Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1611 – 1624
DOI : 10.15672/hujms.548103

Research Article

Nonterminating well–poised hypergeometric series

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Abstract

Two classes of nonterminating well–poised series are examined by means of the modified Abel lemma on summation by parts, that leads to several summation and transformation formulae.

Mathematics Subject Classification (2010). 33C20, 34E05, 41A60

Keywords. classical hypergeometric series, Abel's lemma on summation by parts, Pfaff–Saalschütz theorem, Dougall's formulae for ${}_2H_2$-series and ${}_5H_5$-series

1. Introduction and motivation

As a classical analytic instrument, the Abel lemma on summation by parts has been fundamental in convergence test of infinite series (cf. [3, §80], [19, §43] and [22, §7.36] for example). However, it has not been utilized, until recently, to evaluate finite sums and infinite series. For this purpose, it has been necessary for the author [6, 9] to reformulate it in a more symmetrical form.

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta\tau_k = \tau_k - \tau_{k+1}, \tag{1.1}$$

where Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts for unilateral and bilateral series may be reformulated respectively as

$$\sum_{k=0}^{+\infty} B_k\nabla A_k = [AB]_+ - A_{-1}B_0 + \sum_{k=0}^{+\infty} A_k\Delta B_k, \tag{1.2}$$

$$\sum_{k=-\infty}^{+\infty} B_k\nabla A_k = [AB]_+ - [AB]_- + \sum_{k=-\infty}^{+\infty} A_k\Delta B_k. \tag{1.3}$$

Both formulae just displayed hold for terminating series and nonterminating series, provided, in the latter case, that one of both series in each equation converges and there exist the limits $[AB]_\pm := \lim\limits_{n\to\pm\infty} A_nB_{n+1}$.

The above modified Abel formulae on summation by parts have been shown powerful in dealing with summation and transformation formulae for both unilateral series (cf.[7, 12, 13, 23, 24]) and bilateral series (cf. [8, 10, 11]). The aim of the present paper is to examine two

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Received: 02.04.2019; Accepted: 02.12.2019

classes of well–poised series ${}_6H_6(-1)$ and ${}_7H_7(1)$, as well as their unilateral counterparts. Several transformation formulae will be derived and some important identities will be reviewed.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then for $n \in \mathbb{Z}$ and an indeterminate x, the shifted factorial is defined by the quotient

$$(x)_n = \Gamma(x+n)/\Gamma(x),$$

where the Γ-function is given by the Euler integral

$$\Gamma(x) \;=\; \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u \quad \text{with} \quad \Re(x) > 0.$$

Throughout the paper, we shall adopt the following notations (cf. Bailey [1]) for the generalized hypergeometric series, which has wide applications in mathematics, physics, and computer science (see [20] and [17, §5.5]):

$${}_pF_q\left[\begin{matrix}a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_q\end{matrix}\Big|z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!},$$

$${}_pH_p\left[\begin{matrix}a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_p\end{matrix}\Big|z\right] = \sum_{k=-\infty}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_p)_k}z^k,$$

$${}_pH_p^+\left[\begin{matrix}a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_p\end{matrix}\Big|z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_p)_k}z^k.$$

There exist numerous closed formulae for hypergeometric series in the literature (see e.g. [4, 5, 7, 14, 24]). These series are said to be well–poised (when $q = p-1$ for the unilateral series ${}_pF_q$) if their numerator parameters can be paired off with denominator parameters so that each pair has the same sum. The well–poised series is one of the important classes of hypergeometric series. They have been carefully examined by Whipple [25] almost a century ago. Some extensions with integer parameters can be found in a recent paper [21] by Srivastava et al.

There are several useful properties of the Γ-function. Some of them are recorded below, that will be utilized freely in the paper without explanation:

- Recurrence relation

$$\Gamma(x+n) = (x)_n\Gamma(x) \qquad \text{for} \qquad n \in \mathbb{N}_0.$$

- Asymptotic relation (cf. Rainville [20, §11])

$$\Gamma(x+n) \;\approx\; (n-1)!\,n^x \quad \text{as} \quad n \to +\infty. \tag{1.4}$$

  Using the above relation is sometimes easier than using the Stirling asymptotic formula.
- Euler's reflection formula (cf. Rainville [20, §17])

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

- Gauss' multiplication formula (cf. Rainville [20, §20])

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}}n^{nx-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\Big(x+\frac{k}{n}\Big). \tag{1.5}$$

In addition, we shall make use of the following multi–parameter notations in order to reduce lengthy and complicated expressions:

$$\begin{aligned} \begin{bmatrix} \alpha,\beta,\cdots,\gamma \\ A,B,\cdots,C \end{bmatrix}_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}, \\ \Gamma\begin{bmatrix} \alpha,\beta,\cdots,\gamma \\ A,B,\cdots,C \end{bmatrix} &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}, \\ \sin\pi\begin{bmatrix} \alpha,\beta,\cdots,\gamma \\ A,B,\cdots,C \end{bmatrix} &= \frac{\sin(\pi\alpha)\sin(\pi\beta)\cdots\sin(\pi\gamma)}{\sin(\pi A)\sin(\pi B)\cdots\sin(\pi C)}. \end{aligned}$$

2. Well–poised ${}_7H_7(1)$ series

For the two sequences defined by

$$\begin{aligned} A_k &= \begin{bmatrix} 1+b, & 1+c, & 1+d, & 1+2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d \end{bmatrix}_k, \\ B_k &= \begin{bmatrix} \alpha, & \beta, & \gamma, & 2+2a-\alpha-\beta-\gamma \\ 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma, & \alpha+\beta+\gamma-a-1 \end{bmatrix}_k, \end{aligned}$$

it is almost trivial to factorize their differences

$$\begin{aligned} \nabla A_k &= \begin{bmatrix} b, & c, & d, & 2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d \end{bmatrix}_k \\ &\times \frac{(a+2k)(a-b-c)(a-b-d)(a-c-d)}{bcd(2a-b-c-d)}, \\ \Delta B_k &= \begin{bmatrix} \alpha, & \beta, & \gamma, & 2+2a-\alpha-\beta-\gamma \\ 2+a-\alpha, & 2+a-\beta, & 2+a-\gamma, & \alpha+\beta+\gamma-a \end{bmatrix}_k \\ &\times \frac{(1+a+2k)(1+a-\alpha-\beta)(1+a-\alpha-\gamma)(1+a-\beta-\gamma)}{(1+a-\alpha)(1+a-\beta)(1+a-\gamma)(1+a-\alpha-\beta-\gamma)}, \end{aligned}$$

and determine the boundary condition

$$A_{-1}B_0 = \frac{(a-b)(a-c)(a-d)(b+c+d-a)}{bcd(2a-b-c-d)}.$$

When $1+3a = b+c+d+\alpha+\beta+\gamma$, we can also evaluate, by making use of the asymptotic relation (1.4), the following limits

$$\begin{aligned} [AB]_+ = \lim_{k\to\infty} A_k B_{k+1} &= \lim_{k\to\infty} \begin{bmatrix} 1+b, & 1+c, & 1+d, & 1+2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d \end{bmatrix}_k \\ &\times \begin{bmatrix} \alpha, & \beta, & \gamma, & 2+2a-\alpha-\beta-\gamma \\ 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma, & \alpha+\beta+\gamma-a-1 \end{bmatrix}_{k+1} \\ &= \Gamma\begin{bmatrix} 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d \\ 1+b, & 1+c, & 1+d, & 1+2a-b-c-d \end{bmatrix} \\ &\times \Gamma\begin{bmatrix} 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma, & \alpha+\beta+\gamma-a-1 \\ \alpha, & \beta, & \gamma, & 2+2a-\alpha-\beta-\gamma \end{bmatrix} \\ &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{2a-b-c-d}\Gamma\begin{bmatrix} 1+a-\alpha, 1+a-\beta, 1+a-\gamma \\ 1+b, 1+c, 1+d, \alpha, \beta, \gamma \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
[AB]_- &= \lim_{k\to-\infty}\left[\begin{matrix}1+b, & 1+c, & 1+d, & 1+2a-b-c-d\\ 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d\end{matrix}\right]_k\\
&\qquad\times\left[\begin{matrix}\alpha, & \beta, & \gamma, & 2+2a-\alpha-\beta-\gamma\\ 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma, & \alpha+\beta+\gamma-a-1\end{matrix}\right]_{k+1}\\
&= \lim_{k\to-\infty}\left[\begin{matrix}b-a, & c-a, & d-a, & a-b-c-d\\ -b, & -c, & -d, & b+c+d-2a\end{matrix}\right]_{-k}\\
&\qquad\times\left[\begin{matrix}\alpha-a, & \beta-a, & \gamma-a, & 2+a-\alpha-\beta-\gamma\\ 1-\alpha, & 1-\beta, & 1-\gamma, & \alpha+\beta+\gamma-1-2a\end{matrix}\right]_{-k-1}\\
&= \Gamma\left[\begin{matrix}-b, & -c, & -d, & b+c+d-2a\\ b-a, & c-a, & d-a, & a-b-c-d\end{matrix}\right]\\
&\qquad\times\Gamma\left[\begin{matrix}1-\alpha, & 1-\beta, & 1-\gamma, & \alpha+\beta+\gamma-1-2a\\ \alpha-a, & \beta-a, & \gamma-a, & 2+a-\alpha-\beta-\gamma\end{matrix}\right]\\
&= \frac{1}{b+c+d-2a}\Gamma\left[\begin{matrix}-b,-c,-d,1-\alpha,1-\beta,1-\gamma\\ b-a,c-a,d-a,\alpha-a,\beta-a,\gamma-a\end{matrix}\right].
\end{aligned}
$$

They will be employed, in this section, to investigate two well–poised series: first the unilateral series ${}_7H_7^+(1)$ and then the bilateral one ${}_7H_7(1)$.

2.1. Unilateral series

For the seven complex parameters $\{a,b,c,d,\alpha,\beta,\gamma\}$ satisfying the linear condition $1+3a=b+c+d+\alpha+\beta+\gamma$, define the well–poised series by

$$
\begin{aligned}
&W(a;b,c,d,\alpha,\beta,\gamma)\\
&:= {}_7H_7^+\left[\begin{matrix}1+\frac{a}{2}, & b, & c, & d, & \alpha, & \beta, & \gamma\\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma\end{matrix}\middle|1\right]
\end{aligned}
$$

which is absolutely convergent because the sum of the parameters in denominator exceeds that in numerator by "3".

According to the modified Abel lemma on summation by parts, we can express the W-series as follows:

$$
\begin{aligned}
W(a;b,c,d,\alpha,\beta,\gamma) &= \frac{bcd(2a-b-c-d)}{a(a-b-c)(a-b-d)(a-c-d)}\sum_{k=0}^{\infty}B_k\nabla A_k\\
&= \frac{bcd(2a-b-c-d)}{a(a-b-c)(a-b-d)(a-c-d)}\Big\{[AB]_+ - A_{-1}B_0+\sum_{k=0}^{\infty}A_k\Delta B_k\Big\}.
\end{aligned}
$$

Observing that

$$
\begin{aligned}
\sum_{k=0}^{\infty}A_k\Delta B_k =& W(a+1;b+1,c+1,d+1,\alpha,\beta,\gamma)\\
&\times\frac{(1+a)(1+a-\alpha-\beta)(1+a-\alpha-\gamma)(1+a-\beta-\gamma)}{(1+a-\alpha)(1+a-\beta)(1+a-\gamma)(1+a-\alpha-\beta-\gamma)}
\end{aligned}
$$

we derive the following recurrence relation

$$\begin{aligned}
W(a;b,c,d,\alpha,\beta,\gamma) &= W(a+1;b+1,c+1,d+1,\alpha,\beta,\gamma)\\
&\times \frac{(1+a)bcd(1+a-\alpha-\beta)(1+a-\alpha-\gamma)(1+a-\beta-\gamma)}{a(b+c-a)(b+d-a)(c+d-a)(1+a-\alpha)(1+a-\beta)(1+a-\gamma)}\\
&+\frac{(a-b)(a-c)(a-d)(b+c+d-a)}{a(b+c-a)(b+d-a)(c+d-a)}+\frac{bcd}{a(a-b-c)(a-b-d)(a-c-d)}\\
&\times \Gamma\left[\begin{matrix}1+a-b,1+a-c,1+a-d,1+a-\alpha,1+a-\beta,1+a-\gamma\\ 1+b,1+c,1+d,\alpha,\beta,\gamma\end{matrix}\right].
\end{aligned}$$

Keep in mind that this recursion does not alter the convergence condition because the linear restriction $1+3a=b+c+d+\alpha+\beta+\gamma$ remains invariant. Iterating this relation m-times, we get the transformation formula below that expresses the W-series in terms of another well–poised sum.

Proposition 2.1 ($m\in\mathbb{N}_0$: $1+3a=b+c+d+\alpha+\beta+\gamma$)**.**

$$\begin{aligned}
W(a;b,c,d,\alpha,\beta,\gamma) &= W(a+m;b+m,c+m,d+m,\alpha,\beta,\gamma)\\
&\times\left[\begin{matrix}1+a,b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ a,b+c-a,b+d-a,c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\right]_m\\
&+\frac{(a-b)(a-c)(a-d)(b+c+d-a)}{a(b+c-a)(b+d-a)(c+d-a)}\sum_{k=0}^{m-1}\frac{b+c+d-a+2k}{b+c+d-a}\\
&\times\left[\begin{matrix}b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ 1+b+c-a,1+b+d-a,1+c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\right]_k\\
&+\frac{(a-b)(a-c)(a-d)}{a(b+c-a)(b+d-a)(c+d-a)}\\
&\times\Gamma\left[\begin{matrix}a-b,a-c,a-d,1+a-\alpha,1+a-\beta,1+a-\gamma\\ b,c,d,\alpha,\beta,\gamma\end{matrix}\right]\\
&\times\sum_{k=0}^{m-1}\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ 1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\right]_k.
\end{aligned}$$

By means of the Weierstrass M-test on uniformly convergent series (cf. Stromberg [22, §3.106]), it is not difficult to determine the following limit:

$$\lim_{m\to\infty} W(a+m;b+m,c+m,d+m,\alpha,\beta,\gamma) = {}_3H_3^+\left[\begin{matrix}\alpha, & \beta, & \gamma\\ 1+a-b, & 1+a-c, & 1+a-d\end{matrix}\middle|1\right].$$

Therefore, we have established the reciprocal relation involving two nonterminating well–poised series and two partial ${}_3H_3^+$-series.

Theorem 2.2 $(1+3a=b+c+d+\alpha+\beta+\gamma)$**.**

$$\begin{aligned}&W(a;b,c,d,\alpha,\beta,\gamma)\times a(b+c-a)(b+d-a)(c+d-a)\\=&W(b+c+d-a;b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma)\\&\times(a-b)(a-c)(a-d)(b+c+d-a)\\&+{}_3H_3^+\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\Big|1\right]\\&\times\Gamma\left[\begin{matrix}1+a-b,1+a-c,1+a-d,1+a-\alpha,1+a-\beta,1+a-\gamma\\b,c,d,\alpha,\beta,\gamma\end{matrix}\right]\\&+{}_3H_3^+\left[\begin{matrix}\alpha,\quad\beta,\quad\gamma\\1+a-b,1+a-c,1+a-d\end{matrix}\Big|1\right]\\&\times\Gamma\left[\begin{matrix}1+b+c-a,1+b+d-a,1+c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\\b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\end{matrix}\right].\end{aligned}$$

This theorem has the following three remarkable implications.

- For $c\to a$, we derive an expression for the nonterminating ${}_7F_6$-series

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}a,1+\frac{a}{2},\quad b,\quad d,\quad\alpha,\quad\beta,\quad\gamma\\\frac{a}{2},1+a-b,1+a-d,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\Big|1\right]\\&=\frac{1}{b+d-a}{}_4F_3\left[\begin{matrix}1,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\1+b,1+b+d-a,1+d\end{matrix}\Big|1\right]\\&\quad\times\Gamma\left[\begin{matrix}1+a-b,1+a-d,1+a-\alpha,1+a-\beta,1+a-\gamma\\1+a,1+b,1+d,\alpha,\beta,\gamma\end{matrix}\right]\\&+\frac{1}{b+d-a}{}_3F_2\left[\begin{matrix}\alpha,\quad\beta,\quad\gamma\\1+a-b,1+a-d\end{matrix}\Big|1\right]\\&\quad\times\Gamma\left[\begin{matrix}1+b+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\\1+a,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\end{matrix}\right],\end{aligned}$$

  where the parameters satisfy the condition $1+2a=b+d+\alpha+\beta+\gamma$.
- Letting further $\alpha\to-n$ with $n\in\mathbb{N}_0$ in the above transformation, the first term on the right is annihilated, while the second one can be evaluated by the Pfaff–Saalschütz theorem (cf. Bailey [1, §2.2])

$${}_3F_2\left[\begin{matrix}-n,\quad\beta,\quad\gamma\\1+a-b,1+a-d\end{matrix}\Big|1\right]=\left[\begin{matrix}1+a-b-\gamma,1+a-d-\gamma\\1+a-b,1+a-d\end{matrix}\right]_n.$$

  Consequently, we recover the identity discovered by Dougall [16, :]

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}a,1+\frac{a}{2},\quad b,\quad d,\quad\beta,\quad\gamma,\quad -n\\\frac{a}{2},1+a-b,1+a-d,1+a-\beta,1+a-\gamma,1+a+n\end{matrix}\Big|1\right]\\&=\left[\begin{matrix}1+a,1+a-b-d,1+a-b-\gamma,1+a-d-\gamma\\1+a-b,1+a-d,1+a-\gamma,1+a-b-d-\gamma\end{matrix}\right]_n,\end{aligned}$$

  where the series is 2-balanced because $1+2a+n=b+d+\beta+\gamma$.
- When the series terminates above by $c\to-m$ with $m\in\mathbb{N}_0$, we get the following reciprocal formula $(1+3a+m=b+d+\alpha+\beta+\gamma)$:

$$\begin{aligned}W(a;b,d,-m,\alpha,\beta,\gamma)&=\frac{(a-b)(a+m)(a-d)(b+d-a-m)}{a(b-a-m)(b+d-a)(d-a-m)}\\&\times W(b+d-a-m;b,d,-m,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma).\end{aligned}$$

  Dougall's formula just displayed for terminating ${}_7F_6$-series can also be obtained by letting $d\to a$ in the above transformation, because in this case, there remains the only surviving end term for the W-sum on the right.

2.2. Bilateral series

Under the same parameter restriction $1 + 3a = b + c + d + \alpha + \beta + \gamma$, define the corresponding bilateral series

$$\begin{aligned}&\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma)\\ &:= {}_7H_7\left[\begin{matrix}1+\frac{a}{2}, & b, & c, & d, & \alpha, & \beta, & \gamma\\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-\alpha, & 1+a-\beta, & 1+a-\gamma\end{matrix}\middle|1\right].\end{aligned}$$

In the view of the modified Abel lemma on summation by parts, this $\mathcal{W}$-series can be manipulated as follows:

$$\begin{aligned}\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma) &= \frac{bcd(2a-b-c-d)}{a(a-b-c)(a-b-d)(a-c-d)}\sum_{k=-\infty}^{\infty}B_k\nabla A_k\\ &= \frac{bcd(2a-b-c-d)}{a(a-b-c)(a-b-d)(a-c-d)}\Big\{[AB]_+-[AB]_-+\sum_{k=-\infty}^{\infty}A_k\Delta B_k\Big\}.\end{aligned}$$

Analogously, taking into account

$$\begin{aligned}\sum_{k=-\infty}^{\infty}A_k\Delta B_k =&\mathcal{W}(a+1;b+1,c+1,d+1,\alpha,\beta,\gamma)\\ &\times\frac{(1+a)(1+a-\alpha-\beta)(1+a-\alpha-\gamma)(1+a-\beta-\gamma)}{(1+a-\alpha)(1+a-\beta)(1+a-\gamma)(1+a-\alpha-\beta-\gamma)}\end{aligned}$$

and the difference

$$[AB]_+-[AB]_- = \frac{\Theta(a;b,c,d,\alpha,\beta,\gamma)}{bcd(b+c+d-2a)},$$

where Θ-function is given by the difference of two Γ-function quotients

$$\begin{aligned}\Theta(a;b,c,d,\alpha,\beta,\gamma) &= \Gamma\left[\begin{matrix}1-b,1-c,1-d,1-\alpha,1-\beta,1-\gamma\\ b-a,c-a,d-a,\alpha-a,\beta-a,\gamma-a\end{matrix}\right]\\ &\times\Big\{1-\sin\pi\left[\begin{matrix}b, & c, & d, & \alpha, & \beta, & \gamma\\ a-b,&a-c,&a-d,&a-\alpha,&a-\beta,&a-\gamma\end{matrix}\right]\Big\}\end{aligned}$$

we derive the following recurrence relation

$$\begin{aligned}&\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma) = \mathcal{W}(a+1;b+1,c+1,d+1,\alpha,\beta,\gamma)\\ &\times\frac{(1+a)bcd(1+a-\alpha-\beta)(1+a-\alpha-\gamma)(1+a-\beta-\gamma)}{a(b+c-a)(b+d-a)(c+d-a)(1+a-\alpha)(1+a-\beta)(1+a-\gamma)}\\ &\qquad+\frac{\Theta(a;b,c,d,\alpha,\beta,\gamma)}{a(b+c-a)(b+d-a)(c+d-a)}.\end{aligned}$$

Iterating this relation m-times, we deduce the transformation formula.

Lemma 2.3 ($m \in \mathbb{N}_0$: $1 + 3a = b + c + d + \alpha + \beta + \gamma$)**.**

$$\begin{aligned}&\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma) = \mathcal{W}(a+m;b+m,c+m,d+m,\alpha,\beta,\gamma)\\ &\quad\times\left[\begin{matrix}1+a,b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ a,b+c-a,b+d-a,c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\right]_m\\ &+\frac{\Theta(a;b,c,d,\alpha,\beta,\gamma)}{a(b+c-a)(b+d-a)(c+d-a)}\sum_{k=0}^{m-1}\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ 1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\right]_k.\end{aligned}$$

For this transformation formula, we examine now its limiting case as $\alpha \to -\infty$. In order to avoid confusion, perform the replacement $\alpha \to \alpha - n$ with $n \in \mathbb{N}$. Under the substitution $c = 1+3a-b-d-\alpha-\beta-\gamma+n$ and the condition $1+\Re(2a-b-d-\beta-\gamma) > 0$, we have to make, as $n \to \infty$, term by term estimations. For the two $\mathcal{W}$-series, it is routine to have

$$\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma) \Longrightarrow {}_5H_5\left[\begin{matrix}1+\frac{a}{2}, & b, & d, & \beta, & \gamma\\ \frac{a}{2}, & 1+a-b, & 1+a-d, & 1+a-\beta, & 1+a-\gamma\end{matrix}\,\middle|\,1\right],$$

$$\mathcal{W}(a+m;b+m,c+m,d+m,\alpha,\beta,\gamma)$$
$$\Longrightarrow {}_5H_5\left[\begin{matrix}1+\frac{a+m}{2}, & b+m, & d+m, & \beta, & \gamma\\ \frac{a+m}{2}, & 1+a-b, & 1+a-d, & 1+a-\beta+m, & 1+a-\gamma+m\end{matrix}\,\middle|\,1\right],$$

where both series have the same convergent condition $1+\Re(2a-b-d-\beta-\gamma)>0$. By making use of (1.4), we can reduce the quotient of shifted factorials to

$$\left[\begin{matrix}1+a,b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ a,b+c-a,b+d-a,c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\right]_m$$
$$\Longrightarrow \left[\begin{matrix}1+a,b,d,1+a-\beta-\gamma\\ a,b+d-a,1+a-\beta,1+a-\gamma\end{matrix}\right]_m.$$

Because the sum over $0\le k<m$ is bounded, the ultimate term will vanish:

$$\frac{\Theta(a;b,c,d,\alpha,\beta,\gamma)}{a(b+c-a)(b+d-a)(c+d-a)}\sum_{k=0}^{m-1}\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\ 1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\right]_k$$
$$\Longrightarrow \mathcal{O}\Big(\frac{1}{n^2}\Big)\times\Gamma\left[\begin{matrix}1-\alpha+n,b+d+\alpha+\beta+\gamma-3a-n\\ \alpha-a-n,1+2a-b-d-\alpha-\beta-\gamma+n\end{matrix}\right]$$
$$\approx\ \mathcal{O}\big(n^{-2\Re(1+2a-b-d-\beta-\gamma)}\big)=o(1)\quad\text{for}\quad 1+\Re(2a-b-d-\beta-\gamma)>0.$$

Therefore, we have derived the following transformation formula.

Proposition 2.4 ($m\in\mathbb{N}_0$: $1+\Re(2a-b-d-\beta-\gamma)>0$)**.**

$${}_5H_5\left[\begin{matrix}1+\frac{a}{2}, & b, & d, & \beta, & \gamma\\ \frac{a}{2}, & 1+a-b, & 1+a-d, & 1+a-\beta, & 1+a-\gamma\end{matrix}\,\middle|\,1\right]$$
$$={}_5H_5\left[\begin{matrix}1+\frac{a+m}{2}, & b+m, & d+m, & \beta, & \gamma\\ \frac{a+m}{2}, & 1+a-b, & 1+a-d, & 1+a-\beta+m, & 1+a-\gamma+m\end{matrix}\,\middle|\,1\right]$$
$$\times\left[\begin{matrix}1+a,b,d,1+a-\beta-\gamma\\ a,b+d-a,1+a-\beta,1+a-\gamma\end{matrix}\right]_m.$$

This transformation implies the following important formula for the well–poised ${}_5H_5$-series discovered by Dougall [16, :]

$${}_5H_5\left[\begin{matrix}1+\frac{a}{2}, & b, & c, & d, & e\\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\,\middle|\,1\right]$$
$$=\Gamma\left[\begin{matrix}1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1-b,\ 1-c,\ 1-d,\ 1-e,\ 1+2a-b-c-d-e\\ 1+a,1-a,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e\end{matrix}\right].$$

In fact, by letting $m\to\infty$ in Proposition 2.4, we have the limiting form

$${}_5H_5\left[\begin{matrix}1+\frac{a}{2}, & b, & d, & \beta, & \gamma\\ \frac{a}{2}, & 1+a-b, & 1+a-d, & 1+a-\beta, & 1+a-\gamma\end{matrix}\,\middle|\,1\right]$$
$$=\Gamma\left[\begin{matrix}a,b+d-a,1+a-\beta,1+a-\gamma\\ 1+a,b,d,1+a-\beta-\gamma\end{matrix}\right]\Big\{{}_2H_2\left[\begin{matrix}\beta, & \gamma\\ 1+a-b, & 1+a-d\end{matrix}\,\middle|\,1\right]$$
$$-\Gamma\left[\begin{matrix}b, & d, & 1-\beta, & 1-\gamma\\ b-a, & d-a, & 1+a-\beta, & 1+a-\gamma\end{matrix}\right]{}_2H_2\left[\begin{matrix}b, & d\\ 1+a-\beta, & 1+a-\gamma\end{matrix}\,\middle|\,1\right]\Big\}.$$

Evaluating the both ${}_2H_2$-series by another formula of Dougall [16]

$$ {}_2H_2\left[\begin{matrix}a,b\\c,d\end{matrix}\middle|1\right] = \Gamma\left[\begin{matrix}1-a,1-b,c,d,c+d-a-b-1\\c-a,\ d-a,\ c-b,\ d-b\end{matrix}\right], $$

where $\Re(c+d-a-b)>1$, and then simplifying the result, we reconfirm the identity for ${}_5H_5$-series. □

Furthermore, by following exactly the same procedure for the bilateral well–poised ${}_5H_5$-series as that shown by the author [10, §3], we can evaluate the limit

$$ \begin{aligned} &\lim_{m\to\infty}\mathcal{W}(a+m;b+m,c+m,d+m,\alpha,\beta,\gamma)\\ &= {}_3H_3\left[\begin{matrix}\alpha,\quad \beta,\quad \gamma\\1+a-b,1+a-c,1+a-d\end{matrix}\middle|1\right]\\ &- {}_3H_3\left[\begin{matrix}b,\quad c,\quad d\\1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\middle|1\right]\\ &\times\ \Gamma\left[\begin{matrix}b,c,d,1-\alpha,1-\beta,1-\gamma\\b-a,c-a,d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\right]. \end{aligned} $$

This leads us to the transformation

$$ \begin{aligned} a\mathcal{W}(a;b,c,d,\alpha,\beta,\gamma) &= {}_3H_3\left[\begin{matrix}\alpha,\quad \beta,\quad \gamma\\1+a-b,1+a-c,1+a-d\end{matrix}\middle|1\right]\\ \times\Gamma&\left[\begin{matrix}b+c-a,b+d-a,c+d-a,1+a-\alpha,1+a-\beta,1+a-\gamma\\b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\end{matrix}\right]\\ &- {}_3H_3\left[\begin{matrix}b,\quad c,\quad d\\1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\middle|1\right]\\ \times\Gamma&\left[\begin{matrix}b+c-a,b+d-a,c+d-a,1-\alpha,1-\beta,1-\gamma\\b-a,c-a,d-a,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\end{matrix}\right]\\ +&\frac{\Theta(a;b,c,d,\alpha,\beta,\gamma)}{(b+c-a)(b+d-a)(c+d-a)}{}_3H_3^{+}\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\middle|1\right]. \end{aligned} $$

By making the replacement $k\to -k$, we have the reciprocal relation

$$ \begin{aligned} &-\sum_{k<0}\frac{(1+a-\alpha-\beta)_k(1+a-\alpha-\gamma)_k(1+a-\beta-\gamma)_k}{(b+c-a)_{k+1}(b+d-a)_{k+1}(c+d-a)_{k+1}}\\ &=\sum_{k\ge 0}\frac{(1+a-b-c)_k(1+a-b-d)_k(1+a-c-d)_k}{(\alpha+\beta-a)_{k+1}(\alpha+\gamma-a)_{k+1}(\beta+\gamma-a)_{k+1}}. \end{aligned} $$

Therefore, the difference between the penultimate equation and its reformulated one under the parameter exchanges $\{b\rightleftharpoons\alpha, c\rightleftharpoons\beta, d\rightleftharpoons\gamma\}$ results, after some simplifications, the three term relation among the three bilateral series with the parameter excess "2".

Corollary 2.5 $(1+3a=b+c+d+\alpha+\beta+\gamma)$**.**

$$ \begin{aligned} \Theta(a;b,c,d,\alpha,\beta,\gamma)\times{}_3H_3&\left[\begin{matrix}1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma\\1+b+c-a,1+b+d-a,1+c+d-a\end{matrix}\middle|1\right]\\ &= {}_3H_3\left[\begin{matrix}\alpha,\quad \beta,\quad \gamma\\1+a-b,1+a-c,1+a-d\end{matrix}\middle|1\right]\\ &\times\Theta(b+c+d-a;b,c,d,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma)\\ &-{}_3H_3\left[\begin{matrix}b,\quad c,\quad d\\1+a-\alpha,1+a-\beta,1+a-\gamma\end{matrix}\middle|1\right]\\ &\times\Theta(1+a-\alpha-\beta-\gamma;b-a,c-a,d-a,1+a-\alpha-\beta,1+a-\alpha-\gamma,1+a-\beta-\gamma). \end{aligned} $$

3. Well–poised ${}_6H_6(-1)$ series

For the two sequences $\{A_k, B_k\}$ defined by

$$A_k := \begin{bmatrix} 1+b, & 1+c, & 1+3a-b-4c \\ 1+a-b, & 1+a-c, & 1-2a+b+4c \end{bmatrix}_k (-1)^k,$$
$$B_k := \begin{bmatrix} d, & 2a-b-2c, & 1+2a-2c-d, & 1-2a+b+4c \\ 1+a-d, & 1-a+b+2c, & 2c+d-a, & 3a-b-4c \end{bmatrix}_k;$$

it is routine to compute their differences

$$\nabla A_k = \begin{bmatrix} b, & c, & 3a-b-4c \\ 1+a-b, & 1+a-c, & 1-2a+b+4c \end{bmatrix}_k (-1)^k \times \frac{(a+2k)(2a-b-2c+k)(b+2c-a+k)}{bc(3a-b-4c)},$$
$$\Delta B_k = \begin{bmatrix} d, & 2a-b-2c, & 1+2a-2c-d, & 1-2a+b+4c \\ 2+a-d, & 2-a+b+2c, & 1+2c+d-a, & 1+3a-b-4c \end{bmatrix}_k \times \frac{(1+a+2k)(a-2c)(1-a+b+2c-d)(3a-b-4c-d)}{(1+a-d)(1-a+b+2c)(a-2c-d)(3a-b-4c)};$$

and determine the boundary value

$$A_{-1}B_0 = \frac{(a-b)(a-c)(2a-b-4c)}{bc(3a-b-4c)}.$$

Under the condition $\Re(a-2c) < 0$, we can also show, by making use of (1.4), the two limiting relations

$$[AB]_+ = \lim_{n\to\infty} A_n B_{n+1} = \lim_{n\to\infty} \mathcal{O}\big(n^{3\Re(a-2c)}\big) = 0,$$
$$[AB]_- = \lim_{n\to-\infty} A_n B_{n+1} = \lim_{n\to\infty} \mathcal{O}\big(n^{3\Re(a-2c)}\big) = 0.$$

They will be utilized in this section to establish summation and transformation formulae for the alternating well–poised series ${}_6H_6^+(-1)$ and ${}_6H_6(-1)$.

3.1. Unilateral series

For the four complex parameters $\{a, b, c, d\}$ subject to the condition $\Re(a-2c) < 0$, define the well–poised series by

$$\Omega(a; b, c, d) := {}_6H_6^+ \left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & 1+2a-b-2c, & 1+2a-2c-d \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & b-a+2c, & d-a+2c \end{matrix} \,\Big|\, -1 \right].$$

By applying the modified Abel lemma on summation by parts, we can reformulate the Ω-series as follows:

$$\Omega(a; b, c, d) = \frac{bc(3a-b-4c)}{a(2a-b-2c)(b+2c-a)} \sum_{k=0}^{\infty} B_k \nabla A_k$$
$$= \frac{bc(3a-b-4c)}{a(2a-b-2c)(b+2c-a)} \Big\{ [AB]_+ - A_{-1}B_0 + \sum_{k=0}^{\infty} A_k \Delta B_k \Big\}.$$

Taking into account that

$$\sum_{k=0}^{\infty} A_k \Delta B_k = \Omega(1+a; 1+b, 1+c, d) \times \frac{(1+a)(a-2c)(1-a+b+2c-d)(3a-b-4c-d)}{(1+a-d)(1-a+b+2c)(a-2c-d)(3a-b-4c)}$$

we derive the recurrence relation

$$\Omega(a;b,c,d) = \frac{(a-b)(a-c)(2a-b-4c)}{a(2a-b-2c)(a-b-2c)} + \Omega(1+a;1+b,1+c,d) \times \frac{(1+a)bc(2c-a)(1+b+2c-a-d)(b+4c-3a+d)}{a(1+a-d)(b+2c-a)(1-a+b+2c)(b+2c-2a)(d+2c-a)}.$$

Iterating this relation m-times yields further the transformation formula.

Lemma 3.1 ($m \in \mathbb{N}_0$: $\Re(a-2c) < 0$)**.**

$$\begin{aligned}\Omega(a;b,c,d) &= \Omega(a+m;b+m,c+m,d) \\ &\times \left[\begin{matrix} 1+a, & b, & c, & 2c-a, \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \\ a, 1+a-d, b+2c-2a, d+2c-a, \frac{b+2c-a}{2}, \frac{1+b+2c-a}{2} \end{matrix}\right]_m \\ &+ \frac{(a-b)(a-c)(2a-b-4c)}{a(a-b-2c)(2a-b-2c)} \sum_{k=0}^{m-1} \frac{b+4c-2a+3k}{b+4c-2a} \\ &\times \left[\begin{matrix} b, & c, & 2c-a, \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \\ 1+a-d, 1+b+2c-2a, d+2c-a, \frac{1+b+2c-a}{2}, \frac{2+b+2c-a}{2} \end{matrix}\right]_k.\end{aligned}$$

Its limiting relation as $m \to \infty$ result in the expression.

Proposition 3.2 ($\Re(a-2c) < 0$)**.**

$$\begin{aligned}\Omega(a;b,c,d) &= {}_2H_2^+\left[\begin{matrix} d, 1+2a-2c-d \\ 1+a-b, 1+a-c \end{matrix} \middle| \frac{1}{2}\right] \\ &\times \Gamma\left[\begin{matrix} a, 1+a-d, b+2c-2a, d+2c-a, \frac{b+2c-a}{2}, \frac{1+b+2c-a}{2} \\ 1+a, \quad b, \quad c, \quad 2c-a, \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \end{matrix}\right] \\ &+ \frac{(a-b)(a-c)(2a-b-4c)}{a(a-b-2c)(2a-b-2c)} \sum_{k=0}^{\infty} \frac{b+4c-2a+3k}{b+4c-2a} \\ &\times \left[\begin{matrix} b, & c, & 2c-a, & \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \\ \frac{2+b+2c-a}{2}, 1+b+2c-2a, \frac{1+b+2c-a}{2}, & d+2c-a, & 1+a-d \end{matrix}\right]_k.\end{aligned}$$

The last infinite series is "almost quadratic" and convergent, because the sum of its parameters in denominator exceeds that in numerator by "2".

When $b=a$ and $c=a$, the ${}_2H_2^+$-series on the right can be evaluated respectively by the two formulae for the ${}_2F_1(\frac{1}{2})$-series due to Gauss and Bailey (cf. Bailey [1, §2.4])

$$\begin{aligned}{}_2F_1\left[\begin{matrix} d, 1+2a-2c-d \\ 1+a-c \end{matrix} \middle| \frac{1}{2}\right] &= \Gamma\left[\begin{matrix} \frac{1}{2}, 1+a-c \\ 1+a-c-\frac{d}{2}, \frac{1+d}{2} \end{matrix}\right], \\ {}_2F_1\left[\begin{matrix} d, 1-d \\ 1+a-b \end{matrix} \middle| \frac{1}{2}\right] &= \Gamma\left[\begin{matrix} \frac{1+a-b}{2}, \frac{2+a-b}{2} \\ \frac{1+a-b+d}{2}, \frac{2+a-b-d}{2} \end{matrix}\right].\end{aligned}$$

We recover hence the next two identities found by Whipple [25, Equations 14.1 and 15.73], where the first one can also be found in Bailey [1, Page 97].

$$\begin{aligned}\Omega(a;b,a,d) &= {}_6F_5\left[\begin{matrix} a, 1+\frac{a}{2}, & b, & d, & 1-b, & 1-d \\ \frac{a}{2}, & 1+a-b, & 1+a-d, & a+b, & a+d \end{matrix} \middle| -1\right] \\ &= \Gamma\left[\begin{matrix} a+d, 1+a-d, \frac{a+b}{2}, \frac{1+a+b}{2}, \frac{1+a-b}{2}, \frac{2+a-b}{2} \\ 1+a, a, \frac{a+b+d}{2}, \frac{1+a+b-d}{2}, \frac{1+a-b+d}{2}, \frac{2+a-b-d}{2} \end{matrix}\right],\end{aligned}$$

$$\Omega(a;a,c,d) = {}_6F_5\left[\begin{matrix} a, 1+\frac{a}{2}, & c,\ 1+a-2c, & d,\ 1+2a-2c-d \\ \frac{a}{2}, & 2c,\ 1+a-c, & 1+a-d,\ 2c+d-a \end{matrix}\Big| -1\right]$$
$$= \Gamma\left[\begin{matrix} \frac{1}{2}, c+\frac{1}{2}, 1+a-c, 1+a-d, 2c-a+d \\ 1+a, \frac{1+d}{2}, \frac{1-d}{2}+c, 1+a-c-\frac{d}{2}, 2c-a+\frac{d}{2} \end{matrix}\right].$$

3.2. Bilateral series

Denote further the corresponding bilateral series by

$$\omega(a;b,c,d) := {}_6H_6\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & 1+2a-b-2c, 1+2a-2c-d \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, b-a+2c, d-a+2c \end{matrix}\Big| -1\right].$$

Then by carrying out the same procedure for the bilateral ${}_7H_7$-series, we find the following elegant transformation without remainder terms

$$\omega(a;b,c,d) = \omega(a+m;b+m,c+m,d)$$
$$\times\left[\begin{matrix} 1+a, & b, & c, & 2c-a, \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \\ a, 1+a-d, b+2c-2a, d+2c-a, \frac{b+2c-a}{2}, \frac{1+b+2c-a}{2} \end{matrix}\right]_m.$$

Its limiting form as $m \to \infty$ reads as

$$\omega(a;b,c,d) = \lim_{m\to\infty} \omega(a+m;b+m,c+m,d)$$
$$\times\Gamma\left[\begin{matrix} a, 1+a-d, b+2c-2a, d+2c-a, \frac{b+2c-a}{2}, \frac{1+b+2c-a}{2} \\ 1+a, & b, & c, & 2c-a, \frac{1+b+2c-a-d}{2}, \frac{b+4c-3a+d}{2} \end{matrix}\right].$$

This should lead us to the following summation formula discovered by M. Jackson [18, 1952] (see also Chu et al. [14] by the Cauchy residue method)

$$\omega(a;b,c,d) = \Gamma\left[\begin{matrix} \frac{1+a-b-d}{2}, \frac{1-3a+b+d}{2}+2c \\ \frac{1-a+b-d}{2}+c, \frac{1-a-b+d}{2}+c \end{matrix}\right]\left\{1+\sin\pi\left[\begin{matrix} a-b-c, a-c-d \\ c, c-a \end{matrix}\right]\right\}$$
$$\times\Gamma\left[\begin{matrix} 1+a-b, 1+a-d, 1-b, 1-d, b-a+2c, d-a+2c, b-2a+2c, d-2a+2c \\ 1+a, 1-a, c, c-a, 2c-a, 1+a-b-d, b+d-3a+4c \end{matrix}\right]$$

who derived it by applying Dixon's summation theorem [15] and its bilateral extension for well–poised ${}_3F_2$ and ${}_3H_3$-series to the transformation due to Bailey [2]

$${}_6H_6\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & e, & f \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f \end{matrix}\Big| -1\right]$$
$$= \Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1-d, 1-e, 1-f \\ 1+a, 1-a, 1+a-b-c, d-a, e-a, f-a \end{matrix}\right]$$
$$\times\left\{\Gamma\left[\begin{matrix} 1+2a-d-e-f, d-a, e-a, f-a \\ 1+a-d-e, 1+a-d-f, 1+a-e-f \end{matrix}\right]\right.$$
$$\times\, {}_3H_3\left[\begin{matrix} b, c, 1+2a-d-e-f \\ 1+a-d, 1+a-e, 1+a-f \end{matrix}\Big| 1\right]$$
$$+\ \Gamma\left[\begin{matrix} 1-b, 1-c, 2+2a-d-e-f, d+e+f-1-2a \\ 2+2a-b-d-e-f, 2+2a-c-d-e-f \end{matrix}\right]$$
$$\left.\times\, {}_3F_2\left[\begin{matrix} 1+a-d-e, 1+a-d-f, 1+a-e-f \\ 2+2a-b-d-e-f, 2+2a-c-d-e-f \end{matrix}\Big| 1\right]\right\}.$$

Problem. Even though there is no doubt about the existence for the limit of $\omega(a+m;b+m,c+m,d)$ as $m \to \infty$, it remains, however, an intriguing question to determine this limit directly, but without appealing to M. Jackson's evaluation formula.

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