

Blow up for Porous Medium Equations

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Abstract

In various branches of applied sciences, porous medium equations exist where this basic model occurs in a natural fashion. It has been used to model fluid flow, chemical reactions, diffusion or heat transfer, population dynamics, etc.. Nonlinear diffusion equations involving the porous medium equations have also been extensively studied. However, there has not been much research effort in the parabolic problem for porous medium equations with two nonlinear boundary sources in the literature. This paper addresses the following porous medium equations with nonlinear boundary conditions. Firstly, we obtain finite time blow up on the boundary by using the maximum principle and blow up criteria and existence criteria by using steady state of the equation $k_t = k_{xx}^n$, $(x, t) \in (0, L) \times (0, T)$ with $k_x^n(0, t) = k^\alpha(0, t)$, $k_x^n(L, t) = k^\beta(L, t)$, $t \in (0, T)$ and initial function $k(x, 0) = k_0(x)$, $x \in [0, L]$ where $n > 1$, α and β and positive constants.

Keywords: Heat equation; Nonlinear parabolic equation; Nonlinear boundary condition; Blow up; Maximum principles.

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1. Introduction

In this paper, we examine to get blow up properties for the following porous medium equation:

$$\begin{cases} k_t = (k^n)_{xx}, & x \in (0, L), t \in (0, T), \\ (k^n)_x(0, t) = k^\alpha(0, t), & (k^n)_x(L, t) = k^\beta(L, t), t \in (0, T), \\ k(x, 0) = k_0(x), & x \in [0, L], \end{cases} \quad (1.1)$$

where $n > 1$, $\alpha, \beta, L > 0$ and $T \in (0, \infty)$. The initial function $k_0(x)$ is an positive initial function satisfying the boundary conditions

$$(k^n)_x(0, 0) = k^\alpha(0, 0), \quad (k^n)_x(L, 0) = k^\beta(L, 0),$$

Suppose T_β is a finite blow up time and Ψ is a blow up point since $\Psi \in [0, L]$. The solution of (1.1) is said to blow up such that

$$\lim_{t \rightarrow T_\beta^-} \sup\{k(\Psi, t) : 0 \leq x \leq L\} \rightarrow \infty.$$

The porous medium equations appear in different branches of applied sciences where this simple model appears in a natural way. It has been used to model fluid flow, chemical reactions, heat transfer or diffusion, population dynamics, and so on. Nonlinear diffusion equations involving the porous medium equations have been considered comprehensively (cf. the book by Vazquez [10], [2], [3], [4], [7], [9], [11]). Lately, the authors considered to get blow-up properties of various reaction-diffusion equations with various boundary conditions ([2], [3], [4], [6], [7], [8], [9]). For example, Ozalp and Selcuk [8] studied to get blow-up properties in the following the heat equation

$$\begin{cases} k_t = k_{xx}, & x \in (0, L), t \in (0, T), \\ k_x(0, t) = k^\alpha(0, t), & k_x(L, t) = k^\beta(L, t), t \in (0, T), \\ k(x, 0) = k_0(x), & x \in [0, L], \end{cases} \quad (1.2)$$

where $\alpha, \beta, L > 0$ and $T \in (0, \infty)$. They obtained finite time blow up on the right boundary if $k_0(x)$ is a lower solution, $\beta \geq \alpha$ and $\beta > 1$. They given blow up criteria or non-blow up criteria via steady-state. Jiang, et. al. [7] studied to get blow-up properties in the following porous medium equation

$$\begin{cases} k_t = (k^n)_{xx} + k^\alpha, & x \in (0, L), t \in (0, T), \\ (k^n)_x(0, t) = 0, & (k^n)_x(L, t) = k^\beta(L, t), t \in (0, T), \\ k(x, 0) = k_0(x), & x \in [0, L], \end{cases} \quad (1.3)$$

where $L = 1, n > 1, \alpha, \beta > 0$ and $T \in (0, \infty)$. They obtained finite time blow up and blow up rate by using certain conditions.

Nonlinear partial differential equations are applied in the mathematical modelling of the many problems in biology, chemistry, physics and engineering. The porous medium equations with a nonlinear source in heat flow problems have been extensively studied in the literature. However, porous medium equations with two nonlinear sources in the parabolic problems has not received much attention. The existing research on this topic is sparse. This gap in the literature will be tried to be filled in this paper. Motivated by problems (1.2) and (1.3), we deal with the blow up character of problem (1.1). Firstly, we obtain single blow up point is $x = L$ in finite time by using the certain assumptions and maximum principle. Also, we give blow up criteria and existence criteria by using steady state.

2. Blow up Properties

In this section, we examine to get single blow up point for (1.1).

Definition 2.1. A lower solution for the problem (1.1) is called a function ζ that satisfies the following conditions:

$$\begin{cases} \zeta_t \leq (\zeta^n)_{xx}, & (x, t) \in (0, L) \times (0, T), \\ (\zeta^n)_x(0, t) \geq \zeta^\alpha(0, t), & (\zeta^n)_x(L, t) \leq \zeta^\beta(L, t), t \in (0, T), \\ \zeta(x, 0) \leq k_0(x), & x \in [0, L]. \end{cases}$$

It is an upper solution when the reverse inequalities are satisfied.

Lemma 2.1. Suppose $\mu = \mu(x, t, \mu_0)$ and $\varsigma = \varsigma(x, t, \varsigma_0)$ be solutions of (1.1) with $\mu_0 = \mu_0(x)$ and $\varsigma_0 = \varsigma_0(x)$, respectively. If $\mu_0 \leq \varsigma_0 < 1$, then $\mu \leq \varsigma$ on $[0, L] \times [0, T)$.

The proof of Lemma 2.1 is a trivial modification of Theorem 1 in [1].

If we use the transform $k^n = h$, $\alpha_1 = \alpha/n$ and $\beta_1 = \beta/n$ in (1.1), then we get following problem:

$$\begin{cases} h_t = B(h)h_{xx}, & (x, t) \in (0, L) \times (0, T), \\ h_x(0, t) = h^{\alpha_1}(0, t), & h_x(L, t) = h^{\beta_1}(L, t), t \in (0, T), \\ h_0(x) = k_0^n(x), & x \in [0, L]. \end{cases} \quad (2.1)$$

where $B(h) = nh^{(n-1)/n} > 0$ for $h > 0$. Instead of (1.1), we use (2.1) in this section for convenience.

Remark 2.1. It is easily prove that, if $B(h_0)h_0'(x) \geq 0$ in $[0, L]$, then $h_t \geq 0$ in $[0, L] \times (0, T)$. Also, $h_t > 0$ in $(0, L) \times (0, T)$ using strong maximum principle. Also, since $h_x(0, t) = h^{\alpha_1}(0, t) > 0$ and $B(h)h_{xx} = h_t > 0$, $(0, L) \times (0, T)$. For this, h_x is a nondecreasing function and therefore $h_x(x, t) > 0$, $(0, L) \times (0, T)$.

Theorem 2.1. Let $M_0 = \max_{x \in [0, L]} h_0(x)$, $G(\lambda) = \lambda^{\beta_1}$ and ξ is a positive constant. The solution $h(x, t)$ blows up in a finite time $T_\beta \left(\leq \frac{1}{\xi} \frac{M_0^{-\beta_1+1}}{\beta_1-1} \right)$ if and only if

$$\int_{h_0(x)}^{\infty} \frac{d\lambda}{G(\lambda)} < \infty. \quad (2.2)$$

is satisfies, $\beta_1 \geq \alpha_1$ and $\beta_1 > 1$. Also, a blow up rate is given

$$h(x, t) \leq C(T - t)^{1/(-\beta_1+1)},$$

since C is a positive constant and t sufficiently close to T .

Proof. Let $\sigma \in (0, T)$ and ξ is a positive constant. Define the auxiliary function

$$\Delta(x, t) = h_t(x, t) - \xi h^{\beta_1}(x, t),$$

in $[0, L] \times [\sigma, T)$. Thus, $\Delta(x, t)$ supplies

$$\Delta_t - B(h)\Delta_{xx} = (B'(h)/B(h))h_t^2 + B(h)\xi\beta_1(\beta_1 - 1)h^{\beta_1-2}h_x^2 > 0$$

in $(0, L) \times (\sigma, T)$, where $n > 1$ and $\beta_1 > 1$. $\Delta(x, \sigma) \geq 0$ by Remark 2.1, if ξ is sufficiently small. Further, we have

$$\begin{aligned} \Delta_x(0, t) &\leq \alpha_1 h^{\alpha_1-1}(0, t)\Delta(0, t), \\ \Delta_x(L, t) &= \beta_1 h^{\beta_1-1}(L, t)\Delta(L, t), \end{aligned}$$

since $\beta_1 \geq \alpha_1$ and $t \in (\sigma, T)$. So, it is easily see that $\Delta(x, t) \geq 0$ for $(x, t) \in [0, L] \times [\sigma, T)$ using the maximum principle and Hopf's lemma. Therefore, we have

$$h_t(x, t) \geq \xi h^{\beta_1}(x, t),$$

in $(x, t) \in [0, L] \times [\sigma, T)$. Hence, integrating for t from t to T

$$\int_t^T \frac{h_t(x, t)}{h^{\beta_1}(x, t)} dt \geq \xi(T - t).$$

Since x_{β_1} is a blow up point and $\sup_{x_{\beta_1} \in [0, L]} h(x_{\beta_1}, T) \rightarrow \infty$ as $T \rightarrow \infty$ and $M_0 = \max_{x \in [0, L]} h(x, 0)$, then we have

$$\int_{M_0}^{h(x_{\beta_1}, T)} \frac{d\lambda}{G(\lambda)} \geq \xi(T - t),$$

where $G(\lambda) = \lambda^{\beta_1}$. But if assumption (2.2) is satisfied, we have a contradiction. Therefore, it blows up in a finite time

$$T \leq \frac{1}{\xi} \frac{M_0^{-\beta_1+1}}{\beta_1 - 1}.$$

In addition to, we have a blow up rate

$$h(x, t) \leq C(T - t)^{1/(-\beta_1+1)},$$

where $C = (\xi(\beta_1 - 1))^{1/(-\beta_1+1)}$.

Theorem 2.2. If $B(h_0)h_0''(x) \geq 0$ in $[0, L]$ and $\beta_1 > 1$ are satisfies, then the single blow up point is $x = L$.

Proof. Let $d_1 \in [0, L)$, $d_2 \in (d_1, L]$, $\sigma \in (0, T)$ and $\xi > 0$. Define the auxiliary function

$$\Delta(x, t) = h_x - \xi(x - d_1)h^{\beta_1} \text{ in } [d_1, d_2] \times [\sigma, T).$$

Hence, $\Delta(x, t)$ satisfies

$$\Delta_t - B(h)\Delta_{xx} = (B'(h)/B(h))h_x h_t + B(h)(2\xi\beta_1 h^{\beta_1-1}h_x + \xi\beta_1(\beta_1 - 1)(x - d_1)h^{\beta_1-2}h_x^2) > 0$$

in $(d_1, d_2) \times [0, T)$, where $h_x, h_t > 0$, $n > 1$ and $\beta_1 > 1$. $\Delta(x, \sigma) \geq 0$ by Remark 2.1, if ε is sufficiently small. Further

$$\begin{aligned}\Delta(d_1, t) &= h_x(d_1, t) > 0, \\ \Delta(d_2, t) &= h_x(d_2, t) - \xi(d_2 - d_1)h^{\beta_1} > 0,\end{aligned}$$

for $t \in (\sigma, T)$. So, it is easily see that $\Delta(x, t) \geq 0$ for $(x, t) \in [d_1, d_2] \times [0, T)$ using the maximum principle. That is, $h_x \geq \xi(x - d_1)h^{\beta_1}$ for $(x, t) \in [d_1, d_2] \times [\sigma, T)$. Hence, integrating for x from d_1 to d_2 ,

$$h(d_1, t) \leq \left[\frac{\xi(\beta_1 - 1)(d_2 - d_1)^2}{2} \right]^{-\frac{1}{\beta_1 + 1}} < \infty.$$

So h does not blow up in $[0, L)$.

Theorem 2.3. If $\beta_1 > 1$, $B(h_0)h_0''(x) \geq 0$ and $h_x(x, 0) \geq xh^{\beta_1}(x, 0)$ in $[0, L]$ are satisfies, then the single blow up point is $x = L$ and $L \leq 1$.

Proof. Define the auxilary function

$$\Delta(x, t) = h_x - xh^{\beta_1} \text{ in } [0, L] \times [0, T).$$

Thus, $\Delta(x, t)$ supplies

$$\Delta_t - B(h)\Delta_{xx} = (B'(h)/B(h))h_x h_t + B(h)(2\beta_1 h^{\beta_1 - 1} h_x + \beta_1(\beta_1 - 1)xh^{\beta_1 - 2} h_x^2) > 0$$

in $(0, L) \times (0, T)$, where $h_x, h_t > 0$, $n > 1$ and $\beta_1 > 1$. Besides, $\Delta(x, 0) \geq 0$ from $h_x(x, 0) \geq xh^{\beta_1}(x, 0)$ in $(0, L)$ and

$$\begin{aligned}\Delta(0, t) &= h_x(0, t) > 0, \\ \Delta(L, t) &= (1 - L)h^{\beta_1}(L, t) \geq 0,\end{aligned}$$

since $L \leq 1$ and $t \in (0, T)$. So, it is easily see that $\Delta(x, t) \geq 0$ using the maximum principle, namely $h_x \geq xh^{\beta_1}$ for $(x, t) \in [0, L] \times [0, T)$. Hence, integrating for x from x to L ,

$$h(x, t) \leq \left[(\beta_1 - 1) \frac{L^2 - x^2}{2} \right]^{-\frac{1}{\beta_1 + 1}} < \infty.$$

So h does not blow up in $[0, L)$.

Now, we get blow up criteria and existence criteria using steady state. We follow similar process as lemma and theorem in [1]. We define the steady states of the problem (1.1);

$$K_{xx}^n = 0, K_x^n(0) = K^\alpha(0), K_x^n(L) = K^\beta(L). \quad (2.3)$$

If we transform $K^n = H$, $\alpha_1 = \alpha/n$ and $\beta_1 = \beta/n$ in (2.3), we get the following problem

$$H_{xx} = 0, H_x(0) = H^{\alpha_1}(0), H_x(L) = H^{\beta_1}(L), \quad (2.4)$$

we have $H = b + \lambda x$, where

$$\lambda = b^{\alpha_1}, \lambda = (b + \lambda L)^{\beta_1}.$$

For these, the solution of (2.4) is

$$H = b + b^{\alpha_1} x, \quad (2.5)$$

where $b^{\alpha_1} = (b + b^{\alpha_1} L)^{\beta_1}$. It is esaily see that

$$L(b) = b^{-\alpha_1} \left(b^{\alpha_1/\beta_1} - b \right). \quad (2.6)$$

We obtain

$$\lim_{b \rightarrow 0} L(b) = \lim_{b \rightarrow 0} \frac{b^{\alpha_1/\beta_1} - b}{b^{\alpha_1}} = \infty.$$

But, we obtain by using L'Hôpital's rule two times

$$\lim_{b \rightarrow 0} L(b) = \lim_{b \rightarrow 0} \frac{\left(\frac{\alpha_1}{\beta_1}\right) \left(\frac{\alpha_1}{\beta_1} - 1\right) b^{(\alpha_1/\beta_1)-2}}{\alpha_1(\alpha_1 - 1)b^{\alpha_1-2}} = 0$$

for $\alpha_1 \neq 1$ and $\beta_1 \neq 1$. If γ is a positive number, which is very close to 0, then we get $L(\gamma) = 0$ and $L(1) = 0$. Also, If we select $\alpha_1 > \beta_1$, then we note that $L(b) > 0$ for $\gamma < b < 1$. Now, $L'(b) = 0$ implies

$$b = \left[\frac{\beta_1(1 - \alpha_1)}{\alpha_1(1 - \beta_1)} \right]^{\frac{\beta_1}{\alpha_1 - \beta_1}}. \quad (2.7)$$

We denote this value by M . From (2.6),

$$M = \left[\frac{\beta_1(1 - \alpha_1)}{\alpha_1(1 - \beta_1)} \right]^{\frac{\alpha_1(1 - \beta_1)}{\alpha_1 - \beta_1}} - \left[\frac{\beta_1(1 - \alpha_1)}{\alpha_1(1 - \beta_1)} \right]^{\frac{\beta_1(1 - \alpha_1)}{\alpha_1 - \beta_1}}.$$

Lemma 2.2.

(i) If $\beta_1 \geq \alpha_1$, then the steady-state problem (2.4) does not have a positive solution.

(ii) If $\alpha_1 > \beta_1$, then (2.4) has a solution h if and only if $0 < L \leq M$. Furthermore, if $0 < L < M$, then there exist two positive solutions; if $L = M$, then there exists exactly one positive solution.

Proof. (i) For $L(b) > 0$, we have

$$L(b) = b^{-\alpha_1 + \alpha_1/\beta_1} - b^{-\alpha_1 + 1}$$

which is impossible for $\beta_1 \geq \alpha_1$.

(ii) Since $L(\gamma) = 0 = L(1)$ and $L(b) > 0$ for $\gamma < b < 1$, the graph of $L(b)$ is concave downwards with maximum attained at M . Thus for $\alpha_1 > \beta_1$, the problem (2.4) has a solution if and only if $0 < L \leq M$. To each $L \in (0, M)$, there are exactly two values of b . If $L = M$, then b is given by (2.7).

Theorem 2.4.

(i) If $\alpha_1 > \beta_1$ and $L \in (0, M)$, then h exists globally, provided $h_0 \leq H(0)$.

(ii) Suppose that the assumptions of Theorem 2.1 hold. Then, h blows up in a finite time and $x = L$ is the only blow up point. Further, if $h_x(x, 0) \geq xh^{\beta_1}(x, 0)$ in $(0, L)$, then $L \leq 1$.

Proof. (i) By Lemma 2.1, $h \leq H$. Hence h exists globally.

(ii) By Remark 2.1, $h_t > 0$ in $(0, L) \times (0, T)$. If h does not blow up in a finite time, then h converges to H which by Lemma 2.2 (i), does not exist for $\beta_1 \geq \alpha_1$. This contradiction and Theorem 2.1 shows that h blows up in a finite time for $\beta_1 \geq \alpha_1$. Further, from Theorem 2.2, $x = L$ is the single blow up point. Furthermore, from Theorem 2.3, if $h_x(x, 0) \geq xh^{\beta_1}(x, 0)$ in $(0, L)$, then $L \leq 1$. The theorem is proved.

3. Conclusion

The main results in (1.1) are the following;

(i) the solution $h(x, t)$ blows up in a finite time $T_\beta \left(\leq \frac{1}{\xi} \frac{M_0^{-\beta_1+1}}{\beta_1-1} \right)$ since $\beta_1 \geq \alpha_1$ and $\beta_1 > 1$,

(ii) the single blow up point is $x = L$ since since $\beta_1 \geq \alpha_1$, $\beta_1 > 1$ and $B(h_0)h_0''(x) \geq 0$ in $[0, L]$,

(iii) the single blow up point is $x = L$ and $L \leq 1$ since $\beta_1 > 1$, $B(h_0)h_0''(x) \geq 0$ and $h_x(x, 0) \geq xh^{\beta_1}(x, 0)$ in $[0, L]$,

(iv) $h(x, t)$ exists globally since $h_0 \leq H(0)$, $\alpha_1 > \beta_1$ and $L \in (0, M)$ where $M = \left[\frac{\beta_1(1 - \alpha_1)}{\alpha_1(1 - \beta_1)} \right]^{\frac{\alpha_1(1 - \beta_1)}{\alpha_1 - \beta_1}} - \left[\frac{\beta_1(1 - \alpha_1)}{\alpha_1(1 - \beta_1)} \right]^{\frac{\beta_1(1 - \alpha_1)}{\alpha_1 - \beta_1}}$.

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