface M given by equation (4.1) in H^3. If we get help from definition of Lorentz cross product in Minkowski space R_1^4, then we get

$$x \wedge x_s \wedge x_t = (\cosh t)\,n(s) - (\tau_h(s)\sinh t)\,t(s)$$

and the length of this vector given

$$\|x \wedge x_s \wedge x_t\| = \sqrt{(\cosh t)^2 + (\tau_h(s)\sinh t)^2}$$

and the unit normal vector field of surface M given by (4.1) parametrization is obtained as

$$\xi = \frac{(\cosh t)\,n(s) - (\tau_h(s)\sinh t)\,t(s)}{\sqrt{\Delta_1}} \tag{4.2}$$

and

$$\Delta_1 = (\cosh t)^2 + (\tau_h(s)\sinh t)^2.$$

Let us now examine the case where the surface M given by equation (4.1) is a constant angle surface. In order for the binormal surface M to be a constant angle surface, the normal ξ of the surface and the constant direction of the hyperbolic $3-$space must make a constant angle. We can take this constant direction in the hyperbolic $3-$space to be spacelike. Furthermore, considering the causal character of the angle between the unit normal vector field of the surface and the constant direction of the hyperbolic $3-$space, the spacelike binormal surfaces produced by a spacelike curve are divided into two parts as follows.

4.1. Binormal Ruled Surfaces with Timelike And Spacelike Angles

Definition 4.2. *Let $x : M \to H^3 \subset R_1^4$ is a spacelike immersion and ξ is unit spacelike normal vector of binormal surface M. If there is a ξ_{d_3} spacelike direction as $\theta\left(\xi,\xi_{d_3}\right)$ timelike angle on M surface is constant, then M surface is called a* ***constant timelike angled binormal ruled surface*** *on hyperbolic $3-$space H^3.*

$x : M \to H^3 \subset R_1^4$ is a spacelike immersion and M surface is constant timelike angled binormal surface given by

$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,b(s),\ \ (s,t) \in I \times R.$

In this case, according to the definition of a constant angle surface, the normal vector field ξ of the surface of M and the constant direction ξ_{d_3} will be constant timelike angle, that is

$\left\langle \xi,\xi_{d_3}\right\rangle = \cosh\theta.$

From the equation (4.2), we get

$$\left\langle \xi,\xi_{d_3}\right\rangle = \frac{(\cosh t)}{\sqrt{\Delta_1}}\left\langle b(s),\xi_{d_3}\right\rangle - \frac{(\tau_h(s)\sinh t)}{\sqrt{\Delta_1}}\left\langle t(s),\xi_{d_3}\right\rangle.$$

If both sides of this statement are squared and necessary arrangements are made, then we obtain the second order polynomial equation according to w so that

$$\left[\tau_h^2(s)\left(\cosh^2\theta - \left\langle t(s),\xi_{d_3}\right\rangle^2\right)\right]w^2 + \left(2\tau_h(s)\left\langle n(s),\xi_{d_3}\right\rangle\left\langle t(s),\xi_{d_3}\right\rangle\right)w+ \\ +\left(\cosh^2\theta - \left\langle n(s),\xi_{d_3}\right\rangle^2\right) = 0$$

where

$\tanh t = w.$

Obviously,

$w \neq 0.$

Thus, the coefficients of such a quadratic equation must be zero. In that case, we have

$$\begin{cases} \tau_h^2(s)\left(\cosh^2\theta - \left\langle t(s),\xi_{d_3}\right\rangle^2\right) = 0 \\ 2\tau_h(s)\left\langle n(s),\xi_{d_3}\right\rangle\left\langle t(s),\xi_{d_3}\right\rangle = 0 \\ \cosh^2\theta - \left\langle n(s),\xi_{d_3}\right\rangle^2 = 0. \end{cases} \tag{4.3}$$

The third equation of (4.3) system, we get

$$\left\langle n(s),\xi_{d_3}\right\rangle = \pm\cosh\theta. \tag{4.4}$$

Considering the second equation of the system (4.3), we have

$\tau_h(s)\left\langle n(s),\xi_{d_3}\right\rangle\left\langle t(s),\xi_{d_3}\right\rangle = 0.$

The following conditions hold.
If $\tau_h(s) = 0$, both the first and second equations of the (4.3) system we wrote above are provided. Furthermore, since the hyperbolic torsion of curve α is zero, curve α is a hyperbolic planar curve and M surface is a the Lorentz plane part.
If $\tau_h(s) \neq 0$ and $\left\langle n(s),\xi_{d_3}\right\rangle = 0$, in this case we have

$\cosh\theta = 0$

from (4.4) and $\cosh\theta$ cannot be zero by definition of a constant angle surface.
If $\tau_h(s) \neq 0$, $\left\langle n(s),\xi_{d_3}\right\rangle \neq 0$ and $\left\langle t(s),\xi_{d_3}\right\rangle = 0$, in this case, we get from the first equation of(4.3) system

$\cosh^2\theta\,\tau_h^2(s) = 0$

and

$\tau_h(s) = 0.$

This is a contradiction.
Considering the above three situations, we get

$\tau_h(s) = 0$

and the constant timelike angle binormal ruled surface produced by the hyperbolic planar curve $\alpha(s)$ is the Lorentz plane part in the hyperbolic $3-$space.

So we can give the following theorem.

Theorem 4.3. *The binormal spacelike ruled surfaces of constant timelike angles in hyperbolic space H^3 are Lorentz plane parts.*

Example 4.4. *Let $\alpha : I \to H^3 \subset \mathbb{R}^4_1$ be a regular spacelike curve given by arc-length*

$$\alpha(s) = \left(\sqrt{3}, \cos s, \sin s, 1\right).$$

The binormal ruled surface M generated by α as the surface parametrized by

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,b(s)\ ,\ (s,t) \in I \times \mathbb{R}.$$

It is clear that the spacelike curve $\alpha(s)$ is a hyperbolic planar curve and the binormal ruled surface M generated by α is Lorentz plane parts. Thus, theorem 4 and theorem 5 are provided. The pictures of the Stereografik projection of binormal surface appear in figure-4.1

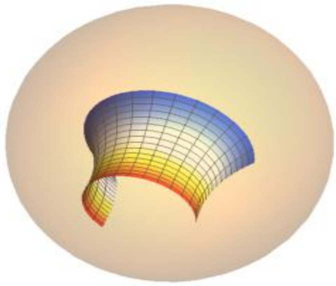

Figure 4.1

Example 4.5. *Let $\alpha : I \to H^3 \subset \mathbb{R}^4_1$ be a regular spacelike curve given by arc-length*

$$\alpha(s) = (\cosh s, \sinh s, 1, 1).$$

Tangent vector, normal vector, binormal vector, hyperbolic curvature and hyperbolic torsion of the spacelike curve $\alpha(s)$ are as follows.

$$t(s) = (\sinh s, \cosh s, 0, 0),$$

$$n(s) = \left(0, 0, -\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right),$$

$$b(s) = \left(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right),$$

$$\kappa_h(s) = \left\| t'(s) - \alpha(s) \right\| = \sqrt{2},$$

$$\tau_h(s) = -\frac{\det(\alpha(s), \alpha'(s), \alpha''(s), \alpha'''(s))}{(\kappa_h(s))^2} = 0.$$

The binormal ruled surface M generated by α as the surface parametrized by

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,b(s)\ ,\ (s,t) \in I \times \mathbb{R}.$$

It is clear that the spacelike curve $\alpha(s)$ is a hyperbolic planar curve and the binormal ruled surface M generated by α is Lorentz plane parts. The pictures of the Stereografik projection of binormal surface appear in figure-4.2.

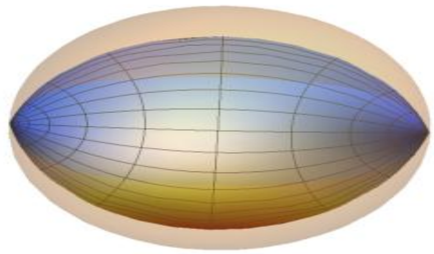

Figure 4.2

Now, if constant spacelike direction ξ_{d_3} of the spacelike binormal ruled surface M in hyperbolic $3-$space H^3 is found similar to the previous one, then we can express the following lemma.

Lemma 4.6. *$x : M \to H^3 \subset R^4_1$ is a spacelike immersion and M is a constant timelike angled binormal surface in hyperbolic space H^3. In this case, constant spacelike direction of M surface is as,*

$$\begin{aligned}\xi_{d_3} = \quad & \frac{(\cosh t)\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}}{\sqrt{\Delta_1}} t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\left|\sinh^2\varphi - \sinh^2\theta\right|}}{\sqrt{\Delta_1}} n(s) \\ & + (\cosh\theta)\,\xi\end{aligned}$$

where φ is the timelike angle between the constant spacelike direction ξ_{d_3} and the timelike vector x.

Definition 4.7. *Let $x : M \to H^3 \subset R^4_1$ is a spacelike immersion and ξ is unit spacelike normal vector of binormal surface M. If there is a ξ_{d_4} spacelike direction as $\theta\left(\xi, \xi_{d_4}\right)$ spacelike angle on M surface is constant, then M surface is called a* ***constant spacelike angled binormal ruled surface*** *on hyperbolic $3-$space H^3.*

Theorem 4.8. *The spacelike binormal surfaces of constant spacelike angles in hyperbolic space H^3 are Lorentz plane parts.*

Proof. The proof of the theorem can be proved in a similar way to the previous proofs above given. □

Now, let us can find the constant spacelike direction ξ_{d_4} of the spacelike binormal ruled surface M in hyperbolic $3-$space H^3. The proof of the lemma can be proved in a similar way to the previous proofs above given.

Lemma 4.9. *$x : M \to H^3 \subset R^4_1$ is a spacelike immersion and M is a constant spacelike angled binormal surface in hyperbolic space H^3. In this case, constant spacelike direction of M surface is as,*

$$\begin{aligned}\xi_{d_4} = \quad & \frac{(\cosh t)\sqrt{\left|\cosh^2\varphi - \cos^2\theta\right|}}{\sqrt{\Delta_1}} t(s) \\ & + \frac{(\tau_h(s)\sinh t)\sqrt{\left|\cosh^2\varphi - \cos^2\theta\right|}}{\sqrt{\Delta_1}} n(s) \\ & + (\cos\theta)\,\xi\end{aligned}$$

where φ is the timelike angle between the constant spacelike direction ξ_{d_4} and the timelike vector x.

5. Conclusion

For many years, many studies have been done on the geometry of regle surfaces. This study has been prepared to contribute to making more detailed studies on special surfaces of hyperbolic 3-space. In the introduction section, a summary of the literature, basic definitions and theorems are given for a better understanding of the subject. In the following sections, special ruled surface in hyperbolic 3-space are examined in detail. As a result, this study has been presented to the literature as a resource that will be used by every scientist who will study regle surface in hyperbolic 3-space.

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