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Numerical Approximation of Highly Oscillatory Integrals with Weak Singularity

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Abstract. In this paper, we design an accurate scheme for the approximation of highly oscillatory integrals having singularity . The interval of integration $[a, b]$ is divided into two subintervals and then approximate the integral over first interval by hybrid function quadrature $(Q_h f[g])$, while for the approximation of integrals over the second interval we use Levin meshless method $(Q_L^m[g])$. For the result, we find sum of both the integrals. To check the effectiveness of method, results of some test problems are calculated by hybrid function quadrature and compared with the results produces by proposed method.

Keywords: Oscillatory integral · singularity · Levin method · hybrid function.

1 Introduction

Highly oscillatory integrals have many practical applications in physical sciences, particularly in the field of acoustics, optics, mechanics and electromagnetics[1-3]. It is possible to represent approximate solution for many differential equations [4-15] as oscillatory integrals. In this work, we discuss high oscillatory integrals of the type

$$I = \int_a^b g(x)e^{i\omega h(x)}dx, \tag{1}$$

where $g(x)$ is a non-oscillatory function and has a singularity at $a = 0, h(x)$ is phase function also known as oscillator. The parameter is a positive real number which is responsible for the frequency of oscillations, and for $\omega \gg 1$the integrand becomes highly oscillatory. Due to singularity at $x = 0$ and highly oscillatory behavior, the integrand cannot be solved analytically mostly. In such case, we choose numerical methods.Many numerical schemes are developed in last few decades, such as Filon-type method [16-18], Levin-type method [19-21], Generalized quadrature rule [22,23], ClenshawCurtis method [24,25], asymptotic method and steepest descent method [26,27] for numerical solutions of highly oscillatory integrals.

In [28] highly oscillatory integrals with algebraic singularities are expended into asymptotic series by two different types of transformations. Then two methods Filon-type and ClenshawCurtisFilon-type method based on asymptotic series are presented. Some examples and error analysis are discussed to check the accuracy of the proposed methods. In [28] the authors have discussed highly oscillatory integrals with algebraic and logarithmic singularities. Singularities at the endpoints, an internal point only and at the end points and an internal point at a time are discussed. The proposed method is based on expanding the amplitude function into a series of Chebyshev polynomials. Some numerical examples are also discussed. Gauss quadrature and Filon-type method are proposed in [29] for the approximation of highly oscillatory integrals with an algebraic singularity. Error bounds and numerical examples are also discussed in the paper. In [26] an efficient numerical method is proposed for the computation of highly oscillatory Fourier-type integrals with Jacobi-type singularities. The target integrand is transformed by using Cauchys integral theorem and then a Mathematica program is presented for evaluation of integral. The effectiveness of the method is shown through the results of some numerical problems. [30] Proposed meshless procedure for the approximation of highly oscillatory integrals with Bessel kernel. To handle the case of singularity an adoptive splitting technique is introduced and then Haar wavelet and hybrid function based multi-resolution quadrature are used as supporting methods. Asymptotic order of convergence of the proposed method is $o(\omega^{(-7/2)})$ which has been proved theoretically. In current work, we follow the same approach of splitting technique with some modification.

Table 1. Nomenclature box.

Symbols	Description
ω	Frequency parameter
ζ	Splitting parameter
N	Quadrature points
m	Meshless points
$Q_{hf}[g]$	Hybrid function quadrature
$Q_L^m[g]$	Levin meshless method
$Q_h^{mL}[g]$	Levin meshless with Hybrid function

2 A Levin Based Meshless Method ($Q_L^m[g]$)

For n centres $x_1, x_2, \ldots, x_n^\varepsilon$, an RBF interpolation is given by [28]

$$P(x) = \sum_{j=1}^{n} \Psi_j \Theta(\|x - x_j^\varepsilon\|_2, \varepsilon), x \in R^d. \tag{2}$$

The coefficients $\Psi_j, j = 1, 2, \ldots, n$ are chosen by the following interpolation condition

$$P(x_i) = g_i, i = 1, 2, \ldots, m.$$

For n=m this leads to the linear system as,

$$A\Psi = g, \tag{3}$$

which can be solved for the coefficients $\Psi_j, j = 1, 2, , n$. A is square matrix or order n with entries

$$A = \Theta_{ij} = \Theta(\|x - x_j^\varepsilon\|_2, i, j = 1, 2, \ldots, n. \tag{4}$$

For better result we choose maltiquadric radial basis function (MQ RBF) define as following,

$$\Theta(r) = \sqrt{r^2 + \epsilon^2} \qquad \text{, where } \epsilon \text{ is a shape parameter.}$$

Now considering the function P(x) satisfying the following ordinary differential equation (ODE).

$$P'(x) + i\omega h'(x)P(x) = g(x). \tag{5}$$

Using (5) in (1) we get,

$$I = \int_a^b \left[P'(x) + i\omega h'(x)P(x)\right] e^{i\omega h(x)} dx, \tag{6}$$

$$= \int_a^b d\left[P'(x)e^{i\omega h(x)}\right],$$

$$= P(b)e^{i\omega h(b)} - P(a)e^{i\omega h(a)}.$$

3 Hybrid Function for Approximating Integral ($Q_{hf}[g]$)

Hybrid function formula of order 8 developed by [29], for integral of the form

$$\int_a^b g(x)dx, \tag{7}$$

is given by,

$$\int_a^b g(x) \approx \frac{(b-a)}{1935360} \sum_{(k=1)}^{N} [295627 g(a + \frac{(b-a)(16k-15)}{16N}) \tag{8}$$

$$+71329 \cdot g(a + \frac{(b-a)(16k-13)}{16N}) + 471771 \cdot g(a + \frac{(b-a)(16k-11)}{16N})$$
$$+128953 \cdot g(a + \frac{(b-a)(16k-9)}{16N})$$
$$+128953 \cdot g(a + \frac{(b-a)(16k-7)}{16N})$$
$$+471771 \cdot g(a + \frac{(b-a)(16k-5)}{16N})$$
$$+71329 \cdot g(a + \frac{((b-a)(16k-3))}{16N})$$
$$+295627 \cdot g(a + \frac{((b-a)(16k-13))}{16N})]$$

4 Interval Splitting Procedure

Due to singularity at $x = 0$, the integrand (1) cannot be evaluated by $Q_L^m[g]$ method. Therefore we introduce a splitting parameter, $\zeta \in (a, b)$ which divides interval of integration into two subintervals. The splitting parameter ζ can be defined as follows $\zeta = \dfrac{(b-a)}{p}, \quad p > b$.

Using this parameter in (1) as following,

$$\int_a^b g(x)e^{i\omega h(x)}dx = \int_a^\zeta g(x)e^{i\omega h(x)}dx + \int_\zeta^b g(x)e^{i\omega h(x)}dx, \tag{9}$$
$$= I_1 + I_2$$

The integral I_1 can be approximated by $Q_{hf}[g]$ method, as hybrid function quadrature, skip the singularity at $x = 0$. The second integral I_2 is free of singularity and can be evaluated by $Q_L^m[g]$ method and the final result can be calculated as,

$$I = Q_{hf}[g] + Q_L^m[g] \tag{10}$$

Eq (10) is a combination of both methods by $Q_{hf}[g]$ and $Q_L^m[g]$ which can be further denoted by $Q_h^{mL}[g]$, i.e. (1) can be solved by $Q_h^{mL}[g]$.

5 Error analysis

5.1 Error Bound of Hybrid Function Quadrature

Theorem 1 According to [30] error bound for I_1 with phase function $h(x) = x^q$ for $q > 1$, let $h^{'}(a) = h^{''}(a) = \cdots = h^{(q-1)}(a) = 0$ and $h^{(q)}(x) \neq 0$ for all $x \in [a, b]$. Suppose $\epsilon = max_{x\in[a,\zeta]}|h^{'}(x) - h^{'}(a)|$ and ζ satisfy $\epsilon p = b - a$, for $p > b$. Then error for evaluating I_1 by $Q_{hf}[g]$ is given by.

$$Absolute error = |I_1 - Q_{hf}[g]| \leqslant \frac{\sigma(\zeta - a)}{4.54 \times 10^8} \tag{11}$$

$$= \frac{\sigma\left(\left(\dfrac{b-a}{p}\right)^{1/q} - a\right)}{4.54 \times 10^8}, \qquad p > b$$

where σ is a constant independent of ζ.

5.2 Error Bound of Levin Meshless Method

Theorem 2 According to [28], let $g(x)$ and $h(x)$ are smooth functions and $h^{'}(x) \neq 0$ for all $x \in (\zeta, b)$, then error for approximating I_2 by $Q_L^m[g]$ is given by,

$$Absolute error = |I_2 - Q_L^m[g]| \leq \frac{3(1+n)\|\Psi^{(n)}\|_\infty |g(b) - g(\zeta)|^n}{n! \quad \omega^2} \tag{12}$$

$$= o(\frac{1}{\omega^2}), \omega \geq 1$$

Where $\Psi(x) = I_2 - Q_L^m[g]$ with $\Psi(x_i) = 0$, $i = 1, 2, \ldots, n$.

6 Test Problems and Discussion

In this section we discuss some test problems, results are given in term of absolute error for different meshless points m and different frequencies . The results are also produced by hybrid function quadrature $Q_{hf}[g]$ and compared with the results evaluated by the proposed method $Q_h^{mL}[g]$. To calculate the absolute error, the result of imaginary part is considered in this paper. Exact solutions are calculated by Maple 16 and numerical evaluations are done by Matlab 15.

Test problem 1. Consider the following oscillatory integral,

$$I_1[g, \omega] = \int_0^1 \frac{1}{\sqrt{x}} e^{i\omega x^3} dx. \tag{13}$$

Oscillatory integral $I_1[g, \omega]$ is approximated by $Q_{hf}[g]$ and $Q_h^{mL}[g]$. A comparison of absolute error produced by both methods is shown in Fig. 1, which shows

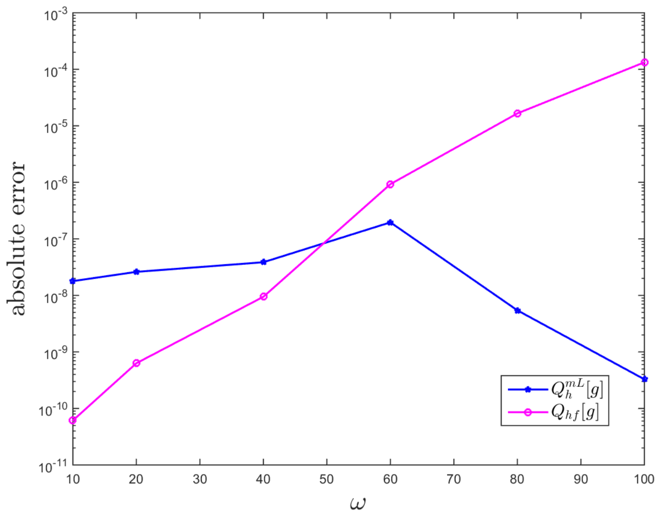

Fig. 1. Comparison of absolute error produced by $Q_{hf}[g]$ and $Q_h^{mL}[g]$ for $I_1[g,\omega]$.

that the method $Q_{hf}[g]$ gives high accuracy for low frequency but absolute error increases with increase in ω. While absolute error produced by the proposed method decreases as increases. CPU time of the proposed method for this problem is shown in Fig. 2. Absolute error is evaluated by the proposed method $Q_h^m L[g]$ for fixed $N = 40, p = 2, 5$ and $m = 5, 10, 15, 20$ and results are given in Tables 2 and 3. From Tables, it is clear that absolute error decreases as ω and m increases.

Table 2. The absolute error produced by $Q_h^{mL}[g]$ for different $m, p = 2$ and fixed $N = 40$, for $I_1[g,\omega]$.

ω	m=5	m =10	m =15	m =20
10	4.37e-4	4.96e-6	5.09e-8	1.79e-8
20	4.36e-5	1.05e-6	2.39e-8	2.61e-8
40	1.47e-4	4.26e-6	3.41e-8	3.86e-8
60	3.70e-5	4.51e-6	1.21e-7	1.96e-7
80	2.32e-5	4.76e-6	3.64e-8	5.44e-9
100	5.75e-5	3.50e-6	7.71e-8	3.28e-10

Table 3. The absolute error produced by $Q_h^{mL}[g]$ for different $m, p = 5$ and fixed $N = 40$, for $I_1[g, \omega]$.

ω	m =5	m =10	m =15	m =20
10	1.30e-2	2.7e-4	1.81e-6	2.36e-7
20	1.73e-2	4.56e-4	8.69e-6	2.89e-8
40	5.05e-3	1.18e-4	1.29e-5	7.88e-7
60	2.47e-2	1.90e-4	7.62e-6	1.95e-8
80	2.87e-2	3.29e-4	2.92e-6	8.16e-7
100	2.35e-2	3.33e-4	8.31e-6	8.63e-7

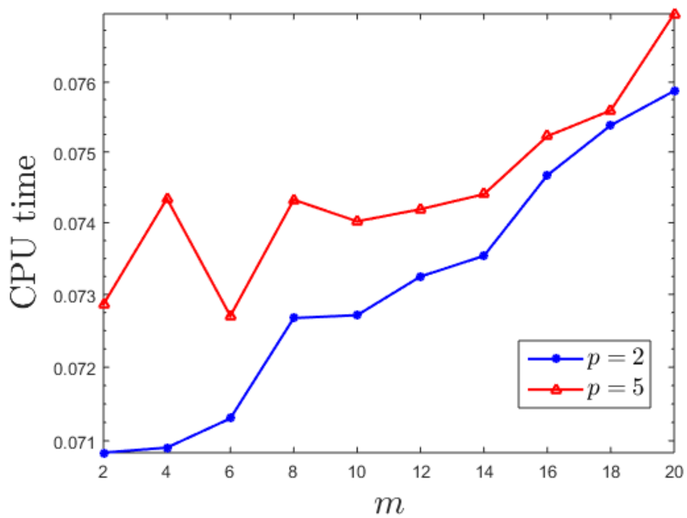

Fig. 2. CPU time (in seconds) of $Q_h^{mL}[g]$ for $I_1[g, \omega]$.

Test problem 2. Consider the following oscillatory integral,

$$I_2[g, \omega] = \int_0^1 \frac{\sqrt{1+x}}{x^0.2} e^{i\omega x^2} dx \tag{14}$$

$I_2[g, \omega]$ is evaluated by $Q_{hf}[g]$ and $Q_h^{mL}[g]$ methods and results in term of absolute error are compared, which are shown in Fig. 3.

From Fig. 3 it is clear that absolute error increases with increase in ω for method $Q_{hf}[g]$, while absolute error decreases as ω increases for the proposed method $Q_h^{mL}[g]$. Absolute error is evaluated by the proposed method $Q_h^{mL}[g]$ for fixed $N = 40, p = 2, 5$ and $m = 5, 10, 15, 20$ and results are given in Tables 4 and 5.

Table 4. The absolute error produced by $Q_h^{mL}[g]$ for different $m, p = 2$ and fixed $N = 40$, for $I_2[g, \omega]$.

ω	m=5	m =10	m =15	m =20
10	9.45e-5	1.41e-6	1.25e-8	2.63e-10
20	2.73e-5	1.61e-6	1.51e-9	5.07e-9
40	2.37e-5	1.41e-6	8.81e-9	5.52e-10
60	2.82e-5	8.94e-7	1.04e-8	5.42e-9
80	1.02e-5	6.07e-8	3.48e-9	8.18e-9
100	5.42e-6	3.89e-7	5.09e-9	3.80e-9

Table 5. The absolute error produced by $Q_h^{mL}[g]$ for different $m, p = 5$ and fixed $N = 40$, for $I_2[g, \omega]$.

ω	m=5	m =10	m =15	m =20
10	4.43e-4	7.88e-5	1.79e-6	6.2e-8
20	8.29e-3	8.33e-5	1.76e-5	1.31e-7
40	4.25e-4	3.19e-4	2.22e-5	4.76e-7
60	2.67e-3	5.83e-6	1.47e-5	7.87e-7
80	1.51e-3	2.98e-4	8.56e-6	4.27e-7
100	3.17e-5	1.63e-4	2.82e-5	1.95e-6

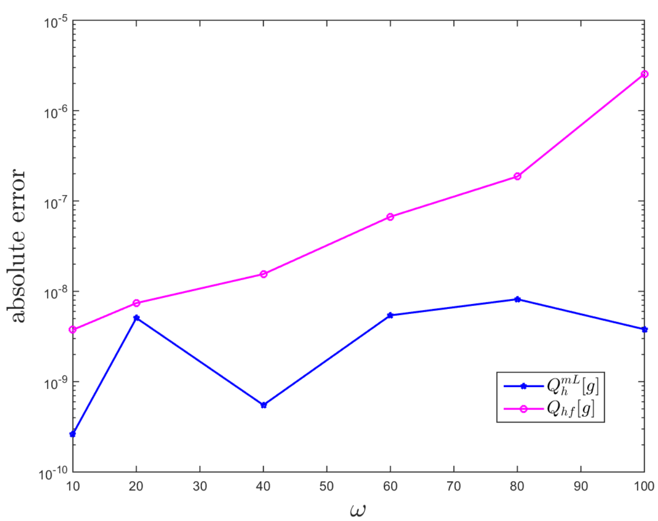

Fig. 3. Comparison of absolute error produced by $Q_{hf}[g]$and $Q_h^{mL}[g]$ for $I_2[g, \omega]$.

From Tables, it is clear that absolute error increases some time with an increase in , but this is due to the oscillatory behavior of the integrand. Considering

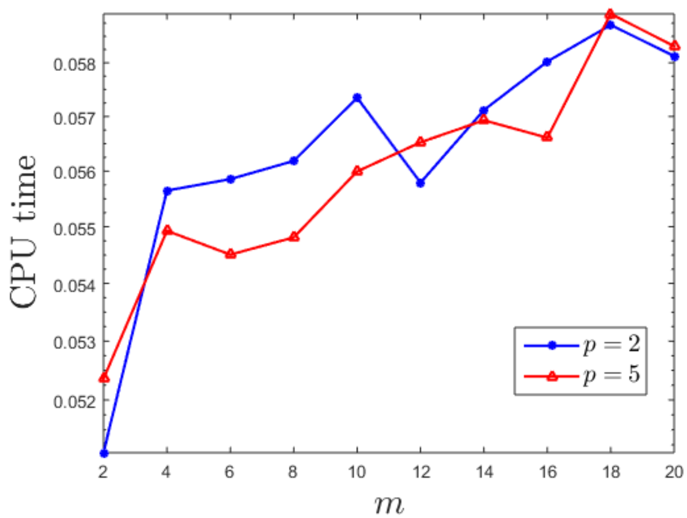

Fig. 4. PU time (in seconds) of $Q_h^{mL}[g]$ for $I_2[g, \omega]$.

the overall results of Tables 4 and 5 we see that the proposed method $Q_h^{mL}[g]$ is effective and accurate versus $Q_{hf}[g]$ method. CPU time (in seconds) is shown in Fig. 4 for proposed method.

7 Conclusion

In this paper, we have combined two methods $Q_{hf}[g]$ and $Q_L^m[g]$ to propose a new effective and accurate method $Q_h^m L[g]$. Integrals of the form (1) cannot be approximated by $Q_L^m[g]$ due to singularity at $x = 0$, and method $Q_{hf}[g]$ is not accurate for high frequency integrals. Comparison of results produced by both methods $Q_h f[g]$ and $Q_h^{mL}[g]$ confirmed the efficiency of the proposed method. In future work the proposed method can be extended to the integrals having oscillatory kernel with singularities.

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