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(m_1, m_2)-Geometric Arithmetically Convex Functions and Related Inequalities

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Abstract
In this manuscript, we introduce and study the concept of (m_1, m_2)-geometric arithmetically (GA) convex functions and their some algebric properties. In addition, we obtain Hermite-Hadamard type inequalities for the newly introduced this type of functions whose derivatives in absolute value are the class of (m_1, m_2)-GA-convex functions by using both well-known power mean and Hölder's integral inequalities.

Keywords: *Convex function; m-convex function; (m_1, m_2)-GA convex function; Hermite-Hadamard inequality.*
AMS Subject Classification (2020): *26A51; 26D10; 26D15.*

1. Preliminaries and fundamentals

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Hermite-Hadamard integral inequality is very important in the convexity theory. Readers can find more informations in [1–6, 8, 9, 12, 13, 16] and references therein regarding both convexity theory and H-H integral inequalities.

Definition 1.1 ([10, 11])**.** $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is called GA-convex on I if

$$f\left(a^{\xi} b^{1-\xi}\right) \leq \xi f(a) + (1-\xi) f(b)$$

holds for all $a, b \in I$ and $\xi \in [0, 1]$.

Definition 1.2 ([14])**.** $f : [0, b] \to \mathbb{R}$ is called m-convex for $m \in (0, 1]$ if the following inequality

$$f(\xi x_1 + m(1-\xi)x_2) \leq \xi f(x_1) + m(1-\xi) f(x_2)$$

holds for all $x_1, x_2 \in [0, b]$ and $\xi \in [0, 1]$.

Definition 1.3 ([7])**.** $f : [0, b] \to \mathbb{R}$, $b > 0$, is caled (m_1, m_2)-convex function, if

$$f(m_1 \xi \theta + m_2(1-\xi)\vartheta) \leq m_1 \xi f(\theta) + m_2(1-\xi) f(\vartheta)$$

for all $\theta, \vartheta \in I$, $\xi \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$.

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The purpose of this manuscript is to give the concept of (m_1, m_2)-geometric arithmetically (GA) convex functions and find some results connected with new inequalities similar to the well-known H-H inequality for these classes of functions.

2. Some properties of (m_1, m_2)-GA convex functions

Here, we will definite a new concept, which is called (m_1, m_2)-GA convex functions and we give by setting some algebraic properties for the (m_1, m_2)-GA convex functions.

Definition 2.1. Let the function $f : [0, b] \to \mathbb{R}$ and $(m_1, m_2) \in (0, 1]^2$. If

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq m_1 t f(a) + m_2 (1-t) f(b). \tag{2.1}$$

for all $[a, b] \subset [0, b]$ and $t \in [0, 1]$, then the function f is called (m_1, m_2)-GA convex function, if this inequality reversed, then the function f is called (m_1, m_2)-GA concave function.

We discuss some connections between the class of the (m_1, m_2)-GA convex functions and other classes of generalized convex functions.

Remark 2.1. When $m_1 = m_2 = 1$, the (m_1, m_2)-GA convex (concave) function becomes a GA convex (concave) function in defined [10, 11].

Remark 2.2. When $m_1 = 1$, $m_2 = m$, the (m_1, m_2)-GA convex (concave) function becomes the (α, m)-GA convex (concave) function defined in [15].

Proposition 2.1. *$f : I \subset (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-GA convex on $I \iff f \circ \exp : \ln I \to \mathbb{R}$ is (m_1, m_2)-convex on the interval $\ln I = \{\ln x | x \in I\}$.*

Proof. $(\Rightarrow)$ Suppose $f : I \subset (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-GA convex function. Then, we get

$$\begin{aligned} (f \circ \exp)(m_1 t \ln a + m_2(1-t) \ln b) &\leq m_1 t (f \circ \exp)(\ln a) + m_2(1-t)(f \circ \exp)(\ln b) \\ f\left(a^{m_1 t} b^{m_2(1-t)}\right) &\leq m_1 t f(a) + m_2 (1-t) f(b). \end{aligned}$$

Therefore, the function $f \circ \exp$ is (m_1, m_2)-convex function on $\ln I$.

$(\Leftarrow)$ Let $f \circ \exp : \ln I \to \mathbb{R}$, (m_1, m_2)-convex function on $\ln I$. Then, we get

$$\begin{aligned} f\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(e^{m_1 t \ln a + m_2(1-t) \ln b}\right) \\ &= (f \circ \exp)(m_1 t \ln a + m_2(1-t) \ln b) \\ &\leq m_1 t f\left(e^{\ln a}\right) + m_2 (1-t) f\left(e^{\ln b}\right) \\ &= m_1 t f(a) + m_2 (1-t) f(b). \end{aligned}$$

□

Theorem 2.1. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$. If f and g are (m_1, m_2)-geometric arithmetically convex functions, then*
(i) $f + g$ is an (m_1, m_2)-geometric arithmetically convex function,
(ii) For $c \in \mathbb{R}$ $(c \geq 0)$ cf is an (m_1, m_2)-geometric arithmetically convex function.

Proof. (i) Let f, g be (m_1, m_2)-geometric arithmetically convex functions, then

$$\begin{aligned} (f+g)\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(a^{m_1 t} b^{m_2(1-t)}\right) + g\left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\leq m_1 t f(a) + m_2(1-t) f(b) + m_1 t g(a) + m_2(1-t) g(b) \\ &= m_1 t (f+g)(a) + m_2(1-t)(f+g)(b) \end{aligned}$$

(ii) Let f be (m_1, m_2)-GA convex function and $c \in \mathbb{R}$ $(c \geq 0)$, then

$$\begin{aligned} (cf)\left(a^{m_1 t} b^{m_2(1-t)}\right) &\leq c\left[m_1 t f(x) + m_2(1-t) f(y)\right] \\ &= m_1 t (cf)(x) + m_2(1-t)(cf)(y). \end{aligned}$$

□

Theorem 2.2. *Let $f, g : I \to \mathbb{R}$ are nonnegative and monotone increasing. If f and g are (m_1, m_2)-GA convex functions, then fg is (m_1, m_2)-GA convex function.*

Proof. If $\vartheta_1 \leq \vartheta_2$ ($\vartheta_2 \leq \vartheta_1$ is similar) then

$$f(\vartheta_1)g(\vartheta_2) + f(\vartheta_2)g(\vartheta_1) \leq f(\vartheta_1)g(\vartheta_1) + f(\vartheta_2)g(\vartheta_2). \tag{2.2}$$

Therefore, for $a, b \in I$ and $t \in [0, 1]$,

$$\begin{aligned}(fg)\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(a^{m_1 t} b^{m_2(1-t)}\right) g\left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\leq [m_1 t f(a) + m_2(1-t) f(a)]\,[m_1 t g(a) + m_2(1-t) g(b)] \\ &= m_1 m_1 t^2 f(a)g(a) + m_1 m_2 t(1-t) f(a)g(b) + m_2 m_1 t(1-t) f(b)g(a) \\ &\quad + m_2 m_2 (1-t)^2 f(b)g(b) \\ &= m_1^2 t^2 f(a)g(a) + m_1 m_2 t(1-t)\,[f(b)g(a) + f(a)g(b)] + m_2^2 (1-t)^2 f(b)g(b).\end{aligned}$$

Using now the inequality (2.2), we obtain,

$$\begin{aligned}(fg)\,(m_1 t a + m_2(1-t)b) &\leq m_1^2 t^2 f(a)g(a) + m_1 m_2 t(1-t)\,[f(a)g(a) + f(b)g(b)] \\ &\quad + m_2^2 (1-t)^2 f(b)g(b) \\ &= m_1 t\,[m_1 t + m_2(1-t)]\, f(a)g(a) + m_2(1-t)\,[m_1 t + m_2(1-t)]\, f(b)g(b).\end{aligned}$$

Since $m_1 t + m_2(1-t) \leq m \leq 1$, where $m = \max\{m_1, m_2\}$. Therefore, we get

$$\begin{aligned}(fg)\,(m_1 t a + m_2(1-t)b) &\leq m_1 t f(a)g(a) + m_2(1-t) f(b)g(b) \\ &= m_1 t\,(fg)\,(a) + m_2(1-t)\,(fg)\,(b).\end{aligned}$$

□

Theorem 2.3. *Let $b > 0$ and $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary family of (m_1, m_2)-geometric arithmetically convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a, b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an (m_1, m_2)-geometric arithmetically convex function on J.*

Proof. Let $t \in [0, 1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned}f\left(a^{m_1 t} b^{m_2(1-t)}\right) &= \sup_\alpha f_\alpha\left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\leq \sup_\alpha\,[m_1 t f_\alpha(a) + m_2(1-t) f_\alpha(b)] \\ &\leq m_1 t \sup_\alpha f_\alpha\,(a) + m_2(1-t) \sup_\alpha f_\alpha\,(b) \\ &= m_1 t f\,(a) + m_2(1-t) f\,(b) < \infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an (m_1, m_2)-GA convex on J. □

Theorem 2.4. *If the function $f : [a^{m_1}, b^{m_2}] \to \mathbb{R}$ is an (m_1, m_2)-GA, then f is bounded on the interval $[a^{m_1}, b^{m_2}]$.*

Proof. Let $M = \max\{m_1 f(a), m_2 f(b)\}$ and $x \in [a^{m_1}, b^{m_2}]$ is an arbitrary point. Then there exist a $t \in [0, 1]$ such that $x = a^{m_1 t} b^{m_2(1-t)}$. Thus, since $m_1 t \leq 1$ and $m_2(1-t) \leq 1$ we have

$$f(x) = f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq m_1 t f(a) + m_2(1-t) f(b) \leq M.$$

Also, for every $x \in [a^{m_1}, b^{m_2}]$ there exist a $\lambda \in \left[\sqrt{\frac{a^{m_1}}{b^{m_2}}}, 1\right]$ such that $x = \lambda\sqrt{a^{m_1} b^{m_2}}$ and $x = \frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}$. Without loss of generality we can suppose $x = \lambda\sqrt{a^{m_1} b^{m_2}}$. So, we get

$$f\left(\sqrt{a^{m_1} b^{m_2}}\right) = f\left(\sqrt{\left[\lambda\sqrt{a^{m_1} b^{m_2}}\right]\left[\frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}\right]}\right) \leq \frac{1}{2}\left[f\,(x) + f\left(\frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}\right)\right].$$

Using M as the upper bound, we get

$$f(x) \geq 2f\left(\sqrt{a^{m_1}b^{m_2}}\right) - f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right) \geq 2f\left(\sqrt{a^{m_1}b^{m_2}}\right) - M = m.$$

□

3. Hermite-Hadamard inequality for (m_1, m_2)-GA-convex function

In this section, we will obtain some inequalities of similar to the H-H type integral inequalities for (m_1, m_2)-GA-convex.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be an (m_1, m_2)-GA-convex function. If $a < b$ and $f \in L[a, b]$, then the following H-H type integral inequalities hold:*

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right) \leq \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du \leq \frac{m_1 f(a)}{2} + \frac{m_2 f(b)}{2}. \tag{3.1}$$

Proof. Firstly, from the property of the (m_1, m_2)-GA convex function of f, we get

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &= f\left(\sqrt{a^{m_1 t}b^{m_2(1-t)}a^{m_1(1-t)}b^{m_2 t}}\right) \\ &\leq \frac{f\left(a^{m_1 t}b^{m_2(1-t)}\right) + f\left(a^{m_1(1-t)}b^{m_2 t}\right)}{2}. \end{aligned}$$

Now, if we take integral in the above inequality with respect to $t \in [0, 1]$, we deduce that

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &\leq \frac{1}{2}\int_0^1 f\left(a^{m_1 t}b^{m_2(1-t)}\right) dt + \frac{1}{2}\int_0^1 \left(a^{m_1(1-t)}b^{m_2 t}\right) dt \\ &= \frac{1}{2}\left[\frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du + \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du\right] \\ &= \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du. \end{aligned}$$

Secondly, from the property of the (m_1, m_2)-GA convex function of f, if the variable is changed as $u = a^{m_1 t}b^{m_2(1-t)}$, then

$$\begin{aligned} \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du &= \int_0^1 f\left(a^{m_1 t}b^{m_2(1-t)}\right) dt \\ &\leq \int_0^1 \left[m_1 t f(a) + m_2(1-t) f(b)\right] dt \\ &= m_1 f(a) \int_0^1 t dt + m_2 f(b) \int_0^1 (1-t) dt \\ &= \frac{m_1 f(a)}{2} + \frac{m_2 f(b)}{2}. \end{aligned}$$

□

4. Some new inequalities for (m_1, m_2)-GA convex functions

The aim of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is (m_1, m_2)-GA convex function. Ji et al. [15] used the following lemma:

Lemma 4.1 ([15]). *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be differentiable function and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then*

$$\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx = \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} f'\left(a^{1-t}b^t\right) dt.$$

Theorem 4.1. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$, then the following integral inequalities hold*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le \frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[b^3 - L\left(a^3, b^3\right)\right], \quad (4.1)$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1 and the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right| = \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right| \le m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|,$$

we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\le\ & \frac{\ln(b/a)}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|dt \\
\le\ & \frac{\ln(b/a)}{2}\int_0^1 a^{3(1-t)}b^{3t}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\right]dt \\
=\ & m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\frac{\ln(b/a)}{2}\int_0^1 (1-t)a^{3(1-t)}b^{3t}dt + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\frac{\ln(b/a)}{2}\int_0^1 t a^{3(1-t)}b^{3t}dt \\
=\ & m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\frac{\ln(b/a)}{2}\left[\frac{b^3 - a^3 - a^3\left(\ln b^3 - \ln a^3\right)}{\left(\ln b^3 - \ln a^3\right)^2}\right] + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\frac{\ln(b/a)}{2}\left[\frac{b^3\left(\ln b^3 - \ln a^3\right) - \left(b^3 - a^3\right)}{\left(\ln b^3 - \ln a^3\right)^2}\right] \\
=\ & \frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[b^3 - L\left(a^3, b^3\right)\right].
\end{aligned}$$

□

Corollary 4.1. *By considering the conditions of Theorem 4.1, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le \frac{m}{6}\left|f'\left(a^{\frac{1}{m}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{1}{6}\left|f'(b)\right|\left[b^3 - L\left(a^3, b^3\right)\right].$$

Corollary 4.2. *By considering the conditions of Theorem 4.1, If we take $m_1 = m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le \frac{|f'(a)|}{6}\left[L\left(a^3, b^3\right) - a^3\right] + \frac{|f'(b)|}{6}\left[b^3 - L\left(a^3, b^3\right)\right].$$

Theorem 4.2. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$ and $q \ge 1$ then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le \frac{\left(b^3 - a^3\right)^{1-\frac{1}{q}}}{6}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^3, b^3\right) - a^3\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^3 - L\left(a^3, b^3\right)\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1, power mean inequality and the (m_1, m_2)-GA convexity of $|f'|^q$ on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right| = \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q \le m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[\int_0^1 a^{3(1-t)}b^{3t}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\int_0^1(1-t)a^{3(1-t)}b^{3t}dt+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\int_0^1 ta^{3(1-t)}b^{3t}dt\right]^{\frac{1}{q}} \\
=\ & \frac{\left(b^3-a^3\right)^{1-\frac{1}{q}}}{6}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^3,b^3\right)-a^3\right)+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^3-L\left(a^3,b^3\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 4.3. *By considering the conditions of Theorem 4.2, If we take $q=1$, then,*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \left[\frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left(L\left(a^3,b^3\right)-a^3\right)+\frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left(b^3-L\left(a^3,b^3\right)\right)\right].$$

This inequality coincides with the inequality (4.1).

Corollary 4.4. *By considering the conditions of Theorem 4.2, If we take $m_1=m$ and $m_2=1$, then,*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\left(b^3-a^3\right)^{1-\frac{1}{q}}}{6}\left[m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q\left(L\left(a^3,b^3\right)-a^3\right)+\left|f'(b)\right|^q\left(b^3-L\left(a^3,b^3\right)\right)\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [15].

Theorem 4.3. *Let the function $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|^q$ is (m_1,m_2)-GA convex on $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$ and $q>1$, then,*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\ln(b/a)}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)A^{\frac{1}{q}}\left(m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q,m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)$$

where L is the logarithmic mean, A is the arithmetic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 4.1, Hölder inequality and the (m_1,m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right|=\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q\leq m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\int_0^1(1-t)dt+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\int_0^1 tdt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)A^{\frac{1}{q}}\left(m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q,m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right).
\end{aligned}$$

$\square$

Corollary 4.5. *By considering the conditions of Theorem 4.3, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \leq \frac{\ln(b/a)}{2} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) A^{\frac{1}{q}}\left(m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q, \left|f'(b)\right|^q\right)$$

Corollary 4.6. *By considering the conditions of Theorem 4.3, If we take $m_1 = m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \leq \frac{\ln(b/a)}{2} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)$$

Theorem 4.4. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$ and $q > 1$, then the following integral inequalities hold*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \leq \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}$$
$$\times\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 4.1, Hölder inequality and the (m_1, m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left(\int_0^1 1 dt\right)^{\frac{1}{p}}\left[\int_0^1 a^{3q(1-t)} b^{3qt}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)q} b^{3tq}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right] dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \int_0^1 (1-t) a^{3q(1-t)} b^{3qt} dt + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \int_0^1 t a^{3q(1-t)} b^{3qt} dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

$\square$

Corollary 4.7. *By considering the conditions of Theorem 4.4, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \leq \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}$$
$$\times\left[m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + \left|f'(b)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.$$

Corollary 4.8. *By considering the conditions of Theorem 4.3, If we take $m_1 = m_2 = 1$, then,*

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \\
\leq\ & \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}\left[\left|f'(a)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + \left|f'(b)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

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