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A Hybrid Third-Order Iterative Process To Solve Nonlinear Equations

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ABSTRACT: In this study, by using the iterative method discussed in (Kang et al., 2013) and adopting a technique given in details (Biazar and Amirteimoori, 2006) introduced a new hybrid third-order iterative method to solve nonlinear equations derived from the Picard-Mann fixed-point iterative method. Some problems have been solved in order to demonstrate the performance of the established iterative method for the solution of the nonlinear equations.

Keywords : Fixed Point Method, Picard-Mann Iterative Method, Newton Iterative Method, Nonlinear Equations, Convergence Analysis.

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INTRODUCTION

For the last many years, numerical techniques have been effectively applied to solve non-linear equations (see for details (Biazar and Amirteimoori, 2006 ; Babolian and Biazar, 2002) and references there in). One of the basic algorithms for solving nonlinear equations is the fact that it is a fixed point iteration method. Fixed point theory becomes one of the most interesting branches in mathematics. Naturally, many mathematical and real world problems are known to be formulated as fixed point problems, that is, in the fixed point theory, for the solution of non-linear equation $f(s) = 0$, the equation is re-arranged as follows

$$\Upsilon(s) = s \tag{1.1}$$

where $\Upsilon(s) \in C^1[a,b]$ for all $s \in [a,b]$ and $|\Upsilon'(s)| \leq L < 1$ for all $s \in (a,b)$.

A fixed point of the mapping Υ satisfying the condition (1.1) is a point s. Moreover, fixed point theory has been effectively applied in various topics, including differential equation, integral equation, matrix equation, convex minimization and split feasibility, as well as for finding zeros of contractive mappings. Then, it is necessary to develop an iterative process which approximate the solution of these equations that has a good rate of convergence. Many studies in the field of fixed point theory concerning the existence and uniqueness of fixed points of single-valued contractions have been developed using basic iterative algorithms, such as : Picard iteration, Krasnoselksii, Mann and Ishikawa iterative processes. Over the years the interest regarding the speed of convergence of such iterations grew very fast. For example, many authors considered numerous iteration processes and studied their rate of convergence, see for details (Abbas and Nazir, 2014; Berinde, 2014; Fukhar-ud-din and Berinde, 2016; Chugh et al., 2015; Karakaya et al., 2013; Karakaya and Dogan, 2014; Dogan and Karakaya, 2018; Karakaya et al., 2017; Phuengrattana and Suantai, 2013). It is clear that the fixed point iteration methods have convergence from the first order.

MATERIALS AND METHODS

The aim of this study is to introduce a new third-order iteration method, derived from the Picard-Mann fixed point iterative method, by adopting a technique given in details (Biazar and Amirteimoori, 2006) and suggesting this iteration method to solve non-linear equations. This study is supported by some numerical examples.

The Picard iteration process (Picard, 1890) is defined by

$$q_{n+1} = \Upsilon q_n \qquad n \in \mathbb{N}, \tag{1.2}$$

where q_0 is an initial point.

In 2013, Khan (Khan, 2013) introduced Picard-Mann Hybrid iterative process as follows:

$$\begin{cases} s_0 \in [a,b] \quad \text{is a initial point}, \\ s_{n+1} = \Upsilon(w_n) \\ w_n = (1-\alpha_n)s_n + \alpha_n \Upsilon(s_n), (n \in \mathbb{N}), \end{cases} \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$.

We establish a sequence $\{s_n\}_{n\in\mathbb{N}}$ for the solution of the nonlinear equation s^* The iterative method will converge to the root s^*. It provides the following conditions

1) $\Upsilon \in C^1[a,b]$, 2) $|\Upsilon'(s)| < 1$ for all $s \in [a,b]$, 3) $\Upsilon(s) \in [a,b]$ for all $s \in [a,b]$ (Isaacson and Keller, 1966).

Definition 1 Let $\{s_n\}_{n\in\mathbb{N}}$ converge to s^*. If $p \in \mathbb{Z}$, and $C \in \mathbb{R}^+$ such that $\lim_{n\to\infty}\left|\frac{s_{n+1}-s^*}{(s_n-s^*)^p}\right| = C$. Then p and C are called order and the constant of convergence, respectively. By the Taylor expansion of the $\Upsilon(s_n)$, the order of convergence of the sequence $\{s_n\}_{n\in\mathbb{N}}$ is determined as follows:

$$\Upsilon(s_n) = \Upsilon(s) + \frac{\Upsilon'(s)}{1!}(s_n - s) + \frac{\Upsilon''(s)}{2!}(s_n - s)^2 + \ldots + \frac{\Upsilon^{(k)}(s)}{k!}(s_n - s)^k + \ldots$$

Taking into account $(1.1)$and (1.2) we obtain

$$s_{n+1} - s = \frac{\Upsilon'(s)}{1!}(s_n - s) + \frac{\Upsilon''(s)}{2!}(s_n - s)^2 + \ldots + \frac{\Upsilon^{(k)}(s)}{k!}(s_n - s)^k + \ldots$$

Theorem 1 (Isaacson and Keller, 1966) Let $\Upsilon(s) \in C^p[a,b]$. We suppose that $\Upsilon^{(k)}(s) = 0$ for $k = 1,2,3,\ldots,p-1$ and $\Upsilon^{(p)}(s) \neq 0$, then the sequence $\{s_n\}_{n\in\mathbb{N}}$ has a order of convergenge p.

2.1 Modified Third-Order Iterative Method

Consider the non-linear equation

$$f(s) = 0. \tag{1.4}$$

We suppose that s^* is a simple zero of (1.4) and s_0 initial point sufficiently near to s^*. In the fixed point theory, for the solution of non-linear equation $f(s) = 0$, the equation is re-arranged as $\Upsilon(s) = s$.

Then

$$s^* = \Upsilon(s^* - s_0 + s_0).$$

Expansion of s_0 by Taylor series, we have

$$s^* = \Upsilon(s_0) + (s^* - s_0)\Upsilon'(s_0) + \frac{(s^* - s_0)^2}{2!}\Upsilon''(s_0) + \ldots.$$

First order approximation is

$$s^* = \Upsilon(s_0) + (s^* - s_0)\Upsilon'(s_0) \tag{1.5}$$

which gives us Newton iteration method, that is,

$$s^* = \frac{\Upsilon(s_0) - s_0\Upsilon'(s_0)}{1 - \Upsilon'(s_0)}.$$

Algorithm 1. (Kang et al., 2013)

$$\begin{cases} s_0 \quad \text{is a initial point,} \\ s_{n+1} = \dfrac{\Upsilon(s_n) - s_n\Upsilon'(s_n)}{1 - \Upsilon'(s_n)}. \end{cases}$$

Algorithm 2. (Ashiq et al., 2015)

Second order approximation is

$$s^* = \Upsilon(s_0) + (s^* - s_0)\Upsilon'(s_0) + \frac{(s^* - s_0)^2}{2!}\Upsilon''(s_0)$$

by simplification

$$s^* = \frac{\Upsilon(s_0) - s_0\Upsilon'(s_0)}{1 - \Upsilon'(s_0)} + \frac{(s^* - s_0)^2}{2\left(1 - \Upsilon'(s_0)\right)}\Upsilon''(s_0). \tag{1.6}$$

By using the value of $(s^* - s_0)$ in (1.5), we get

$$s^* = \frac{\Upsilon(s_0) - s_0\Upsilon'(s_0)}{1 - \Upsilon'(s_0)} + \frac{(\Upsilon(s_0) - s_0)^2}{2\left(1 - \Upsilon'(s_0)\right)^3}\Upsilon''(s_0).$$

Consequently, we obtain

$$\begin{cases} s_0 \quad \text{is a initial point,} \\ s_{n+1} = \frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-s_n)^2}{2(1-\Upsilon'(s_n))^3}\Upsilon''(s_n), \qquad \Upsilon'(s_n) \neq 0 \end{cases} \tag{1.7}$$

which has the convergence order at least 3. Inspired by the above iteration method, we have defined the following third-order hybrid iteration method and proved that the order is 3.

Algorithm3.

$$\begin{cases} s_0 \quad \text{initial point, } \Upsilon'(s_n) \neq 0 \\ s_{n+1} = \frac{\Upsilon(w_n)-w_n\Upsilon'(w_n)}{1-\Upsilon'(w_n)} + \frac{(\Upsilon(w_n)-w_n)^2}{2(1-\Upsilon'(w_n))^3}\Upsilon''(w_n) \\ w_n = (1-a)s_n + a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-s_n)^2}{2(1-\Upsilon'(s_n))^3}\Upsilon''(s_n)\right). \end{cases} \tag{1.8}$$

RESULTS AND DISCUSSION

Now, let us prove the following theorem which constitutes the main results.

Theorem 2 Let $f: E \subset R \to R$ and consider the equation (1.4) having simple root $s^* \in E$, where $T: E \subset R \to R$ sufficiently smooth in the neighborhood of s^*, then Algorithm 3 (1.8) is of the order of convergence which is at least 3.

Proof. The iterative method (1.8) is taken into consideration rewritten as follows:

$$s_{n+1} = \frac{\begin{array}{c}\Upsilon\left((1-a)s_n + a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-x_n)^2}{2(1-\Upsilon'(s_n))^3}T''(s_n)\right)\right) \\ -\left((1-a)s_n + a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-s_n)^2}{2(1-\Upsilon'(s_n))^3}T''(s_n)\right)\right) \\ \times\Upsilon'\left((1-a)s_n + a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-s_n)^2}{2(1-\Upsilon'(s_n))^3}\Upsilon''(s)\right)\right)\end{array}}{1-\Upsilon'\left((1-a)s_n + a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)} + \frac{(\Upsilon(s_n)-s_n)^2}{2(1-\Upsilon'(s_n))^3}\Upsilon''(s_n)\right)\right)}$$

Hence $K(s)$ defined as follows:

$$+\frac{\left(\Upsilon\left((1-a)s_n+a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)}+\frac{(\Upsilon(s_n)-s_n)^2}{2\left(1-\Upsilon'(s_n)\right)^3}\Upsilon''(s_n)\right)\right)-\left((1-a)s_n+a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)}+\frac{(\Upsilon(s_n)-s_n)^2}{2\left(1-\Upsilon'(s_n)\right)^3}\Upsilon''(s_n)\right)\right)\right)^2}{2\left(1-\Upsilon'\left((1-a)s_n+a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)}+\frac{(\Upsilon(s_n)-s_n)^2}{2\left(1-\Upsilon'(s_n)\right)^3}\Upsilon''(s_n)\right)\right)\right)^3}$$
$$\times\Upsilon''\left((1-a)s_n+a\left(\frac{\Upsilon(s_n)-s_n\Upsilon'(s_n)}{1-\Upsilon'(s_n)}+\frac{(\Upsilon(s_n)-s_n)^2}{2\left(1-\Upsilon'(s_n)\right)^3}\Upsilon''(s_n)\right)\right).$$

$$K(s)=\frac{\begin{array}{c}\Upsilon\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)-\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)\\ \times\Upsilon'\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)\end{array}}{1-\Upsilon'\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)}$$

$$+\frac{\left(\Upsilon\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)-\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)\right)^2}{2\left(1-\Upsilon'\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(s)\right)\right)\right)^3}$$
$$\times\Upsilon''\left((1-a)s+a\left(\frac{\Upsilon(s)-s\Upsilon'(s)}{1-\Upsilon'(s)}+\frac{(\Upsilon(s)-s)^2}{2\left(1-\Upsilon'(s)\right)^3}\Upsilon''(x)\right)\right).$$

By using $K(s^*)=s^*$, we have $K(s)=\mathrm{s}$.

If the first derivative of the $K(s)$ nonlinear equation is taken and $K(s^*)=s^*$ put in $K'(s)$ we obtain

$$K'(x^*)=\frac{-s^*\Upsilon''(s^*)(1-a)}{1-\Upsilon'(s^*)}+\frac{\left(\Upsilon(s^*)-s^*\Upsilon'(s^*)\right)\Upsilon''(s^*)(1-a)}{\left(1-\Upsilon'(s^*)\right)^2}+3\frac{\left(-s^*+\Upsilon(s^*)\right)^2\Upsilon''(s^*)^2(1-a)}{2\left(1-\Upsilon'(s^*)\right)^4}$$
$$+\frac{\left(-s^*+\Upsilon(s^*)\right)\Upsilon''(s^*)\left(-1+a+\Upsilon'(s^*)(1-a)\right)}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\left(-s^*+\Upsilon(s^*)\right)^2(1-a)\Upsilon'''(s^*)}{2\left(1-\Upsilon'(s^*)\right)^3}$$
$$=\frac{-s^*\Upsilon''(s^*)(1-a)}{1-\Upsilon'(s^*)}+\frac{s^*\left(1-\Upsilon'(s^*)\right)\Upsilon''(s^*)(1-a)}{\left(1-\Upsilon'(s^*)\right)^2}=\frac{-s^*\Upsilon''(s^*)(1-a)}{1-\Upsilon'(s^*)}+\frac{s^*\Upsilon''(s^*)(1-a)}{1-\Upsilon'(s^*)}=0$$

If the second derivative of the $K(s)$ equation is taken and $K(s^*) = s^*$ put in the operator $K''(s)$, we have

$$K''(s^*) = \frac{\Upsilon''(s^*)(1-a)^2}{1-\Upsilon'(s^*)} - 2\frac{x^*\Upsilon''(s^*)^2(1-a)^2}{(1-\Upsilon'(s^*))^2} + 2\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon''(s^*)^2(1-a)}{(1-\Upsilon'(s^*))^3}$$
$$+6\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)^3(1-a)}{(1-\Upsilon'(s^*))^5} + 6\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)^2(1-a)(-1+a+\Upsilon'(s^*)(1-a))}{(1-\Upsilon'(s^*))^4}$$
$$+\frac{\Upsilon''(s^*)(-1+a+\Upsilon'(s^*)(1-a))^2}{(1-\Upsilon'(s^*))^3} - \frac{s^*\Upsilon'''(s^*)(1-a)^2}{1-\Upsilon'(s^*)} + \frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon'''(s^*)(1-a)^2}{(1-\Upsilon'(s^*))^2}$$
$$+9\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)(1-a)^2\Upsilon'''(s^*)}{2(1-\Upsilon'(s^*))^4} + \frac{(-s^*+\Upsilon(s^*))(1-a)\Upsilon'''(s^*)(-1+a+\Upsilon'(s^*)(1-a))}{(1-\Upsilon'(s^*))^3}$$
$$-a\frac{s^*\Upsilon''(s^*)\times 0}{1-\Upsilon'(s^*)} + a\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon''(s^*)\times 0}{(1-\Upsilon'(s^*))^2} + 3a\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)^2\times 0}{2(1-\Upsilon'(s^*))^4} + \frac{(-s^*+\Upsilon(s^*))^2\Upsilon'''(s^*)\times 0}{2(1-\Upsilon'(s^*))^3} +$$
$$\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)\Upsilon''(s^*)(1-a)^2}{(1-\Upsilon'(s^*))^3} + \frac{(-s^*+\Upsilon(s^*))^2\Upsilon^{(4)}(s^*)(1-a)^2}{2(1-\Upsilon'(s^*))^3}$$
$$= 0.$$

If the third derivative of the $K(s)$ equation is taken and $K(s^*) = s^*$ put in $K'''(s)$, we obtain

$$K'''(s^*) = -3\frac{\Upsilon''(s^*)^2(1-a)^3}{(1-\Upsilon'(s^*))^2} - 6\frac{s^*\Upsilon''(s^*)^3(1-a)^3}{(1-\Upsilon'(s^*))^3} + 6\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon''(s^*)^3(1-a)^3}{(1-\Upsilon'(s^*))^4}$$
$$+30\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)^4(1-a)^3}{(1-\Upsilon'(s^*))^6} + 36\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)^3(1-a)^2(-1+a+\Upsilon'(s^*)(1-a))}{(1-\Upsilon'(s^*))^5}$$
$$+9\frac{\Upsilon''(s^*)^2(1-a)(-1+a+\Upsilon'(s^*)(1-a))^2}{(1-\Upsilon'(s^*))^5} - 2\frac{\Upsilon'''(s^*)(1-a)^3}{1-\Upsilon'(s^*)} - 6\frac{s^*\Upsilon''(s^*)(1-a)^3\Upsilon'''(s^*)}{(1-\Upsilon'(s^*))^2}$$
$$+6\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon''(s^*)(1-a)^3\Upsilon'''(s^*)}{(1-\Upsilon'(s^*))^3} + 36\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)^2(1-a)^3\Upsilon'''(s^*)}{(1-\Upsilon'(s^*))^5}$$
$$+27\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)(1-a)^2\Upsilon'''(s^*)(-1+a+\Upsilon'(s^*)(1-a))}{(1-\Upsilon'(s^*))^4} + 3\frac{\Upsilon'''(s^*)(1-a)(-1+a+\Upsilon'(s^*)(1-a))^2}{(1-\Upsilon'(s^*))^3} +$$
$$9\frac{(-s^*+\Upsilon(s^*))^2\Upsilon'''(s^*)^2(1-a)^3}{2(1-\Upsilon'(s^*))^4}$$
$$-3a\frac{\Upsilon''(s^*)(1-a)\times 0}{1-\Upsilon'(s^*)} - 6a\frac{s^*\Upsilon''(s^*)^2(1-a)\times 0}{(1-\Upsilon'(s^*))^2} + 6a\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon''(s^*)^2(1-a)\times 0}{(1-\Upsilon'(s^*))^3}$$
$$+18a\frac{(-s^*+\Upsilon(s^*))^2\Upsilon''(s^*)^3(1-a)\times 0}{(1-\Upsilon'(s^*))^5} + 9a\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)^3(-1+a+\Upsilon'(s^*)(1-a))\times 0}{(1-\Upsilon'(s^*))^4}$$
$$-3a\frac{s^*\Upsilon''(s^*)(1-a)\times 0}{1-\Upsilon'(s^*)} + 3a\frac{(\Upsilon(s^*)-s^*\Upsilon'(s^*))\Upsilon'''(s^*)(1-a)\times 0}{(1-\Upsilon'(s^*))^2}$$
$$+27a\frac{(-s^*+\Upsilon(s^*))^2\Upsilon'''(s^*)\Upsilon''(s^*)(1-a)\times 0}{2(1-\Upsilon'(s^*))^4} - \frac{s^*(1-a)^3\Upsilon^{(4)}(s^*)}{1-\Upsilon'(s^*)}$$
$$+3a\frac{(-s^*+\Upsilon(s^*))\Upsilon'''(s^*)(-1+a+\Upsilon'(s^*)(1-a))(1-a)\times 0}{(1-\Upsilon'(s^*))^3} + 9\frac{(-s^*+\Upsilon(s^*))\Upsilon''(s^*)^3(1-a)^3}{(1-\Upsilon'(s^*))^4}$$

$$+3\frac{\Upsilon''(s^*)^2\left(-1+a+\Upsilon'(s^*)(1-a)\right)(1-a)^2}{\left(1-\Upsilon'(s^*)\right)^3}+3\frac{\left(-s^*+\Upsilon(s^*)\right)\Upsilon''(s^*)\Upsilon'''(s^*)(1-a)^3}{\left(1-\Upsilon'(s^*)\right)^3}$$
$$+\frac{\left(\Upsilon(s^*)-s^*\Upsilon'(s^*)\right)(1-a)^3\Upsilon^{(4)}(s^*)}{\left(1-\Upsilon'(s^*)\right)^2}+6\frac{\left(-s^*+\Upsilon(s^*)\right)^2(1-a)^3\Upsilon''(s^*)\Upsilon^{(4)}(s^*)}{\left(1-\Upsilon'(s^*)\right)^4}$$
$$+3\frac{\left(-s^*+\Upsilon(s^*)\right)(1-a)^2\left(-1+a+\Upsilon'(s^*)(1-a)\right)\Upsilon^{(4)}(s^*)}{\left(1-\Upsilon'(s^*)\right)^3}+3a\frac{\left(-s^*+\Upsilon(s^*)\right)^2(1-a)\Upsilon^{(4)}(s^*)\times 0}{2\left(1-\Upsilon'(s^*)\right)^3}$$
$$-\frac{as^*\Upsilon''(s^*)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)}{1-\Upsilon'(s^*)}+\frac{a\left(\Upsilon(s^*)-s^*\Upsilon'(s^*)\right)\Upsilon''(s^*)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)}{\left(1-\Upsilon'(s^*)\right)^2}$$
$$+3\frac{a(\Upsilon(s^*)-s^*)^2\Upsilon''(s^*)^2\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)}{2\left(1-\Upsilon'(s^*)\right)^4}+\frac{a(\Upsilon(s^*)-s^*)^2\Upsilon'''(s^*)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)}{2\left(1-\Upsilon'(s^*)\right)^3}$$
$$+\frac{(\Upsilon(s^*)-s^*)\Upsilon''(s^*)\begin{bmatrix}(1-a)^3\Upsilon'''(x^*)+3a\Upsilon''(x^*)(1-a)\left(\frac{9\Upsilon''(x^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(x^*)}{1-\Upsilon'(s^*)}\right)\\ -a\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)+a\Upsilon'(s^*)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)\end{bmatrix}}{\left(1-\Upsilon'(s^*)\right)^3}$$
$$+\frac{(\Upsilon(s^*)-s^*)^2(1-a)^3\Upsilon^{(5)}(s^*)}{2\left(1-\Upsilon'(s^*)\right)^3}=\left(\frac{(1-a)^3}{\left(1-\Upsilon'(s^*)\right)^2}-a\right)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)\neq 0.$$

Thus, we obtain $K(s^*)=s^*$, $K'(s^*)=0$, $K''(s^*)=0$ and

$$K'''(s^*)=\left(\frac{(1-a)^3}{\left(1-\Upsilon'(s^*)\right)^2}-a\right)\left(\frac{9\Upsilon''(s^*)^2}{\left(1-\Upsilon'(s^*)\right)^2}+\frac{\Upsilon'''(s^*)}{1-\Upsilon'(s^*)}\right)\neq 0.$$

Hence, it is concluded that Algorithm 3 has third-order convergence for the non-linear equations.

Now, let's give some examples showing the advantages of Algorithm 3.

In the following examples, KNHM (Khan-Modified New Hybrid Method), MNIM (Modified New Iterative Method), NIM (New Iterative Method) and FPM (Fixed Point Method) iteration methods are compared.

Example 1 For initial guess $s_0=1$, we consider the equation $f(s)=s^3-3s-2=0$. Then we have $\Upsilon(s)=(3s+2)^{1/3}=s$, $\Upsilon'(s)=(3s+2)^{-2/3}$ and $\Upsilon''(s)=2(3s+2)^{(-5/3)}$. The numerical solution of this equation is 2.000000 for six decimal. The following tables and graphs show the accuracy of the result.

Table 1: Comparison the rate of convergence

Iteration	KNHM	MNIM	NIM	FPM
s_0	1.000000	1.000000	1.000000	1.000000
s_1	1.999168	1.957965	2.078983	1.709976
s_2	2.000000	1.999999	2.000075	1.989436
s_3	2.000000	2.000000	2.000000	2.000000

The following figures are graphical presentations of the above results:

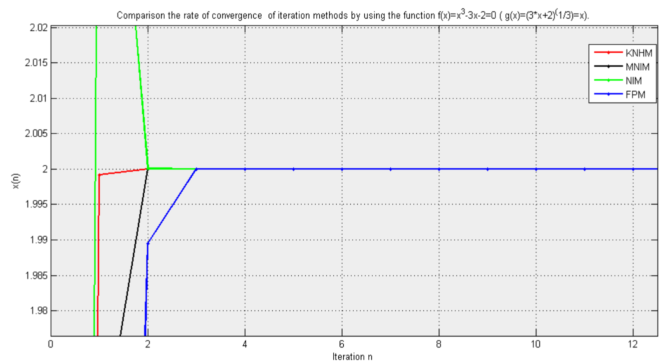

Figure 1: The convergence rate comparison among KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 - 3s - 2 = 0$

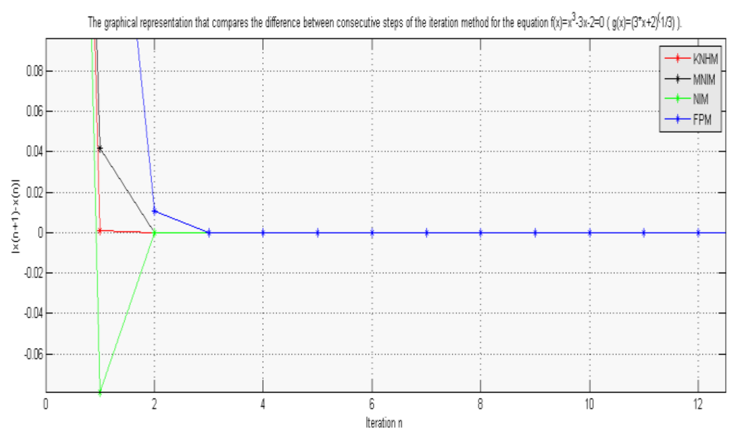

Figure 2: Differences among successive steps of KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 - 3s - 2 = 0$

Example 2 For initial guess $s = 0.1$, we consider the equation $\mathrm{f(x)} = \ln s + s$. Then we have $\Upsilon(s) = e^{(-s)} = s$, $\Upsilon'(s) = -e^{(-s)}$ and $\Upsilon''(s) = e^{(-s)}$. The numerical solution of this equation is 0.567102 for 12 decimal. The following table and graphs show the accuracy of the result.

Table 2: Comparison the rate of convergence

Iteration	KNHM	MNIM	NIM	FPM
s_0	0.100000000000	0.100000000000	0.100000000000	0.100000000000
s_1	0.567102582345	0.564924580551	0.522522893773	0.904837418036
s_2	0.567143290410	0.567143290353	0.566778157786	0.404607661664
s_3	0.567143290410	0.567143290410	0.567143266283	0.667238542771
s_4	0.567143290410	0.567143290410	0.567143290410	0.513123591989
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
s_{49}	0.567143290410	0.567143290410	0.567143290410	0.567143290410

The following figures are graphical presentations of the above results:

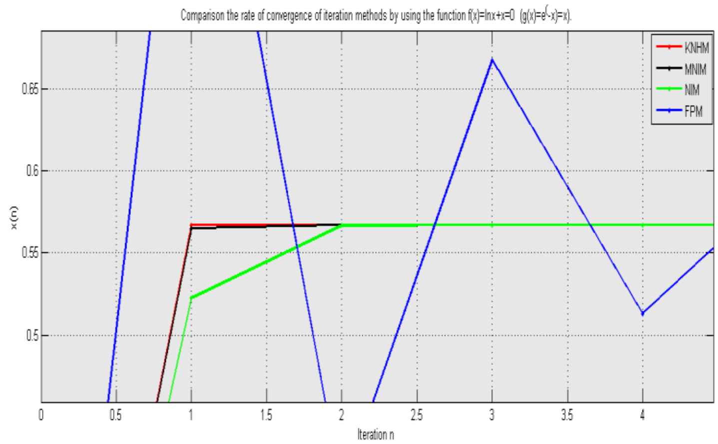

Figure 3: The convergence rate comparison among KNHM, MNIM, NIM and FPM for the equation $f(s) = lns + s = 0$

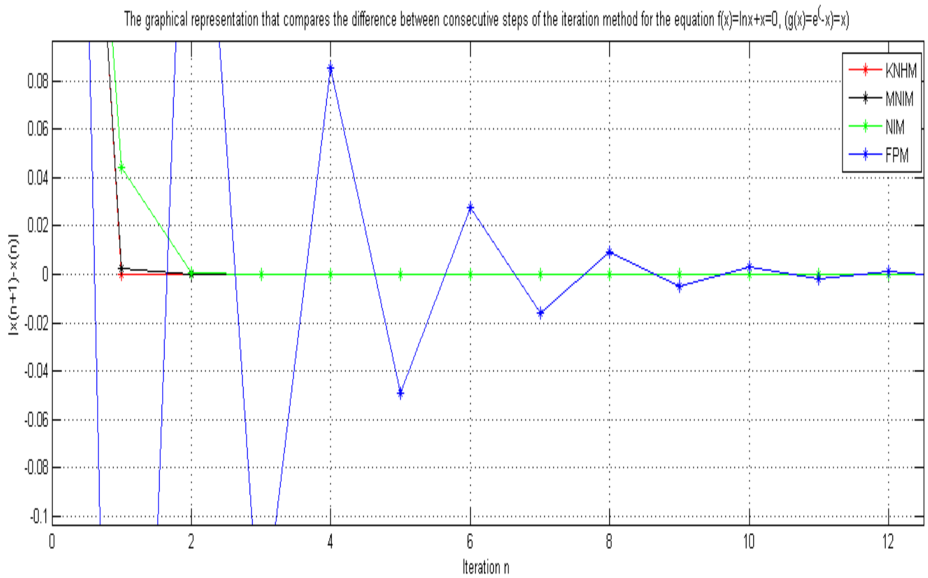

Figure 4: Differences among successive steps of KNHM, MNIM, NIM and FPM for the equation $f(s) = lns + s = 0$

Example 3 For initial guess $s_0 = 2$, we consider the equation $f(s) = s^2 - 2s(1 + \sin(s)) = 0$. Then we have $\Upsilon(s) = 3 + 2\sin(s) = s$, $\Upsilon'(s) = 2\cos(s)$ and $\Upsilon''(s) = -2\sin(s)$. The numerical solution of this equation is 3.094383 for six decimal. The following table and graphs show the accuracy of the result.

Table 3: Comparison the rate of convergence

$Iteration$	KNHM	MNIM	NIM	FPM
s_0	2.000000	2.000000	2.000000	2.000000
s_1	2.904072	2.363969	3.538288	4.818595
s_2	3.094371	3.000281	3.077417	1.011269
s_3	3.094383	3.094288	3.094389	4.695012
s_4	3.094383	3.094383	3.094383	1.000302

The following figures are graphical presentations of the above results:

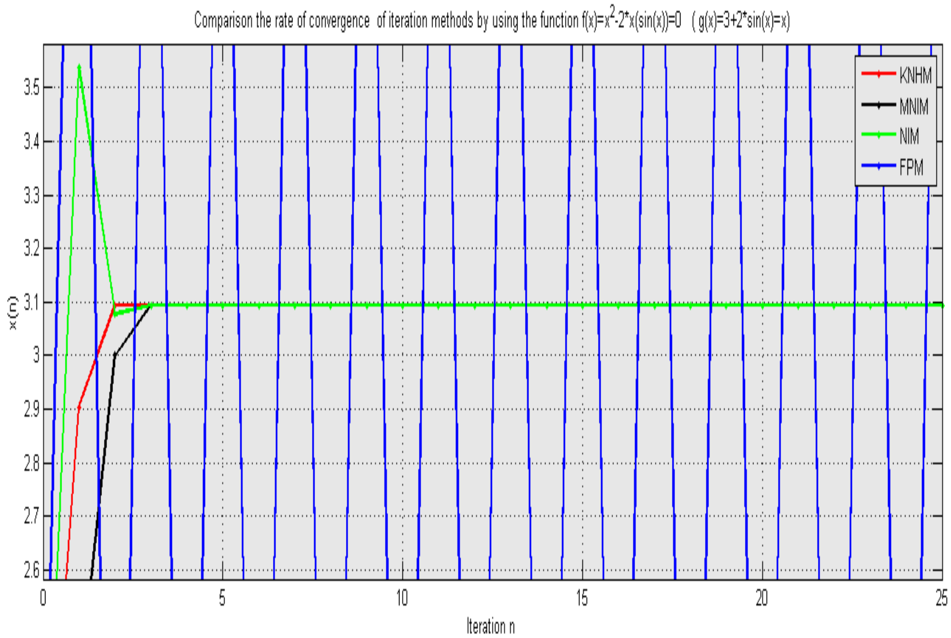

Figure 5: The convergence rate comparison among KNHM, MNIM, NIM and FPM for the equation $f(s) = s^2 - 2s(1 + sin(s)) = 0$

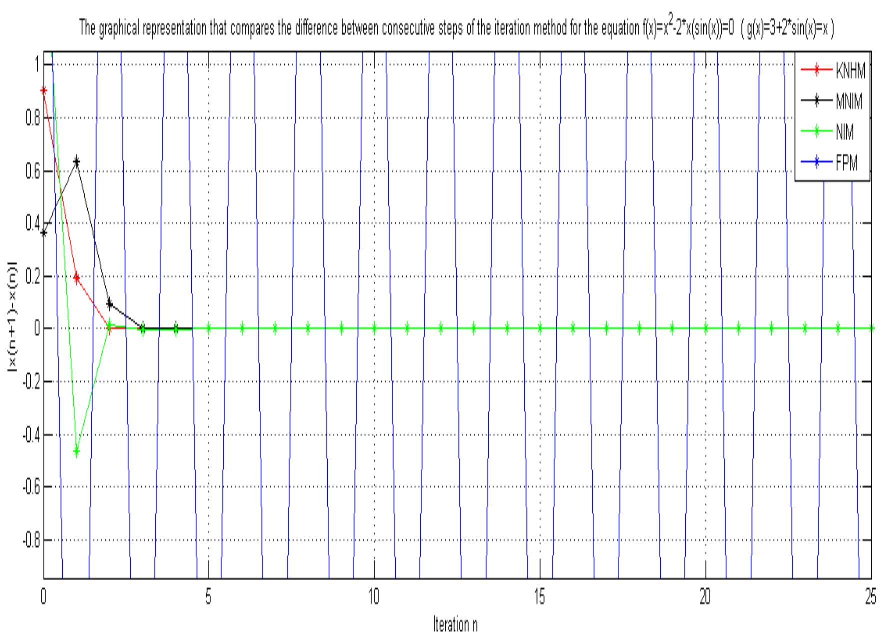

Figure 6: Differences among successive steps of KNHM, MNIM, NIM and FPM for the equation $f(s) = s^2 - 2s(1 + sin(s)) = 0$

Example 4 For initial guess $s_0 = 3$, we consider the equation $f(s) = s^3 + 4s^2 - 10 = 0$. Then we have $\Upsilon(s) = (10/(4+s))^{(1/2)} = s$, $\Upsilon'(s) = -(10/(4(4+x)^3))^{(1/2)}$ and $\Upsilon''(s) = (10/(8(4+s)^5))^{(1/2)}$. The numerical solution of this equation is 1.365230 for six decimal. The following table and graphs show the accuracy of the result.

Table 4: Comparison the rate of convergence

Iteration	**KNHM**	**MNIM**	**NIM**	**FPM**
s_0	3.000000	3.000000	3.000000	3.000000
s_1	1.363059	1.348173	1.337189	1.195229
s_2	1.365230	1.365228	1.365218	1.387387
s_3	1.365230	1.365230	1.365230	1.362420
s_4	1.365230	1.365230	1.365230	1.365588
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
s_8	1.365230	1.365230	1.365230	1.365230

The following figures are graphical presentations of the above results:

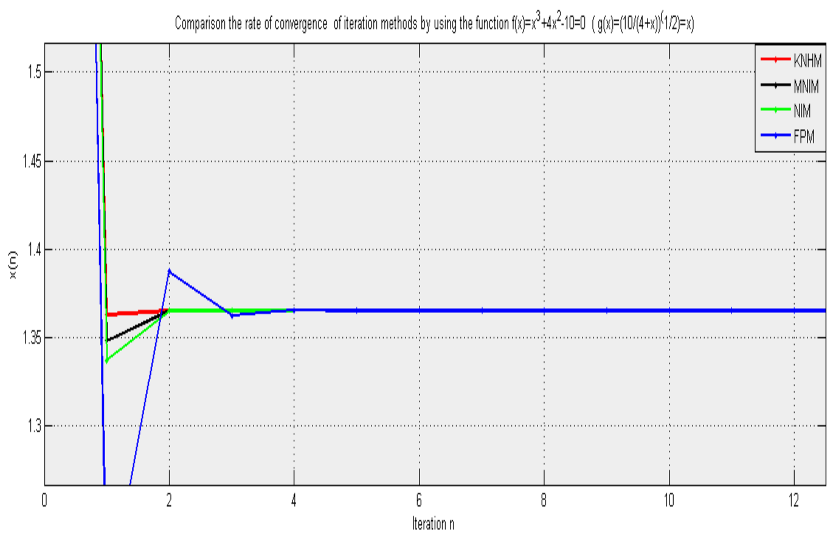

Figure 7: The convergence rate comparison among KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 + 4s^2 - 10 = 0$

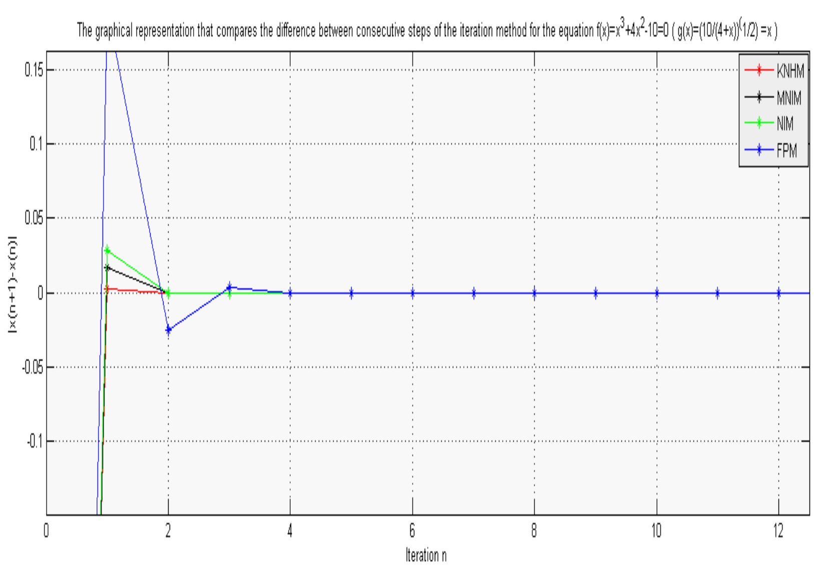

Figure 8: Differences among successive steps of KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 + 4s^2 - 10 = 0$

Example 5 For initial guess $s_0 = 0.1$, we consider the equation $f(s) = s^3 + 5s - 5 = 0$. Then we have $\Upsilon(s) = ((5 - s^3)/5) = s$, $\Upsilon'(s) = (-3s^2)/5$ and $\Upsilon''(s) = (-6s)/5$. The numerical solution of this equation is 0.86883002 for six decimal. The following table and graphs show the accuracy of the result.

Table 5: Comparison rate of convergence

Iteration	KNHM	MNIM	NIM	FPM
s_0	0.100000	0.100000	0.100000	0.100000
s_1	0.868244	0.946719	0.994433	0.999800
s_2	0.868830	0.868892	0.874489	0.800119
s_3	0.868830	0.868830	0.868842	0.897554
s_4	0.868830	0.868830	0.868830	0.855386
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
s_{17}	0.868830	0.868830	0.868830	0.868830

The following figures are graphical presentations of the above results:

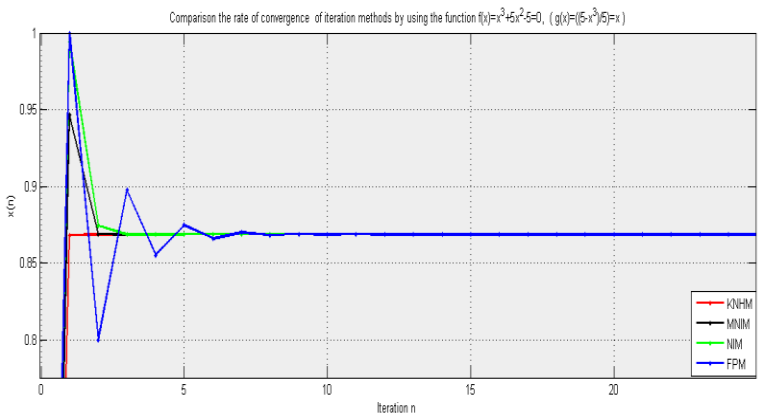

Figure 9: The convergence rate comparison among KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 + 5s - 5 = 0$

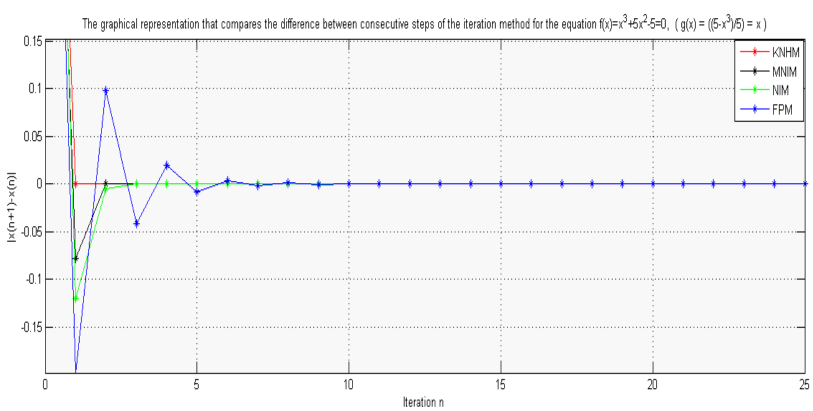

Figure 10: Differences among successive steps of KNHM, MNIM, NIM and FPM for the equation $f(s) = s^3 + 5s - 5 = 0$

CONCLUSION

In this study, a new hybrid iterative method derived from the Picard-Mann fixed point iteration method and the modified Newton method with the 3rd order convergence ratio were obtained. It was later shown that this iteration method was faster than other iteration methods and had a convergence ratio of at least 3rd order. Our results were supported by 5 examples.The results of these examples are given by the tables, the rate convergence graphs and the derivative graphs showing the difference between consecutive terms. Therefore, our results have improved the results for Algorithm1 (Kang et al., 2013) and Algorithm 2 (Ashiq et al., 2015).

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