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Research Article

Rota-Baxter bialgebra structures arising from (co-)quasi-idempotent elements

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Abstract

In this note, we construct Rota-Baxter (coalgebras) bialgebras by (co-)quasi-idempotent elements and prove that every finite dimensional Hopf algebra admits nontrivial Rota-Baxter bialgebra structures and tridendriform bialgebra structures. We give all the forms of (co)-quasi-idempotent elements and related structures of tridendriform (co, bi)algebras and Rota-Baxter (co, bi)algebras on the well-known Sweedler's four-dimensional Hopf algebra.

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1. Introduction

Rota-Baxter algebras were introduced in [11] in the context of differential operators on commutative Banach algebras and since [1], intensively studied in probability and combinatorics, and more recently in mathematical physics, such as free Rota-Baxter algebras, Lie algebras, multiple zeta values, differential algebras and Connes-Kreimer renormalization theory in quantum field theory, see ([2–7], etc.). One can refer to the book [2] for the detailed theory of Rota-Baxter algebras.

In 2014, based on the dual method in the Hopf algebra theory, Jian and Zhang in [8] defined the notion of Rota-Baxter coalgebras and also provided various examples of the new object. Then Rota-Baxter bialgebras were presented in [9] whose examples can be constructed from the well-known Radford biproduct. In 2017, Jian construct quasi-idempotent Rota-Baxter operators by quasi-idempotent elements and show that every finite dimensional Hopf algebra admits nontrivial Rota-Baxter algebra structures and tridendriform algebra structures (see [7]).

So it is natural to consider if every finite dimensional Hopf algebra admits nontrivial Rota-Baxter bialgebra structure and tridendriform bialgebra structure. In this paper, we give a positive answer to this question. This is the motivation to write this paper.

This paper is organized as follows. In Section 2, we list some definitions that will be used later. In Section 3, we present the notions of tridendriform coalgebras, tridendriform

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bialgebras, and co-quasi-idempotent element in a coalgebra. We use (co-)quasi-idempotent element to construct Rota-Baxter coalgebras and bialgebras. And then we prove that every finite dimensional Hopf algebra admits nontrivial Rota-Baxter bialgebra structures and tridendriform bialgebra structures. All the forms of (co)-quasi-idempotent elements and related structures of tridendriform (co, bi)algebras and Rota-Baxter (co, bi)algebras on the well-known Sweedler's four-dimensional Hopf algebra are provided in Section 4.

2. Preliminaries

For simplicity, we fix our ground field to be the complex number field $\mathbb{C}$ throughout this paper. All the objects we discuss are defined over $\mathbb{C}$ unless otherwise specified. For an algebra A, we denote its multiplication μ_A (or simply μ) by $\mu_A(a \otimes b) = ab$.

In what follows, we recall some useful definitions which will be used later (see [2,7,9]).

Definition 2.1. For $\lambda \in \mathbb{C}$, a **Rota-Baxter algebra of weight** λ is an associative algebra A together with a linear map $R : A \longrightarrow A$ such that

$$R(a)R(b) = R(aR(b)) + R(R(a)b) + \lambda R(ab) \tag{2. 1}$$

for all $a, b \in A$. Such a linear operator is called a **Rota-Baxter operator of weight** λ on A.

Remark 2.2. If R is a Rota-Baxter operator of weight 1, then λR is a Rota-Baxter operator of weight λ. Conversely, if R is a Rota-Baxter operator of weight λ and λ is invertible, then $\lambda^{-1}R$ is a Rota-Baxter operator of weight 1.

Definition 2.3. Let C be a vector space and $\Delta_C : C \longrightarrow C \otimes C$ (here we use Sweedler's notation and denote $\Delta_C(c)$ by $c_1 \otimes c_2$), $\varepsilon_C : C \longrightarrow \mathbb{C}$ two linear maps. Then C is a coassociative coalgebra if

$$c_{11} \otimes c_{12} \otimes c_2 = c_1 \otimes c_{21} \otimes c_{22} \text{ and } \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = c$$

hold for all $c \in C$.

Let γ be an element in $\mathbb{C}$. A pair (C, Q) is called a **Rota-Baxter coalgebra of weight** γ if C is a coassociative coalgebra and Q is a linear endomorphism of C satisfying that for all $c \in C$,

$$Q(c_1) \otimes Q(c_2) = Q(c)_1 \otimes Q(Q(c)_2) + Q(Q(c)_1) \otimes Q(c)_2 + \gamma Q(c)_1 \otimes Q(c)_2. \tag{2. 2}$$

The map Q is called a **Rota-Baxter operator weight** γ on C.

Remark 2.4. If Q is a Rota-Baxter operator of weight 1, then γQ is a Rota-Baxter operator of weight γ. Conversely, if Q is a Rota-Baxter operator of weight γ and γ is invertible, then $\gamma^{-1}Q$ is a Rota-Baxter operator of weight 1.

Definition 2.5. Let H be a vector space. H is a bialgebra if (H, μ_H) is an associative algebra and (H, Δ_H) is a coassociative coalgebra such that Δ_H and ε_H are algebra maps.

Let λ, γ be elements in $\mathbb{C}$ and H a bialgebra (maybe without unit and counit). A triple (H, R, Q) is called a **Rota-Baxter bialgebra of weight** (λ, γ) if (H, R) is a Rota-Baxter algebra of weight λ and (H, Q) is a Rota-Baxter coalgebra of weight γ.

Remark 2.6. If (H, R, Q) is a Rota-Baxter bialgebra of weight $(1, 1)$, then $(H, \lambda R, \gamma Q)$ is a Rota-Baxter bialgebra of weight (λ, γ). Conversely, if (H, R, Q) is a Rota-Baxter bialgebra of weight (λ, γ) and λ, γ are invertible, then $(H, \lambda^{-1}R, \gamma^{-1}Q)$ is a Rota-Baxter bialgebra of weight $(1, 1)$.

Definition 2.7. Let A be an associative algebra and $\lambda \in \mathbb{C}$. A linear endomorphism ϕ of A is called a **quasi-idempotent operator of weight** λ **on** A if $\phi^2 = -\lambda\phi$. A nonzero element $\xi \in A$ is called a **quasi-idempotent element of weight** λ if $\xi^2 = -\lambda\xi$.

Definition 2.8. Let V be a vector space, and $\prec, \succ, \cdot : V \otimes V \longrightarrow V$ be three linear maps. The quadruple $(V, \prec, \succ, \cdot)$ is called a **tridendriform algebra** if the following conditions are satisfied: for all $x, y, z \in V$,

$$(x \prec y) \prec z = x \prec (y * z), \quad (x \succ y) \prec z = x \succ (y \prec z),$$
$$(x * y) \succ z = x \succ (y \succ z), \quad (x \succ y) \cdot z = x \succ (y \cdot z),$$
$$(x \prec y) \cdot z = x \cdot (y \succ z), \quad (x \cdot y) \prec z = x \cdot (y \prec z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

where $x * y = x \prec y + x \succ y + x \cdot y$.

Remark 2.9. Given a Rota-Baxter algebra (A, R) of weight 1, we define

$$a \prec b = a \cdot R(b), \qquad a \succ b = R(a) \cdot b,$$

for all $a, b \in A$. Then $(V, \prec, \succ, \mu_A)$ is a tridendriform algebra.

3. Construction of tridendriform co(bi)algebra and Rota-Baxter bialgebras

In this section, based on the dual method in Hopf algebra theory, we define tridendriform co(bi)algebras, co-quasi-idempotent elements, then construct tridendriform co(bi)algebras and Rota-Baxter co(bi)algebras through (co-)quasi-idempotent elements.

Definition 3.1. Let V be a vector space, and $\Delta_\prec, \Delta_\succ, \Delta_\cdot : V \longrightarrow V \otimes V$ be three linear maps (write $\Delta_\prec(x) = x^1 \otimes x^2, \Delta_\succ(x) = x^{(1)} \otimes x^{(2)}, \Delta_\cdot(x) = x^{[1]} \otimes x^{[2]}$). The quadruple $(V, \Delta_\prec, \Delta_\succ, \Delta_\cdot)$ is called a **tridendriform coalgebra** if the following conditions are satisfied: for all $x \in V$,

$$x^{11} \otimes x^{12} \otimes x^2 = x^1 \otimes (x^{21} \otimes x^{22} + x^{2(1)} \otimes x^{2(2)} + x^{2[1]} \otimes x^{2[2]}),$$
$$x^{1(1)} \otimes x^{1(2)} \otimes x^2 = x^{(1)} \otimes x^{(2)1} \otimes x^{(2)2},$$
$$(x^{(1)1} \otimes x^{(1)2} + x^{(1)(1)} \otimes x^{(1)(2)} + x^{(1)[1]} \otimes x^{(1)[2]}) \otimes x^{(2)} = x^{(1)} \otimes x^{(2)(1)} \otimes x^{(2)(2)},$$
$$x^{[1](1)} \otimes x^{[1](2)} \otimes x^{[2]} = x^{(1)} \otimes x^{(2)[1]} \otimes x^{(2)[2]},$$
$$x^{[1]1} \otimes x^{[1]2} \otimes x^{[2]} = x^{[1]} \otimes x^{[2](1)} \otimes x^{[2](2)},$$
$$x^{1[1]} \otimes x^{1[2]} \otimes x^2 = x^{[1]} \otimes x^{[2]1} \otimes x^{[2]2},$$
$$x^{[1][1]} \otimes x^{[1][2]} \otimes x^{[2]} = x^{[1]} \otimes x^{[2][1]} \otimes x^{[2][2]}.$$

Rota-Baxter coalgebras are closely related to tridendriform coalgebras.

Lemma 3.2. Given a Rota-Baxter coalgebra (C, Q) of weight 1, we define

$$\Delta_\prec(c) = c_1 \otimes Q(c_2), \qquad \Delta_\succ(c) = Q(c_1) \otimes c_2.$$

Then $(C, \Delta_\prec, \Delta_\succ, \Delta_C)$ is a tridendriform coalgebra.

Proof. It can be proved by direct computation. □

Definition 3.3. Let V be a vector space. A seven-tuple $(V, \prec, \succ, \cdot, \Delta_\prec, \Delta_\succ, \Delta_\cdot)$ is called a **tridendriform bialgebra** if $(V, \prec, \succ, \cdot)$ is a tridendriform algebra and at the same time $(V, \Delta_\prec, \Delta_\succ, \Delta_\cdot)$ is a tridendriform coalgebra.

Proposition 3.4. Let H be a bialgebra and (H, R, Q) a Rota-Baxter bialgebra of weight $(1, 1)$. Define

$$x \prec y = xR(y), \qquad x \succ y = R(x)y,$$
$$\Delta_\prec(x) = x_1 \otimes Q(x_2), \qquad \Delta_\succ(x) = Q(x_1) \otimes x_2,$$

for all $x, y \in H$. Then $(V, \prec, \succ, \mu_H, \Delta_\prec, \Delta_\succ, \Delta_H)$ is a tridendriform bialgebra.

Proof. It is a consequence of Lemma 3.2 and the Remark 2.9. □

Definition 3.5. Let C be a coassociative coalgebra and $\gamma \in \mathbb{C}$. A linear endomorphism ϑ of C is called a **quasi-idempotent operator of weight** γ **on** C if $\vartheta^2 = -\gamma\vartheta$. A nonzero element $\tau \in C^*$ is called a *co-quasi-idempotent element of weight* γ if $\tau(c_1)\tau(c_2) = -\gamma\tau(c)$ for all $c \in C$.

Proposition 3.6. Let C be a coalgebra. Given a co-quasi-idempotent element $\tau \in C^*$ of weight $\gamma \neq 0$. Three linear maps $\Delta_{\prec}, \Delta_{\succ}, \Delta_{\cdot} : C \longrightarrow C \otimes C$ defined below endow a tridendriform coalgebra structure on C: for all $c \in C$,

$$\Delta_{\prec}(c) = \gamma^{-1}c_1 \otimes \tau(c_2)c_3,\ \Delta_{\succ}(c) = \gamma^{-1}\tau(c_1)c_2 \otimes c_3,\ \Delta_{\cdot}(c) = c_1 \otimes c_2.$$

Proof. We only check the first equality in the definition of tridendreform coalgebra as follows. For all $c \in C$, we can get

$$\begin{aligned}
&c^1 \otimes (c^{21} \otimes c^{22} + c^{2(1)} \otimes c^{2(2)} + c^{2[1]} \otimes c^{2[2]})\\
&\quad= \gamma^{-2}c_1\tau(c_2)\tau(c_{32}) \otimes c_{31} \otimes c_{33} + \gamma^{-2}c_1\tau(c_2)\tau(c_{31}) \otimes c_{32} \otimes c_{33}\\
&\qquad+\gamma^{-1}c_1\tau(c_2) \otimes c_{31} \otimes c_{32}\\
&\quad= \gamma^{-2}c_1\tau(c_2)\tau(c_{32}) \otimes c_{31} \otimes c_{33} - \gamma^{-1}c_1\tau(c_2) \otimes c_{31} \otimes c_{32}\\
&\qquad+\gamma^{-1}c_1\tau(c_2) \otimes c_{31} \otimes c_{32}\\
&\quad= \gamma^{-2}c_1\tau(c_2)\tau(c_{32}) \otimes c_{31} \otimes c_{33}\\
&\quad= c^{11} \otimes c^{12} \otimes c^2,
\end{aligned}$$

finishing the proof. □

Theorem 3.7. Let H be a bialgebra. Given a quasi-idempotent element $\xi \in H$ of weight $\lambda \neq 0$ and a co-quasi-idempotent element $\tau \in H^*$ of weight $\gamma \neq 0$. Six linear maps $\prec, \succ, \cdot : H \otimes H \longrightarrow H$ and $\Delta_{\prec}, \Delta_{\succ}, \Delta_{\cdot} : H \longrightarrow H \otimes H$ defined below endow a tridendriform bialgebra structure on H: for all $x, y \in H$,

$$x \prec y = \lambda^{-1}x\xi y,\ x \succ y = \lambda^{-1}\xi xy,\ x \cdot y = xy,$$

and

$$\Delta_{\prec}(x) = \gamma^{-1}x_1 \otimes \tau(x_2)x_3,\ \Delta_{\succ}(x) = \gamma^{-1}\tau(x_1)x_2 \otimes x_3,\ \Delta_{\cdot}(x) = x_1 \otimes x_2.$$

Proof. We can finish the proof by [7, Corollary 2.4] and Proposition 3.6. □

Now we use co-quasi-idempotent elements to construct quasi-idempotent Rota-Baxter operators.

Proposition 3.8. For a fixed co-quasi-idempotent element $\tau \in C^*$ of weight γ, we define linear map $Q_\tau : C \longrightarrow C$ by $Q_\tau(c) = \tau(c_1)c_2$ for any $c \in C$. Then Q_τ is a quasi-idempotent Rota-Baxter operator of weight γ on C.

Proof. It is direct to prove that $Q_\tau^2 = -\gamma Q_\tau$ by the definition of co-quasi-idempotent element. Next for any $c \in C$, we have

$$\begin{aligned}
&Q_\tau(c)_1 \otimes Q_\tau(Q_\tau(c)_2) + Q_\tau(Q_\tau(c)_1) \otimes Q_\tau(c)_2 + \gamma Q_\tau(c)_1 \otimes Q_\tau(c)_2\\
&\quad= \tau(c_1)c_{21} \otimes \tau(c_{221})c_{222} + \tau(c_1)\tau(c_{211})c_{212} \otimes c_{22} + \gamma\tau(c_1)c_{21} \otimes c_{22}\\
&\quad= \tau(c_1)c_{21} \otimes \tau(c_{221})c_{222} - \gamma\tau(c_1)c_{21} \otimes c_{22} + \gamma\tau(c_1)c_{21} \otimes c_{22}\\
&\quad= \tau(c_{11})c_{12} \otimes \tau(c_{21})c_{22}\\
&\quad= Q_\tau(c_1) \otimes Q_\tau(c_2),
\end{aligned}$$

finishing the proof. □

Theorem 3.9. Let H be a bialgebra. Suppose that $\xi \in H$ is a quasi-idempotent of weight of λ and $\tau \in H^*$ is a co-quasi-idempotent element of weight γ, then (H, R_ξ, Q_τ) is a Rota-Baxter bialgebra of weight (λ, γ), where

$$R_\xi(x) = \xi x, \qquad Q_\tau(x) = \tau(x_1)x_2,$$

for all $x \in H$.

Proof. By [7, Prosition 2.2] and Proposition 3.8, we can finish the proof. $\square$

Let recall the following result from [10] on finite dimensional Hopf algebra. As we know, a Hopf algebra H is a bialgebra H with an antipode S, where the linear map $S : H \longrightarrow H$ is the convolution inverse of identity map id_H in convolution algebra $Hom(H, H)$.

Let H be a finite dimensional Hopf algebra. Then there is a unique element x_H such that

$$\langle a^*, x_H \rangle = Tr(l_{a^*}),\ \forall\ a^* \in H^*.$$

Furthermore, the element x_H has the following properties.

$$\varepsilon(x_H) = \dim(H), \qquad x_H^2 = \varepsilon(x_H)x_H.$$

that is to say, $x_H \in H$ is a quasi-idempotent element of weight $-\dim(H)$ on H.

When H is finite dimensional, H^* is also a finite dimensional Hopf algebra and $\dim(H^*)$=$\dim(H)$. So using the above result to finite dimensional Hopf algebra H^*, we can get: there is a unique element $\chi_H \in H^*$ such that

$$\langle \chi_H, a \rangle = Tr(l_a),\ \forall\ a \in H.$$

Furthermore, the element χ_H has the following properties.

$$\varepsilon_{H^*}(\chi_H) = \langle \chi_H, 1_H \rangle = \dim(H), \qquad \chi_H^2 = \varepsilon_{H^*}(\chi_H)\chi_H$$

$$i.e., \chi_H(a_1)\chi_H(a_2) = \langle \chi_H, 1_H \rangle \chi_H(a) = \dim(H)\chi_H(a),$$

that is to say, $\chi \in H^*$ is a co-quasi-idempotent element of weight $-\dim(H)$ on H.

Also we know the integral Λ and cointegral $\bigwedge$ (i.e. integral of H^*) for finite dimensional Hopf algebra H must exist, and Λ is a quasi-idempotent element and $\bigwedge$ is a co-quasi-idempotent element.

By combining the discussions above, we see that R_{x_H}, R_Λ and Q_χ, $Q_{\bigwedge}$ are Rota-Baxter operators on H. As a consequence, we have

Theorem 3.10. Every finite dimensional Hopf algebra admits nontrivial Rota-Baxter coalgebra and bialgebra structures and tridendriform coalgebra and bialgebra structures.

4. An example

The well-known Sweedler's four-dimensional Hopf algebra H_4 is a very popular example in the theory of Hopf algebras, and many researchers pay their attention to it because there are many nice properties on it. In this section, we will apply the above results in Section 3 to H_4, and give all the forms of (co)-quasi-idempotent elements and related structures of tridendriform (co, bi)algebras and Rota-Baxter (co, bi)algebras.

Let H_4 be the algebra generated by two elements x and y subject to

$$x^2 = 1, \quad y^2 = 0, \quad yx = -xy.$$

Then H_4 is a four-dimensional algebra with a linear basis $\{1, x, y, xy\}$ (see [10, 12]), explicitly, its multiplication is

μ_{H_4}	1	x	y	xy
1	1	x	y	xy
x	x	1	xy	y
y	y	$-xy$	0	0
xy	xy	$-y$	0	0

.

Moreover it is a Hopf algebra equipped with the following operations:

$$\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x,$$

$$\varepsilon(x) = 1, \quad \varepsilon(y) = 0,$$

$$S(x) = x, \quad S(y) = xy.$$

Denote by $\{f_1, f_2, f_3, f_4\}$ the dual basis of $\{1, x, y, xy\}$, i.e.,

	1	x	y	xy
f_1	1	0	0	0
f_2	0	1	0	0
f_3	0	0	1	0
f_4	0	0	0	1

Then the multiplication of ${H_4}^*$ is

$\mu_{H_4{}^*}$	f_1	f_2	f_3	f_4
f_1	f_1	0	f_3	0
f_2	0	f_2	0	f_4
f_3	0	f_3	0	0
f_4	f_4	0	0	0

Thus by the definitions of (co-)quasi-idempotent element, we have

	quasi-idempotent element ξ	weight λ
ξ_1	$l_1(1+x) + l_2y + l_3xy$	$-2l_1$
ξ_2	$l_1(1-x) + l_2y + l_3xy$	$-2l_1$
ξ_3	$l_1 1$	$-l_1$

	co-quasi-idempotent element τ	weight γ
τ_1	$k_1f_2 + k_2f_3 + k_3f_4$	$-k_1$
τ_2	$k_1f_1 + k_2f_3 + k_3f_4$	$-k_1$
τ_3	$k_1f_1 + k_1f_2$	$-k_1$
τ_4	$k_1f_3 + k_2f_4$	0

where $k_i, l_j \in \mathbb{C}, i, j = 1, 2, 3$.

Next we assume that $k_1 \neq 0$ and $l_1 \neq 0$.

By [7, Corollary 2.4], if we set $l = (-2l_1)^{-1}$, then the tridendriform algebra structures on H_4 are given by $(H_4, \prec_i, \succ_i, \mu_{H_4}), i = 1, 2, 3$, where

$\prec_1$	1	x	y	xy
1	$l\xi_1$	$l(l_1(1+x) - l_3y - l_2xy)$	$-\frac{1}{2}(y+xy)$	$-\frac{1}{2}(y+xy)$
x	$l(l_1(1+x) + l_3y + l_2xy)$	$l(l_1(1+x) - l_2y - l_3xy)$	$-\frac{1}{2}(y+xy)$	$-\frac{1}{2}(y+xy)$
y	$-\frac{1}{2}(y-xy)$	$-\frac{1}{2}(y-xy)$	0	0
xy	$\frac{1}{2}(y+xy)$	$\frac{1}{2}(y+xy)$	0	0

$\succ_1$	1	x	y	xy
1	$l\xi_1$	$l(l_1(1+x) - l_3y - l_2xy)$	$-\frac{1}{2}(y+xy)$	$-\frac{1}{2}(y+xy)$
x	$l(l_1(1+x) - l_3y - l_2xy)$	$l\xi_1$	$-\frac{1}{2}(y+xy)$	$-\frac{1}{2}(y+xy)$
y	$-\frac{1}{2}(y+xy)$	$\frac{1}{2}(y+xy)$	0	0
xy	$-\frac{1}{2}(y+xy)$	$\frac{1}{2}(y+xy)$	0	0

$\prec_2$	1	x	y	xy
1	$l\xi_2$	$l(l_1(-1+x)-l_3y-l_2xy)$	$-\frac{1}{2}(y-xy)$	$-\frac{1}{2}(-y+xy)$
x	$l(l_1(-1+x)+l_3y+l_2xy)$	$l(l_1(1-x)-l_2y-l_3xy)$	$\frac{1}{2}(y-xy)$	$-\frac{1}{2}(y-xy)$
y	$-\frac{1}{2}(y+xy)$	$\frac{1}{2}(y+xy)$	0	0
xy	$-\frac{1}{2}(y+xy)$	$\frac{1}{2}(y+xy)$	0	0

$\succ_2$	1	x	y	xy
1	$l\xi_2$	$l(l_1(-1+x)-l_3y-l_2xy)$	$-\frac{1}{2}(y-xy)$	$\frac{1}{2}(y-xy)$
x	$l(l_1(-1+x)-l_3y-l_2xy)$	$l\xi_2$	$\frac{1}{2}(y-xy)$	$-\frac{1}{2}(y-xy)$
y	$-\frac{1}{2}(y-xy)$	$-\frac{1}{2}(y-xy)$	0	0
xy	$\frac{1}{2}(y-xy)$	$\frac{1}{2}(y-xy)$	0	0

and $\prec_3=\succ_3=\mu_{H_4}$.

By Proposition 3.6, if we set $k=(-k_1)^{-1}$, then the tridendriform coalgebra structures on H_4 are given by $(H_4,\Delta_{\prec_j},\Delta_{\succ_j},\Delta_{H_4}), j=1,2,3$, where

$$\left|\begin{array}{c|c} \Delta_{\prec_1}(1)=0 & \Delta_{\succ_1}(1)=0 \\ \Delta_{\prec_1}(x)=-x\otimes x & \Delta_{\succ_1}(x)=-x\otimes x \\ \Delta_{\prec_1}(y)=lk_2 1\otimes x-y\otimes x & \Delta_{\succ_1}(y)=lk_2 x\otimes x \\ \Delta_{\prec_1}(xy)=-x\otimes xy+lk_3 x\otimes 1 & \Delta_{\succ_1}(xy)=-x\otimes xy-xy\otimes 1+lk_3 1\otimes 1 \end{array}\right|$$

$$\left|\begin{array}{c|c} \Delta_{\prec_2}(1)=-1\otimes 1 & \Delta_{\succ_2}(1)=-1\otimes 1 \\ \Delta_{\prec_2}(x)=0 & \Delta_{\succ_2}(x)=0 \\ \Delta_{\prec_2}(y)=-1\otimes y+lk_2 1\otimes x & \Delta_{\succ_2}(y)=-1\otimes y-y\otimes x+lk_2 x\otimes x \\ \Delta_{\prec_2}(xy)=lk_3 x\otimes 1-xy\otimes 1 & \Delta_{\succ_2}(xy)=lk_3 1\otimes 1 \end{array}\right|$$

and

$$\begin{aligned} \Delta_{\prec_3}(1)&=\Delta_{\succ_3}(1)=-1\otimes 1, \\ \Delta_{\prec_3}(x)&=\Delta_{\succ_3}(x)=-x\otimes x, \\ \Delta_{\prec_3}(y)&=\Delta_{\succ_3}(y)=-1\otimes y-y\otimes x, \\ \Delta_{\prec_3}(xy)&=\Delta_{\succ_3}(xy)=-x\otimes xy-xy\otimes 1. \end{aligned}$$

With notations above, then by Theorem 3.7, the tridendriform bialgebra structures on H_4 are given by $(H_4,\prec_i,\succ_i,\mu_{H_4},\Delta_{\prec_j},\Delta_{\succ_j},\Delta_{H_4}), i,j=1,2,3$.

By [7, Prosition 2.2], $(H,R_{\xi_i}), i=1,2,3$ are Rota-Baxter algebras of weight $\lambda_i, i=1,2,3$, where $\lambda_1=\lambda_2=-2l_1$, $\lambda_3=-l_1$ and

	R_{ξ_1}	R_{ξ_2}	R_{ξ_3}
1	ξ_1	ξ_2	ξ_3
x	$l_1(1+x)-l_3y-l_2xy$	$l_1(-1+x)-l_3y-l_2xy$	l_1x
y	$l_1(y+xy)$	$l_1(y-xy)$	l_1y
xy	$l_1(y+xy)$	$l_1(-y+xy)$	l_1xy

By Proposition 3.8, $(H,Q_{\tau_j}), j=1,2,3,4$ are Rota-Baxter coalgebras of weight $\gamma_j, j=1,2,3,4$, where $\gamma_1=\gamma_2=\gamma_3=-k_1$, $\gamma_4=0$ and

	Q_{τ_1}	Q_{τ_2}	Q_{τ_3}	Q_{τ_4}
1	0	$k_1 1$	$k_1 1$	0
x	k_1x	0	k_1x	0
y	k_2x	k_1y	k_1y	0
xy	$k_1xy+k_3 1$	$k_3 1$	k_1xy	$k_2 1$

With notations above, then by Theorem 3.9, $(H, R_{\xi_i}, Q_{\tau_j}), i = 1, 2, 3, j = 1, 2, 3, 4$ are Rota-Baxter bialgebras of weight $(\lambda_i, \gamma_j), i = 1, 2, 3, j = 1, 2, 3, 4$.

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