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Existence and uniqueness of solutions for Steklov problem with variable exponent

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Abstract

In this article, we give some results on the existence and uniqueness of solutions concerned a class of elliptic problems involving $p(x)-$Laplacian with Steklov boundary condition. We give also some sufficient conditions to assure the existence of a positive solution.

Keywords: Uniqueness of solution, $p(x)$âĹŠLaplacian operator, Boundary value problem.
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1. Introduction

In this work, we are concerned with the following elliptic problem

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2} u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= f(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p \in C_+(\overline{\Omega})$ where $C_+(\overline{\Omega}) := \Big\{ p \in C(\overline{\Omega}) : p^- := \inf_{x \in \overline{\Omega}} p(x) > 1 \Big\}$, $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ denotes the $p(x)$-Laplace operator, ν is the outward normal vector on $\partial\Omega$ and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

For the function f, we assume the following conditions:

(H_0) $f \in C(\overline{\Omega} \times \mathbb{R})$, and satisfies

$$|f(x,u)| \leq C(|u|^{r(x)} + 1)$$

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for all $x \in \partial\Omega$, where

$$1 \leq r(x) < p(x).$$

(H_1) $f(x,u) = o(|u|^{p(x)-1})$,

(H_2) $f \in C(\overline{\Omega} \times \mathbb{R})$, is nonincreasing with respect to the second variable, for all $x \in \overline{\Omega}$.

(H_3) $\limsup_{|t|\to+\infty} \frac{p(x)F(x,t)}{|\, t\,|^{p(x)}} < \lambda_1$, uniformly for a.e $x \in \Omega$, with

$$\lambda_1 = \inf_{u\in W^{1,p(x)}(\Omega), u\neq 0} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx}{\int_\Omega \frac{1}{p(x)}|u|^{(x)}\,dx} > 0.$$

The study of differential equations and variational problems involving $p(x)-$ growth conditions has attracted a special interest. This attention reflects directly into a various range of applications. There are applications concerning elastic materials, image restoration (see [10, 23]), thermorheological and electrorheological fluids (see [1, 21]) and also mathematical biology [15].

Problems like (1.1) have been extensively considered in the literature in the recent years. In [11], the authors have studied the case $f(x,u) = \lambda|u|^{p(x)-2}u$.

When $p(.) = p$, the case $f(x,t) = \lambda m(x)|u|^{p-2}u$ was considered by [7, 9, 22].

The existence and multiplicity for (1.1) have been studied by many authors we just quote [3, 4, 5, 6, 7, 8, 9, 11, 16, 17, 20]. However, very little is dealt with the uniqueness for (1.1).

When $f(x,u)$ is independent of the second variable u, the uniqueness for (1.1) holds easily, accurately for the case p is constant. When $f(x,u)$ depends on the second variable and it is nonincreasing in u, the uniqueness for (1.1) can be reached (see [2] for example).

Our purpose is to give results on the existence of solutions under the typical growth condition in which we will deal with the existence of constant sign solutions, in fact, since the non-homogeneity of the $p(x)-$Laplacian. A distinguish feature is that we consider the variable exponent problem when the nonlinear term f neither satisfy *sub* p^-growth condition, nor satisfy *sup* p^+growth condition, for instance, when the nonlinearity satisfies the following typical growth condition

$$0 < \liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^{\alpha(x)}} \leq \limsup_{|u|\to\infty} \frac{F(x,u)}{|u|^{\alpha(x)}} < \infty, uniformely \;\; in \;\; \Omega,$$

with $p^- \leq \alpha(x) \leq p^+$

On the other side, we prove the uniqueness for (1.1) which seems rarely studied for those problems.

So, we state our main results as form as three theorems.

Theorem 1.1. *Assume that* $p \in (1,N)$, (H_0) *and* (H_1) *hold. If* f *satisfies*

(H_0') *There exists* $u \geq 0$, $u^* \in W^{1,p(x)}(\Omega)$ *such that* $\phi(u^*) < 0$, *where*

$$\phi(u^*) = \int_\Omega \frac{1}{p(x)}|\nabla u^*|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u^*|^{p(x)}\,dx - \int_\Omega F(x,u^*)dx.$$

Then (1.1) *has a nontrivial solution.*

Theorem 1.2. *Suppose that the following assumption hold:*

(H_2') $f(x,0) \geq 0$ with $f(x,0) \not\equiv 0$ for all $x \in \overline{\Omega}$.

Under the condition (H_2), *the problem* (1.1) *has a unique solution which is nontrivial.*

Theorem 1.3. *Under the assumptions* (H_0) *and* (H_3), *the problem* (1.1) *has a nontrivial solution in* $W^{1,p(x)}(\Omega)$. *Further, if* f *verifies*

$$\Big(f(x,t) - f(x,s)\Big)\Big(t-s\Big) \leq 0, \;\; \forall\; s,t \in \mathbb{R}, \tag{2}$$

then such solution is unique.

The rest of this paper is organized as follows. In Section 2, we recall some preliminaries on variable exponent spaces. In Section 3, by the Galerkin method and other approaches, we shall establish the results of existence and uniqueness of a solution for problem (1.1).

2. Preliminaries

For the integrity of the paper, we recall some basic facts which will be used later. For more detail we refer to [13, 14, 19]. For $p \in C_+(\overline{\Omega})$, we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\}$$

equipped with the so called Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\Big\}.$$

Proposition 2.1. *If $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a carathéodory function and satisfies $|f(x,t)| \le a(x)+b|t|^{\frac{p_1(x)}{p_2(x)}}$ for any $(x,t) \in \overline{\Omega}\times\mathbb{R}$, where $p_i \in C_+(\overline{\Omega}, i = 1,2,\ a \in L^{p_2(x)}(\overline{\Omega}), a(x) \ge 0$ and $b \ge 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\overline{\Omega})$ to $L^{p_2(x)}(\overline{\Omega})$ defined by $N_f(u)(x) = f(x,u(x))$ is a continuous and bounded operator.*

As in the constant exponent case, the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\},$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

With such norms, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2. *Let $\rho(u) = \int_\Omega \Big(|\nabla u|^{p(x)} + |u|^{p(x)}\Big)dx$. For $u, u_n \in W^{1,p(x)}(\Omega), n = 1,2,...,$ we have*

1. $\rho\Big(u\Big/|u|_{p(x)}\Big) = 1$.
2. $\|u\| < 1 (= 1, > 1) \iff \rho(u) < 1 (= 1 > 1)$.
3. $\|u\| < 1 \Longrightarrow \|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$.
4. $\|u\| > 1 \Longrightarrow \|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$.
5. *Then the following statements are equivalent each other:*
   (a) $\lim_{n\to\infty} \|u_n - u\| = 0$.
   (b) $\lim_{n\to\infty} \rho(u_n - u) = 0$.
   (c) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \Phi(u)$.

Let $a : \partial\Omega \to \mathbb{R}$ be a measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) = \Big\{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |a(x)||u(x)|^{p(x)}d\sigma < +\infty\Big\}$$

with the norm

$$|u|_{p(x),a(x)} = \inf\Big\{\tau > 0 : \int_{\partial\Omega} |a(x)||\frac{u(x)}{\tau}|^{p(x)}d\sigma \le 1\Big\}.$$

Then, $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space.

In particular, when $a(x) \equiv 1$ on $\partial\Omega$, $L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$ and $|u|_{p(x),a(x)} = |u|_{p(x),\partial\Omega}$.

Define

$$p^\partial(x) = \frac{(N-1)p(x)}{Np(x)}$$

and

$$p^\partial_{r(x)}(x) := \frac{r(x)-1}{r(x)}p^\partial(x),$$

where $x \in \partial\Omega$, $r \in C(\partial\Omega)$ with $r^-(\partial\Omega) > 1$.

Recall the following embedding theorem.

Theorem 2.1 ([11, Theorem 2.1]). *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$ *with* $p^- > 1$. *Suppose that* $a \in L^{r(x)}(\partial\Omega), r \in C(\partial\Omega)$ *with* $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ *for all* $x \in \partial\Omega$. *If* $q \in C(\partial\Omega)$ *and*

$$1 \leq q(x) < p^\partial_{r(x)}(x), \quad \forall x \in \partial\Omega.$$

Then, there exists a compact embedding $W^{1,p(x)}(\partial\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$. *In particular, there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\partial\Omega)$ *where* $1 \leq q_0(x) < p^\partial(x), \forall x \in \partial\Omega$.

Let us recall the following interesting results:

Proposition 2.3. *Let* X *a Banach space. If* $J \in C^1(X, \mathbb{R})$ *is bounded from below and satisfies (PS) condition, then* $c = \inf_X J$ *is a critical value of* J.

Lemma 2.1. *([18])) Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *be a continuous function with* $\langle F(x), x\rangle \geq 0$, *for all* x *verifying* $|x| = \rho > 0$, *where* $\langle .,.\rangle$ *is the usual inner product of* $\mathbb{R}^N$. *Then there exists* $\gamma \in B_\rho(0)$ *such that* $F(\gamma) = 0$.

Proof of Theorem 1.1*:* Define

$$f^+(x,u) = \begin{cases} f(x,u) \ \ if \ \ u \geq 0, \\ -f(x,-u) \ \ if \ \ u \leq 0, \end{cases}$$

$F^+(x,u) = \int_0^u f^+(x,t)\,dt$,
$\phi^+(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - \int_{\partial\Omega} F^+(x,u)\,dx, \quad \forall u \in W^{1,p(x)}(\Omega)$.
In view of the assumption (H_1), we have that

$$\phi^+(u) \geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - \varepsilon\int_{\partial\Omega}|u|^{p(x)}\,dx - C_1|\partial\Omega|,$$

then ϕ^+ is coercive and even, so its minimizer can be achieved at certain point $v \geq 0$.
From (H_0'), we get that $\phi^+(u^*) < 0$, thereby, $\phi^+(v) < 0$ and then $v > 0$.
On other hand, denote

$$\widetilde{f}^+(x,u) = \begin{cases} f^+(x,v) \ \ if \ \ u > v, \\ f(x,u) \ \ if \ \ 0 \leq u \leq v, \\ f(x,v) \ \ if \ \ u < 0. \end{cases}$$

In virtue of (H_0) and (H_1) we have that

$$|\widetilde{F}^+(x,u)| \leq \varepsilon|u|^{p(x)} + C(\varepsilon)|u|^{p^\partial(x)},$$

where $\widetilde{F}^+(x,u) = \int_0^u \widetilde{f}^+(x,t)\,dt$.
Let set

$$\widetilde{\phi}^+(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - \int_{\partial\Omega}\widetilde{F}^+(x,u)\,dx, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Accordingly , for $\|u\| > 1$, it is easy to see that

$$\widetilde{\phi}^+(u) \geq \frac{1}{2p^+}\|u\|^{p^-} - C(\varepsilon)\int_{\partial\Omega}|u|^{p^\partial(x)}\,dx. \tag{3}$$

There exists Ω_l such that

$$p_l^+ < p\partial^- = p_l\partial^- = \min\{ess\inf_{\overline{\Omega}}(p_1\partial),\ ess\inf_{\overline{\Omega}}(p_2\partial),\ ...,\ ess\inf_{\overline{\Omega}}(p_k\partial)\},$$

where $\overline{\Omega} = \bigcup_{l=1}^{k} \overline{\Omega_l}$.
Thus, for $\|u\| < 1$, we obtain

$$\widetilde{\phi}^{+}(u) \geq \frac{1}{2p^{+}}\|u\|^{p^{-}} - C_4\|u\|^{p\partial^{-}},\ \forall u \in W^{1,p(x)}(\Omega).$$

Since $p^{+} < p\partial^{-}$, the function $t \mapsto (\frac{1}{2p^{+}} - C_4 t^{p\partial^{-}-p^{+}})$ is strictly positive in a neighborhood of zero. Thereby, we may find positive constants $\alpha > 0$ and $\beta > 0$ such that

$$\widetilde{\phi}^{+}(u) \geq \beta,\ \forall u \in W^{1,p(x)}(\Omega),\quad for\ \ \|u\| = \alpha.$$

Now, we are ready to claim that $\widetilde{\phi}^{+}$ verifies (PS), the Palais Smale condition. In fact, let (u_n) be a sequence in $W^{1,p(x)}(\Omega),\ \|.\|)$ such that $\widetilde{\phi}^{+}(u_n)$ is bounded and $(\widetilde{\phi}^{+})'(u_n) \to 0$ in $(W^{1,p(x)}(\Omega))^{*}$.

Because the operator $u \to \int_{\partial\Omega} \widetilde{F}^{+}(x,u)\, d\sigma_x$ is weakly continuous and its derivative is compact, then we only prove that (u_n) is bounded in $W^{1,p(x)}(\Omega)$. Otherwise, we may find a subsequence still denoted by (u_n) such that $\|u_n\| \to \infty$ when $n \to \infty$. From the coercivity, $\widetilde{\phi}^{+}(u_n) \to \infty$, which is a contradiction with the fact that $\widetilde{\phi}^{+}(u_n)$ is bounded. By regarding Proposition 2.3, it follows that $\widetilde{\phi}^{+}$ has a critical point ω which is nontrivial (because $\widetilde{\phi}^{+}(\omega) > 0 = \widetilde{\phi}^{+}(0)$). What means that ω is a nontrivial solution for the following problem

$$\begin{aligned} \Delta_{p(x)}u &= |u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= \widetilde{f}^{+}(x,u) \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

Next, we check that $0 \leq \omega \leq v$. Because v (resp.ω) is a critical point of ϕ^{+} (resp. $\widetilde{\phi}^{+}$), thus

$$\begin{aligned} 0 &= \Big(\widetilde{\phi}^{+'}(\omega) - \phi^{+'}(v)\Big)(\omega - v)^{+} \\ &= \int_{\Omega}\Big(|\nabla\omega|^{p(x)-2}\nabla\omega - |\nabla v|^{p(x)-2}\nabla v\Big)\nabla(\omega - v)^{+} + \int_{\Omega}\Big(|\omega|^{p(x)-2}\omega - |v|^{p(x)-2}v\Big)(\omega - v)^{+} \\ &- \int_{\partial\Omega}(\widetilde{f}^{+}(x,\omega) - f^{+}(x,v))(\omega - v)^{+}\, dx \\ &= \int_{\Omega}\Big(|\nabla\omega|^{p(x)-2}\nabla\omega - |\nabla v|^{p(x)-2}\nabla v\Big)\nabla(\omega - v)^{+} + \int_{\Omega}\Big(|\omega|^{p(x)-2}\omega - |v|^{p(x)-2}v\Big)(\omega - v)^{+} \\ &- \int_{\partial\Omega}(f^{+}(x,v) - f^{+}(x,v))(\omega - v)^{+}\, dx \ \ (\text{"} = 0\text{"} \quad \text{from} \quad \text{the} \quad \text{definition} \quad \text{of} \ \ \widetilde{f}^{+}) \\ &= \int_{\Omega}\Big(|\nabla\omega|^{p(x)-2}\nabla\omega - |\nabla v|^{p(x)-2}\nabla v\Big)\nabla(\omega - v)^{+} + \int_{\Omega}\Big(|\omega|^{p(x)-2}\omega - |v|^{p(x)-2}v\Big)(\omega - v)^{+}, \end{aligned}$$

where $(\omega - v)^{+} = \max\{(\omega - v), 0\}$.
We conclude that $\omega \leq v\ \ a.e\ in\ \Omega$.

Meanwhile, we have

$$\begin{aligned} 0 &= \widetilde{\phi}^{+'}(\omega).\omega^{-} \\ &= \int_{\Omega}|\nabla\omega|^{p(x)-2}\nabla\omega.\nabla\omega^{-}\, dx + \int_{\Omega}|\omega|^{p(x)-2}\omega\omega^{-}\, dx - \int_{\partial\Omega} f(x,\omega)\omega^{-}\, d\sigma, \\ &= \int_{\Omega}|\nabla\omega|^{p(x)-2}\nabla\omega\nabla\omega^{-}\, dx + \int_{\Omega}|\omega|^{p(x)-2}\omega\omega^{-}\, dx. \end{aligned} \tag{5}$$

Hence $\omega^{-} = 0\ \ a.e\ \ in\ \Omega$, so $\omega \geq 0$.
Then ω is a solution of (1.1) and the proof is achieved. □

Proof of Theorem 1.2. :
a) **Existence:** By (H_2) there exists a constant m such that

$$f(x,0) \leq m, \ \forall x \in \partial\Omega.$$

Then

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2} u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= m \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

has unique L^∞ solution u_1 which is nonegative (see [12]and [20]).

Denote

$$\widetilde{f}(x,u) = \begin{cases} f(x,0) & if \quad u < 0, \\ f(x,u) & if \quad 0 \leq u \leq u_1, \\ f(x,u_1) & if \quad u > u_1 \end{cases}$$

Hence, $-\infty < \widetilde{f}(x,u) \leq m, \ \forall x \in \partial\Omega$ and $u \in \mathbb{R}$. Thus,

$$|\widetilde{F}(x,u)| \leq K|u|, \quad for \ \ x \in \partial\Omega,$$

where $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,s)\, ds$.

Let us consider

$$\psi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx - \int_{\partial\Omega} \widetilde{F}(x,u)\, dx,$$

for $u \in W^{1,p(x)}(\Omega)$.

A standard argument shows that $\psi \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$, since $p^- > 1$ and $\widetilde{f}$ is bounded.

When $\|u\| > 1$ we have

$$\psi(u) \geq \frac{1}{p^+} \|u\|^{p^-} - K_1 \|u\|,$$

with K_1 is a positive constant. Then ψ is coercive, and since it is sequentially weakly lower continuous, we conclude that ψ has a global minimizer $\widetilde{u} \in W^{1,p(x)}(\Omega)$ s.t $\psi'(\widetilde{u}) = 0$. Thereby, $\widetilde{u}$ verifies

$$\int_\Omega \left(|\nabla \widetilde{u}|^{p(x)-2} \nabla \widetilde{u}.\nabla v + |\widetilde{u}|^{p(x)-2} \widetilde{u} v \right) dx - \int_{\partial\Omega} \widetilde{f}(x,\widetilde{u}) v\, dx,$$

for all $v \in W^{1,p(x)}(\Omega)$.
Taking $\widetilde{u}^-$ as test function and keeping in mind that $\widetilde{f}(x,u) = f(x,0)$, for $u < 0$, then we have

$$\int_\Omega \left(|\nabla \widetilde{u}^-|^{p(x)} + |\widetilde{u}^-|^{p(x)} \right) dx - \int_{\partial\Omega} \widetilde{f}(x,\widetilde{u}^-) \widetilde{u}^-\, dx = 0$$

As we have $\widetilde{f}(x,\widetilde{u}^-)\widetilde{u}^- \leq 0$, so we get

$$\int_\Omega \left(|\nabla \widetilde{u}^-|^{p(x)} + |\widetilde{u}^-|^{p(x)} \right) dx \leq 0,$$

which implies that $\widetilde{u}^- = 0$ and then $\widetilde{u} \geq 0$.

Meanwhile, $\widetilde{f}(x,\widetilde{u}) \leq m$, according to comparison principle see shao gao deng..., we have $\widetilde{u} \leq u_1$. Accordingly,

$$\widetilde{f}(x,\widetilde{u}) = f(x,\widetilde{u})$$

and then $\widetilde{u}$ is a solution of (1.1), which is nontrivial because $f(x,0) \not\equiv 0$.

b) **Uniqueness:**

Let recall the following formulas:

$\forall x, y \in \mathbb{R}^N$

$$| x - y |^\gamma \leq 2^\gamma (| x |^{\gamma-2} x - | y |^{\gamma-2} y).(x - y) \quad if \quad \gamma \geq 2,$$

$$| x - y |^2 \leq \frac{1}{\gamma - 1}(| x | + | y |)^{2-\gamma}(| x |^{\gamma-2} x - | y |^{\gamma-2} y).(x - y) \quad if \quad 1 < \gamma < 2,$$

where $x.y$ is the inner product in $\mathbb{R}^N$.

Let u and v two solutions of (1.1), viewing the last inequalities, we have

$$\begin{aligned} 0 \quad &\leq \int_{[u>v]} \Big|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\Big)\Big(\nabla u - \nabla v\Big)\, dx + \\ &\int_{[u>v]} \Big|u|^{p(x)-2}u - |v|^{p(x)-2}v\Big)\Big(u - v\Big)\, dx \\ &\leq \int_\Omega \Big|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\Big)\nabla(u - v)^+\, dx + \\ &\int_\Omega \Big|u|^{p(x)-2}u - |v|^{p(x)-2}v\Big)\Big(u - v\Big)^+\, dx \\ &= \int_{\partial\Omega} \Big(f(x, u) - f(x, v)\Big)(u - v)^+\, dx \leq 0. \end{aligned} \tag{7}$$

Thus, $\nabla u(x) = \nabla v(x)$ and $u(x) = v(x)$ for a.e $[u > v] = \Omega_1$.

Let $x \in \Omega \setminus \Omega_1$, then $(u - v)^+(x) = 0$ and $\nabla(u - v)^+(x) = 0$ for a.e $\Omega \setminus \Omega_1$ thereby, $(u - v)^+(x) = 0$ and $\nabla(u - v)^+(x) = 0$ for a.e Ω, so $(u - v)^+ = 0$ for a.e $x \in \Omega$, that means $u \leq v \quad for \quad a.e \quad x \in \Omega$.

Similarly, we prove $v \leq u \quad a.e \quad x \in \Omega$, hence, $u = v$.

□

Remark 2.1. *If we suppose that* $f(x, u)u \geq 0$*, so the problem has no solution which is* $\neq 0$.

Proof of Theorem 1.3: Since $W^{1,p(x)}(\Omega)$ is a reflexive and separable Banach space, there exist $\{e_k\} \subset W^{1,p(x)}(\Omega)$ and $\{e_k^*\} \subset (W^{1,p(x)}(\Omega))^*$ such that

$$Y = \overline{span\{e_k : \quad k = 1, 2...\}}, \qquad Y^* = \overline{span\{e_k^* : \quad k = 1, 2...\}},$$

and

$$\langle e_i, e_j \rangle = \delta_{i,j}.$$

Define $V_n = span\{e_1, ..., e_n\}$. It is known that V_n and $\mathbb{R}^N$ are isomorphic and for $\eta \in \mathbb{R}^N$, we have an unique $v \in V_n$ by the identification $\eta \mapsto^\phi \sum_{i=1}^N \eta_i e_i = v$.

Define the function $A = (A_1, ..., A_N) : \mathbb{R}^N \to \mathbb{R}$ by

$$A_i(u) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla e_i\, dx + \int_\Omega |u|^{p(x)-2}u e_i - \int_{\partial\Omega} (f(x, u)e_i\, d\sigma, \quad u \in V_i.$$

Our method consists in considering a class of auxiliary problems,

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla e_i\, dx + \int_\Omega |u_n|^{p(x)-2}\nabla u_n \nabla e_i\, dx = \int_{\partial\Omega} f(x, u_n)e_i\, d\sigma, \tag{8}$$

Now, by applying the Lemma 2.1, let check the existence of solution u_n for the problem (2.6).

For $\| u \| > 1$, by (H_3), it yields

$$\begin{aligned}\langle A(u),u\rangle &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx-\int_{\partial\Omega}F(x,u)\,d\sigma\\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx-(\lambda_1-\epsilon)\int_\Omega \frac{|\ u\ |^{p(x)}}{p(x)}dx\\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx\\ &\quad -\frac{(\lambda_1-\epsilon)}{\lambda_1}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx\\ &\geq \frac{1}{2p^+}(1-\frac{(\lambda_1-\epsilon)}{\lambda_1}\parallel u\parallel^{p^-}-c.\end{aligned}$$

This shows that there is $R>1$ such that $\langle A_n u,u\rangle \geq 0$ if $\|u\|=R$, and then (2.6) has a solution $u_n\in V_n$ such that

$$\|u_n\|\leq R,\ \forall n\in\mathbb{N}.$$

From this point, passing to a subsequence if necessary, then

$$u_n \rightharpoonup u\ in\ W^{1,p(x)}(\Omega),$$

$$u_n\to u\ \ a.e.\ \ in\Omega.$$

By using the fact of the continuity of the Nemytskii operator from $L^{r(x)}(\Omega)\to L^{r'(x)}(\Omega)$, it follows that

$$\int_{\partial\Omega} f(x,u_n)e_i\,d\sigma \to \int_{\partial\Omega} f(x,u_n)e_i\,d\sigma.$$

Since $W^{1,p(x)}(\Omega)\hookrightarrow L^{r(x)}(\Omega)$ compactly, tending $n\to\infty$, so we infer that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla e_i\,dx+\int_\Omega |u|^{p(x)-2}\nabla u\nabla e_i\,dx=\int_{\partial\Omega} f(x,u)e_i\,d\sigma.$$

Thus, u is a solution of problem (1.1).

Now let u and v be two solutions of problem (1.1), we have

$$\begin{aligned}0 &\leq \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\Big)\nabla(u-v)\,dx+\\ &\quad \int_\Omega \Big(|u|^{p(x)-2}u-|v|^{p(x)-2}v\Big)\Big(u-v\Big)\,dx\\ &= \int_{\partial\Omega}\Big[f(x,u)-f(x,v)\Big](u-v)\,dx\\ &\leq 0,\end{aligned}$$

which implies that $(u-v)=0$, and the proof will be completed.

□

Remark 2.2. *Under the hypothesis (H_3). If we suppose that there exists $a_0>0$ and $\delta>0$ such that*

$$F(x,t)\geq a_0|t|^{q_0},\ \forall x\in\Omega,\ |t|<\delta,$$

where $q_0\in C(\overline{\Omega})$ with $q_0<p-$.

Then we get that the Energy functional associated to (1.1) ϕ is coercive and has a global minimizer u_1 which is non trivial. Indeed, fix $v_0\in X\setminus\{0\}$ and $t>0$ is small enough, so from (H_3) we have that

$$\begin{aligned}\phi(tv_0) &\le C_2\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|v_0|^{p(x)}dx + \int_\Omega \frac{t^{p(x)}}{p(x)}|v_0|^{p(x)}dx\right) - \int_\Omega F(x,tv_0)dx \\ &\le C_3t^{p^-} - C_4t^{q_0} < 0, \end{aligned} \tag{9}$$

because $q_0 < p^-$. By using the Mountain Pass Theorem and the fact that ϕ is coercive, we construct a continuous curve

$$\gamma :\ [0,1] \to X,\quad \gamma(0) = 0,\ \gamma(1) = u_1.$$

From this point of view, we can show the existence of an other critical point $u_2 \in X$ of ϕ such that $u_2 \neq u_1$ and u_2 is nontrivial.

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