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A study on absolute summability factors

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Abstract

In this study we proved theorems dealing with summability factors giving relations between absolute Cesàro and absolute weighted summability methods. So we deduced some results in the special cases.

Keywords: Summability factors, Absolute Cesàro summability, Absolute weighted summability.

1. INTRODUCTION

Let $\sum x_n$ be an infinite series with sequence of partial sums (s_n) and (θ_n) a sequence of positive real constants. Let $A = (a_{nv})$ be an infinite matrix of complex numbers. We define the A- transform of the sequence $s = (s_n)$ as the sequence $A(s) = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{\infty} a_{nv} s_v$$

provided the series on the right side converges for $n \geq 0$. Then, the series $\sum x_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [13])

$$\sum_{n=1}^{\infty} (\theta_n)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty. \quad (1.1)$$

In particular, if A is chosen to be the matrix of weighted mean $(\bar{N}, p_n)$, then $|A, \theta_n|_k$ summability reduces to $|\bar{N}, p_n, \theta_n|_k$ summability [14]. Also, it may be mentioned that on putting $\theta_n = P_n/p_n$, we obtain $|\bar{N}, p_n|_k$ summability (see[2]). A weighted mean matrix has the entries

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq n \\ 0, & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, $P_{-i} = p_{-i} = 0, i \geq 1$. If we take A as matrix of Cesàro means (C, α) of order $\alpha > -1$, then we get $|C, \alpha|_k$ summability in Flett's notation [3].

Also for $\alpha = -1$, if we get $A_n(s) = T_n$, the nth Cesàro $(C, -1)$ mean, which is defined by Thorpe in [16], with $\theta_n = n$ in (1.1), we obtain the $|C, -1|_k$ summability defined and studied by Hazar and Sarıgöl in [5], where

$$T_n = \sum_{v=0}^{n-1} a_v + (n+1) a_n.$$

Throughout this paper, k^* denotes the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

Let X and Y be summability methods. If $\sum \varepsilon_n x_n$ is summable Y whenever $\sum x_n$ is summable X, then the sequence $\varepsilon = (\varepsilon_n)$ is said to be a summability factor of type (X, Y) and it is denoted by $\varepsilon \in (X, Y)$. In the special case when $\varepsilon = 1$, then $1 \in (X, Y)$ gives the comparisons of these methods, where $1 = (1,1, \ldots)$

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i.e., $X \subset Y$. In this context, Sarıgöl [12] has established the result dealing with summability factor of type $\varepsilon \in (|C,\alpha|_k, |\bar{N},p_n|)$, for $\alpha > -1$ and $k > 1$ on absolute summability factors , which extends some well-known results of [8-11].

Also, Hazar Güleç [4] has recently extended these studies to the range $\alpha \geq -1$ using $|C,-1|_k$ summability method.

For other studies on absolute summability factors and comparisons of the methods, see [1,5,6,11,14,15].

In order to establish our results, we require the following lemmas.

Lemma 1.1. [12] Let $1 < k < \infty$. Then, $A(x) \in \ell$ whenever $x \in \ell_k$ if and only if

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*} < \infty$$

where $\ell_k = \{x = (x_v) : \sum |x_v|^k < \infty\}$.

Lemma 1.2. [7] Let $1 \leq k < \infty$. Then, $A(x) \in \ell_k$ whenever $x \in \ell$ if and only if

$$\sup_v \sum_{n=0}^{\infty}|a_{nv}|^k < \infty.$$

2. MAIN RESULTS

In this paper we characterize summability factors dealing with the methods $|C,-1|$ and $|\bar{N},p_n,\theta_n|_k$. Also, in the special case, we obtain the inclusion relations between the methods.

Theorem 2.1. Let (θ_n) be a sequence of positive real constants and $1 < k < \infty$. Then the necessary and sufficient condition that $\sum \varepsilon_n x_n$ is summable $|C,-1|$ whenever $\sum x_n$ is summable $|\bar{N},p_n,\theta_n|_k$ is

$$\sum_{r=1}^{\infty}\frac{1}{\theta_r}\left(\frac{rP_r|\varepsilon_r| + rP_{r-1}|\varepsilon_{r+1}|}{p_r}\right)^{k^*} < \infty. \quad (2.1)$$

Proof. Let (t_n) and (T_n) denote the sequences the nth weighted mean of the series $\sum x_n$ and the nth Cesàro mean $(C,-1)$ of the series $\sum \varepsilon_n x_n$, respectively. Then we define the sequences $\bar{y} = (\bar{y}_n)$ and $y = (y_n)$ as

$$\bar{y}_n = \theta_n^{1/k^*}(t_n - t_{n-1}) = \frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}x_v, \quad \bar{y}_0 = x_0 \quad (2.2)$$

and

$$y_n = T_n - T_{n-1} = (n+1)x_n\varepsilon_n - (n-1)x_{n-1}\varepsilon_{n-1}.$$

It is clear that $x = (x_n) \in |\bar{N},p_n,\theta_n|_k$ iff $\bar{y} = (\bar{y}_n) \in \ell_k$, and $\varepsilon x = (\varepsilon_n x_n) \in |C,-1|$ iff $y = (y_n) \in \ell$. By virtue of (2.2) we write inverse of $\bar{y}_n$ as

$$x_n = \frac{\theta_n^{-1/k^*}P_n}{p_n}\bar{y}_n - \frac{\theta_{n-1}^{-1/k^*}P_{n-2}}{p_{n-1}}\bar{y}_{n-1}, \quad x_0 = \bar{y}_0 . \quad (2.3)$$

Then, using (2.3), we get for $n \geq 1$,

$$\begin{aligned} y_n &= (n+1)x_n\varepsilon_n - (n-1)x_{n-1}\varepsilon_{n-1} \\ &= (n+1)\varepsilon_n\left(\frac{\theta_n^{-1/k^*}P_n}{p_n}\bar{y}_n - \frac{\theta_{n-1}^{-1/k^*}P_{n-2}}{p_{n-1}}\bar{y}_{n-1}\right) \\ &\quad - (n-1)\varepsilon_{n-1}\left(\frac{\theta_{n-1}^{-1/k^*}P_{n-1}}{p_{n-1}}\bar{y}_{n-1} - \frac{\theta_{n-2}^{-1/k^*}P_{n-3}}{p_{n-2}}\bar{y}_{n-2}\right) \\ &= (n+1)\varepsilon_n\frac{\theta_n^{-\frac{1}{k^*}}P_n}{p_n}\bar{y}_n - \left[(n+1)\varepsilon_n\frac{\theta_{n-1}^{-\frac{1}{k^*}}P_{n-2}}{p_{n-1}} + (n-1)\varepsilon_{n-1}\frac{\theta_{n-1}^{-\frac{1}{k^*}}P_{n-1}}{p_{n-1}}\right]\bar{y}_{n-1} \\ &\quad + (n-1)\varepsilon_{n-1}\frac{\theta_{n-2}^{-\frac{1}{k^*}}P_{n-3}}{p_{n-2}}\bar{y}_{n-2} \\ &= \sum_{r=n-2}^{n}c_{nr}\bar{y}_r \end{aligned}$$

where

$$c_{nr} = \begin{cases} (n+1)\varepsilon_n\frac{\theta_n^{-1/k^*}P_n}{p_n}, & r = n \\ -\left[\frac{(n+1)\varepsilon_n\theta_{n-1}^{-\frac{1}{k^*}}P_{n-2}}{p_{n-1}} + \frac{(n-1)\varepsilon_{n-1}\theta_{n-1}^{-\frac{1}{k^*}}P_{n-1}}{p_{n-1}}\right], & r = n-1 \\ (n-1)\varepsilon_{n-1}\frac{\theta_{n-2}^{-1/k^*}P_{n-3}}{p_{n-2}}, & r = n-2. \end{cases}$$

Then, $\sum \varepsilon_n x_n$ is summable $|C,-1|$ whenever $\sum x_n$ is summable $|\bar{N},p_n,\theta_n|_k$ if and only if $y=(y_n)\in \ell$, whenever $\bar{y}=(\bar{y}_n)\in \ell_k$, or equivalently, the matrix $C=(c_{nr})$ maps ℓ_k into ℓ, i.e., $C\in(\ell_k,\ell)$. Thus, it follows from Lemma 1.1 that $C\in(\ell_k,\ell)$ iff

$$\sum_{r=1}^{\infty}\left(\sum_{n=r}^{r+2}|c_{nr}|\right)^{k^*} = \sum_{r=1}^{\infty}\left(|c_{rr}|+|c_{r+1,r}|+|c_{r+2,r}|\right)^{k^*} = \sum_{r=1}^{\infty}\frac{1}{p_r^{k^*}\theta_r}\Big(|(r+1)\varepsilon_r P_r| + |(r+2)\varepsilon_{r+1}P_{r-1}+r\varepsilon_r P_r| + |(r+1)\varepsilon_{r+1}P_{r-1}|\Big)^{k^*}<\infty.$$

which is equivalent to the condition (2.1). This completes the proof of the Theorem.

The following result is immediate from of the Theorem 2.1.

Corollary 2.2. Let (θ_n) be a sequence of positive real constants and $1<k<\infty$. Then, $|\bar{N},p_n,\theta_n|_k \subset |C,-1|$ if and only if

$$\sum_{r=1}^{\infty}\frac{1}{\theta_r}\left(\frac{r(P_r+P_{r-1})}{p_r}\right)^{k^*}<\infty.$$

Now, we prove the following.

Theorem 2.3. Let (θ_n) be a sequence of positive real constants and $1\le k<\infty$. Then the necessary and sufficient condition that $\sum\varepsilon_n x_n$ is summable $|\bar{N},p_n,\theta_n|_k$ whenever $\sum x_n$ is summable $|C,-1|$, is

$$\sup_r \sum_{n=r}^{\infty}\left|\frac{r\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=r}^{n}\frac{P_{v-1}\varepsilon_v}{v(v+1)}\right|^k<\infty.$$

Proof. Let (t_n) and (T_n) denote the nth weighted mean of the series $\sum\varepsilon_n x_n$ and the nth Cesàro $(C,-1)$ mean of the series $\sum x_n$, respectively. As in proof of Theorem 2.1, we define the sequences $\bar{y}=(\bar{y}_n)$ and $y=(y_n)$ as

$$\bar{y}_n=\theta_n^{1/k^*}(t_n-t_{n-1})=\frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\varepsilon_v x_v\ ,\bar{y}_0=x_0\varepsilon_0$$

and

$$y_n=T_n-T_{n-1}=(n+1)x_n-(n-1)x_{n-1}, \qquad (2.4)$$

respectively.

It is clear that $\varepsilon x=(\varepsilon_n x_n)\in|\bar{N},p_n,\theta_n|_k$ iff $\bar{y}=(\bar{y}_n)\in\ell_k$ and $x=(x_n)\in|C,-1|$ iff $y=(y_n)\in\ell$. By virtue of (2.4), we write inverse of y_n as

$$x_n=\frac{1}{n(n+1)}\sum_{v=1}^{n}vy_v\ ,x_0=y_0\,. \qquad (2.5)$$

Then, using (2.5), we get for $n\ge 1$

$$\bar{y}_n=\frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\varepsilon_v x_v =\frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\varepsilon_v\frac{1}{v(v+1)}\sum_{r=1}^{v}ry_r =\frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{r=1}^{n}r\left(\sum_{v=r}^{n}\frac{P_{v-1}\varepsilon_v}{v(v+1)}\right)y_r=\sum_{r=1}^{n}c_{nr}y_r$$

where

$$c_{nr}=\begin{cases}\dfrac{r\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\displaystyle\sum_{v=r}^{n}\frac{P_{v-1}\varepsilon_v}{v(v+1)}, & 1\le r\le n,\\ 0, r>n. & \end{cases}$$

Then, $\sum \varepsilon_n x_n$ is summable $|\bar{N}, p_n, \theta_n|_k$ whenever $\sum x_n$ is summable $|C, -1|$ if and only if $\bar{y} = (\bar{y}_n) \in \ell_k$ whenever $y = (y_n) \in \ell$, or equivalently, the matrix $C = (c_{nr})$ maps ℓ into ℓ_k, *i.e.*, $C \in (\ell, \ell_k)$. Thus, it follows from Lemma 1.2 that

$$\sup_r \sum_{n=1}^{\infty} |c_{nr}|^k = \sup_r \sum_{n=r}^{\infty} \left| \frac{r\theta_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{v=r}^{n} \frac{P_{v-1}\varepsilon_v}{v(v+1)} \right|^k < \infty.$$

This completes the proof of the Theorem.

In the special case $\varepsilon_v = 1$ for all v, Theorem 2.3 is reduced to the following result.

Corollary 2.4. Let (θ_n) be a sequence of positive real constants and $1 \leq k < \infty$. Then, $|C, -1| \subset |\bar{N}, p_n, \theta_n|_k$ if and only if

$$\sup_r \sum_{n=r}^{\infty} \left(\frac{r\theta_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{v=r}^{n} \frac{P_{v-1}}{v(v+1)} \right)^k < \infty.$$

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