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Research Article

An explicit version of the correspondence between Noetherian modules with Cartier maps and Artinian modules with Frobenius maps

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Abstract

In this paper, we introduce an explicit version of the correspondence between Noetherian modules with Cartier maps and Artinian modules with Frobenius maps which recovers the correspondence given by R. Y. Sharp and Y. Yoshino, and the correspondence given by M. Blickle and G. Böckle.

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1. Introduction

Throughout this paper our general assumption on R is to be a commutative Noetherian regular ring of prime characteristic p. Let e be a positive integer. Let $f : R \to R$ be the Frobenius homomorphism defined by $f(r) = r^p$ for all $r \in R$, whose e-th iteration is denoted by f^e. If the Frobenius homomorphism is finite, R is said to be F-finite. Let M be an R-module. $F^e_*M = \{F^e_*m \mid m \in M\}$ denotes the Abelian group M with the induced R-module structure via the e-th iterated Frobenius and it is given by

$$rF^e_*m = F^e_*r^{p^e}m \text{ for all } m \in M \text{ and } r \in R.$$

In particular, F^e_*R is the Abelian group R with the induced R-module structure

$$rF^e_*s = F^e_*r^{p^e}s \text{ for all } r, s \in R.$$

An e-th Frobenius map on M is an R-linear map $\phi : M \to F^e_*M$, equivalently an additive map $\phi : M \to M$ such that $\phi(rm) = r^{p^e}\phi(m)$ for all $r \in R$ and $m \in M$. Let $R[X; f^e]$ be the skew-polynomial ring whose multiplication is subject to the rule $Xr = f^e(r)X = r^{p^e}X$ for all $r \in R$. Notice that defining an e-th Frobenius map on M is equivalent to endowing M with a left $R[X; f^e]$-module structure extending the rule $Xm = \phi(m)$ for all $m \in M$, where $X(rm) = \phi(rm) = r^{p^e}\phi(m) = r^{p^e}Xm = f^e(r)Xm = (Xr)m$ for all $r \in R$ and $m \in M$.

An e-th Cartier map on M is an R linear map $C : F^e_*M \to M$, equivalently an additive map $C : M \to M$ such that $rC(m) = C(r^{p^e}m)$ for all $r \in R$ and $m \in M$. An R-module M is called a Cartier module if it is equipped with a Cartier map. Notice that defining an e-th Cartier map on M is equivalent to endowing M with a right $R[X; f^e]$-module structure

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extending the rule $mX = C(m)$ for all $m \in M$, where $(mX)r = C(m)r = C(mr^{p^e}) = mr^{p^e}X = mf^e(r)X = m(Xr)$ for all $r \in R$ and $m \in M$.

When R is complete local and F-finite, it was shown by R. Y. Sharp and Y. Yoshino in [8] and independently by M. Blickle and G. Böckle in [1] that there is a bijective correspondence between the category of left $R[X; f^e]$-modules that are Artinian as R-modules and the category of right $R[X; f^e]$-modules that are Noetherian as R-modules. Sharp and Yoshino used this correspondence to translate known results about Artinian modules equipped with Frobenius maps into results about Noetherian modules equipped with Cartier maps, yet Blickle and Böckle used this correspondence to show how their results about Cartier modules globalize known results about left $R[X; f^e]$-modules that are Artinian as R-modules. (When $e = 1$, we simply drop it from notations.)

In this paper, we consider the case that $R = \Bbbk[\![x_1, ..., x_n]\!]$ is a power series ring over a perfect field $\Bbbk$ of prime characteristic p, and we introduce an explicit correspondence between Noetherian R-modules equipped with a Cartier map and Artinian R-modules equipped with a Frobenius map which coincides with the correspondences in [8] and [1], more importantly, extends to a computational level (see Theorem 5.3). To do this, we introduce an explicit isomorphism between two modules which are well-known isomorphic modules but an isomorphism has not been given explicitly before (see Lemma 4.2).

2. Noetherian modules with Cartier maps

In this section, we give an explicit formulation for Cartier maps over finitely generated modules. To do this, we start with explaining the F^e_*R-module structure of $\mathrm{Hom}_R(F^e_*R^\alpha, R^\alpha)$, where α is a positive integer. Let $R = \Bbbk[\![x_1, ..., x_n]\!]$ be a power series ring over an F-finite field $\Bbbk$ of prime characteristic p.

Remark 2.1. Let $\mathcal{C}$ be a base for $\Bbbk$ as a $\Bbbk^{p^e}$-vector space which includes the identity element of $\Bbbk$. Then F^e_*R are free R-modules with the basis set

$$\mathcal{B} = \{F^e_*\lambda x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \le \alpha_1, \dots, \alpha_n < p^e, \lambda \in \mathcal{C}\}.$$

In particular, if $\Bbbk$ is perfect then F^e_*R are free R-modules with the basis set

$$\mathcal{B} = \{F^e_*x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \le \alpha_1, \dots, \alpha_n < p^e\}.$$

Henceforth, in this section $R = \Bbbk[\![x_1, ..., x_n]\!]$ will denote a power series ring over a perfect field $\Bbbk$ of prime characteristic p.

Lemma 2.2 ([2, Example 3.0.5])**.** *Let $\pi_e : F^e_*R \to R$ be the projection map onto the free summand $RF^e_*x_1^{p^e-1} \dots x_n^{p^e-1}$. Then $\mathrm{Hom}_R(F^e_*R, R)$ is generated by π_e as an F^e_*R-module.*

Proof. For each basis element $F^e_*x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathcal{B}$, the projection map onto the free summand $RF^e_*x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is defined by the rule

$$F^e_*z.\pi_e(-) = \pi_e(F^e_*z.-),$$

where $z = x_1^{p^e-1-\alpha_1} \dots x_n^{p^e-1-\alpha_n}$. Since we can obtain all of the projections in this way, the map

$$\Phi : F^e_*R \to \mathrm{Hom}_R(F^e_*R, R) \text{ defined by } \Phi(F^e_*u) = \phi_u,$$

where $\phi_u : F^e_*R \to R$ is the R-linear map $\phi_u(-) = \pi_e(F^e_*u-)$, is surjective. On the other hand, if $\Phi(F^e_*u) = 0$ for some $u \in R$, then we have

$$\phi_u(F^e_*r) = \pi_e(F^e_*ur) = F^e_*u.\pi_e(F^e_*r) = 0 \text{ for all } r \in R.$$

This means that F^e_*u must be zero, and so Φ is injective. Hence, Φ is an F^e_*R isomorphism. In other words, π_e generates $\mathrm{Hom}_R(F^e_*R, R)$ as an F^e_*R-module. □

The following lemma extends Lemma 2.2 to free modules.

Lemma 2.3. *For any $\Psi \in \mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha)$, there exists an $\alpha \times \alpha$ matrix U with entries in R such that $\Psi(-) = \Pi_e(F^e_* U -)$ where*

$$\Pi_e \begin{pmatrix} F^e_* v_1 \\ \vdots \\ F^e_* v_\alpha \end{pmatrix} = \begin{pmatrix} \pi_e(F^e_* v_1) \\ \vdots \\ \pi_e(F^e_* v_\alpha) \end{pmatrix}$$

for all $(F_* v_1, \ldots, F^e_* v_\alpha)^t \in F^e_* R^\alpha$, and $\pi_e \in \mathrm{Hom}_R(F^e_* R, R)$ as in Lemma 2.2.

Proof. If $\alpha = 1$, by Lemma 2.2, $\mathrm{Hom}_R(F^e_* R, R)$ is generated as an $F^e_* R$-module by the map π_e. If $\alpha > 1$, we first need to describe elements of $\mathrm{Hom}_R(F^e_* R, R^\alpha)$. Since $\mathrm{Hom}_R(F^e_* R, R^\alpha) \cong \mathrm{Hom}_R(F^e_* R, R)^\alpha$, any R-linear map $\varphi \in \mathrm{Hom}_R(F^e_* R, R^\alpha)$ can be expressed as a direct sum of elements of $\mathrm{Hom}_R(F^e_* R, R)$. Therefore, we have $\varphi(-) = (\phi_1(-), \ldots, \phi_\alpha(-))^t$ for some $\phi_i \in \mathrm{Hom}_R(F^e_* R, R)$ where $1 \le i \le \alpha$, and by Lemma 2.2, $\varphi(-) = (\pi_e(F^e_* u_1 -), \ldots, \pi_e(F^e_* u_\alpha -))^t$ for some $u_1, \ldots, u_\alpha \in R$.

Since $\mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha) \cong \mathrm{Hom}_R(F^e_* R, R^\alpha)^\alpha$, any $\Psi \in \mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha)$ can be expressed as a direct sum of elements of $\mathrm{Hom}_R(F^e_* R, R^\alpha)$. Therefore, for any element $(v_1, \ldots, v_\alpha)^t \in R^\alpha$, we have

$$\Psi((F^e_* v_1, \ldots, F^e_* v_\alpha)^t) = \sum_{1 \le j \le \alpha} \varphi_j(F^e_* v_j)$$

for some $\varphi_j \in \mathrm{Hom}_R(F^e_* R, R^\alpha)$. By the previous observation of $\mathrm{Hom}_R(F^e_* R, R^\alpha)$, for each j, we also have

$$\varphi_j(F^e_* v_j) = (\pi_e(F^e_* u_{1j} v_j), \ldots, \pi_e(F^e_* u_{\alpha j} v_j))^t$$

for some elements $u_{1j}, \ldots, u_{\alpha j} \in R$. Thus,

$$\Psi \begin{pmatrix} F^e_* v_1 \\ \vdots \\ F^e_* v_\alpha \end{pmatrix} = \sum_{1 \le j \le \alpha} \begin{pmatrix} \pi_e(F^e_* u_{1j} v_j) \\ \vdots \\ \pi_e(F^e_* u_{\alpha j} v_j) \end{pmatrix}.$$

Hence, for any $\Psi \in \mathrm{Hom}_R(F^e_* R^\alpha, R^\alpha)$, there exists an $\alpha \times \alpha$ matrix U with entries $u_{ij} \in R$ such that $\Psi(-) = \Pi_e(F^e_* U -)$ where Π_e takes the components of elements in $F^e_* R^\alpha$ to their images under the map π_e. □

Definition 2.4. Let the notation and situation be as in Lemma 2.3. We call the map Π_e (π_e when $\alpha = 1$) in Lemma 2.3 the trace map on $F^e_* R^\alpha$, or just the trace map when the content is clear.

Proposition 2.5. *Let M be a finitely generated Cartier module equipped with a Cartier map C which is isomorphic to $R^\alpha / \mathrm{Im}\, A$ for some positive integer α and matrix A with entries in R. Then there exists a Cartier module structure C' on R^α such that the diagram below is commutative*

$$\begin{array}{ccc} F_* R^\alpha & \xrightarrow{C'} & R^\alpha \\ \downarrow & & \downarrow \\ F_*(R^\alpha / \mathrm{Im}\, A) & \xrightarrow[\tilde{C}]{} & R^\alpha / \mathrm{Im}\, A \end{array} \tag{2.1}$$

where the vertical arrows are natural surjections and $\tilde{C}$ is isomorphic to C.

Proof. Since $F_* R$ is a free R-module, there exists an R-linear map $C' : F_* R^\alpha \to R^\alpha$ such that the Diagram 2.1 is commutative. □

Let M be a given finitely generated Cartier module with a Cartier map C which is isomorphic to $R^\alpha / \mathrm{Im}\, A$ for some positive integer α and matrix A with entries in R. By Proposition 2.5, there exists a Cartier module structure C' on R^α such that $C'(F_* \mathrm{Im}\, A) \subseteq$

$\operatorname{Im} A$. Then by Lemma 2.3, there exists an $\alpha \times \alpha$ matrix U with entries in R such that $C'(-) = \Pi(F_* U -)$ whose restriction on M is C.

3. Artinian modules with Frobenius maps

In this section, we give an explicit formulation for Frobenius maps over Artinian modules. Let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field of R.

Remark 3.1. Following [3, 13.5.3 Example], E is isomorphic to the module of inverse polynomials $\Bbbk[x_1^-, \dots, x_s^-]$ whose R-module structure is extended from the following rule

$$(\lambda x_1^{\alpha_1} \dots x_s^{\alpha_s})(\mu x_1^{-\nu_1} \dots x_s^{-\nu_s}) = \begin{cases} \lambda\mu x_1^{-\nu_1+\alpha_1} \dots x_s^{-\nu_s+\alpha_s} & \text{if } \alpha_i < \nu_i \text{ for all } i \\ 0 & \text{if } \alpha_i \geq \nu_i \text{ for any } i \end{cases}$$

for all $\lambda, \mu \in \Bbbk$, non-negative integers $\alpha_1, \dots, \alpha_s$, and positive integers $\nu_1, \dots, \nu_s$. Therefore, E has a natural left $R[X; f^e]$-module structure by extending additively the action

$$T(\lambda x_1^{\alpha_1} \dots x_s^{\alpha_s}) = \lambda^{p^e} x_1^{-p^e\nu_1} \dots, x_s^{-p^e\nu_s},$$

where $T = X$, for all $\lambda \in \Bbbk$ and positive integers $\nu_1, \dots, \nu_s$. Notice that $T : E \to E$ defines and e-th Frobenius map. We can also extend this natural left $R[X; f^e]$-module structure on E to E^α which is given by

$$T \begin{pmatrix} a_1 \\ \vdots \\ a_\alpha \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_\alpha \end{pmatrix}$$

for all $a_1, \dots, a_\alpha \in E$, and we call this T as the natural Frobenius on E^α.

Let M be an R-module. The Frobenius functor F_R from the category of R-modules to itself is defined by $F_R(M) := F_* R \otimes_R M$ where $F_R(M)$ acquires its R-module structure via the identification of $F_* R$ with R. The resulting $(F_* R, R)$-bimodule structure on $F_R(M)$ satisfies

$$s \cdot (F_* r \otimes m) := F_* sr \otimes m \text{ and } (F_* r \otimes m) \cdot s := F_* s^p r \otimes m = F_* r \otimes sm$$

for all $r, s \in R$ and $m \in M$. The e-th iteration of F_R is denoted by F_R^e, and it is clearly given by $F_R^e(M) = F_*^e R \otimes_R M$. Regularity of R implies that the Frobenius functor is exact.

Remark 3.2. Following [5, Section 3], let $\mathfrak{C}^e$ be the category of Artinian left $R[X; f^e]$-modules and $\mathfrak{D}^e$ be the category of R-linear maps $M \to F_R^e(M)$ where M is a Noetherian R-module and a morphism between $M \to F_R^e(M)$ and $N \to F_R^e(N)$ is a commutative diagram of R-linear maps

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ F_R^e(M) & \xrightarrow{F_R^e(\phi)} & F_R^e(N) \end{array}$$

We define the functor $\Delta^e : \mathfrak{C}^e \to \mathfrak{D}^e$ as follows: given an e-th Frobenius map $\theta : M \to M$, we can obtain an R-linear map $\phi : F_*^e R \otimes M \to M$ such that $\phi(F_* r \otimes m) = r\theta(m)$ for all $r \in R$, $m \in M$. Applying the Matlis duality functor to this map gives the R-linear map $M^\vee \to (F_*^e R \otimes M)^\vee \cong F_*^e R \otimes M^\vee$ where the last isomorphism is described in [7, Lemma 4.1]. Conversely, we define the functor $\Psi^e : \mathfrak{D}^e \to \mathfrak{C}^e$ as follows: given a Noetherian R-module N with an R-linear map $N \to F_R^e(N)$. Applying the Matlis duality functor to this map gives the R-linear map $\varphi : F_R^e(N^\vee) \cong F_R^e(N)^\vee \to N^\vee$ where the first isomorphism is the composition $F_R^e(N^\vee) \cong F_R^e(N^\vee)^{\vee\vee} \cong F_R^e(N^{\vee\vee})^\vee \cong F_R^e(N)^\vee$. Then we define the action of θ on $N^\vee$ by defining $\theta(n) = \varphi(1 \otimes n)$ for all $n \in N^\vee$. The mutually inverse exact

functors Δ^e and Ψ^e are extensions of the Matlis duality functor which also keeps track of Frobenius actions.

Given an Artininan R-module M, we can embed M in E^α for some positive integer α, we can then embed $\operatorname{Coker}(M \hookrightarrow E^\alpha)$ in E^β for some positive integer β. Continuing in this way, we get an injective resolution

$$0 \to M \to E^\alpha \xrightarrow{A^t} E^\beta \to \cdots$$

of M, where A is an $\alpha \times \beta$ matrix with entries in R since $\operatorname{Hom}_R(E^\alpha, E^\beta) \cong \operatorname{Hom}_R(S^\alpha, R^\beta)$, and so $M \cong \operatorname{Ker} A^t$.

Proposition 3.3 ([6, Proposition 2.1])**.** *Let $M \cong \operatorname{Ker} A^t$ be an Artininan R-module where A is an $\alpha \times \beta$ matrix with entries in R. For a given e-th Frobenius map on M, $\Delta^e(M) \in \operatorname{Hom}_R(\operatorname{Coker} A, \operatorname{Coker} A^{[p^e]})$ and is given by an $\alpha\times\alpha$ matrix U such that $U \operatorname{Im} A \subseteq \operatorname{Im} A^{[p^e]}$, conversely any such U defines an $R[X; f^e]$-module structure on M which is given by the restriction to M of the Frobenius map $X : E^\alpha \to E^\alpha$ defined by $X(a) = U^t T(a)$ for all $a \in E^\alpha$.*

4. An explicit isomorphism

Henceforth $R = \Bbbk[\![x_1, \ldots, x_n]\!]$ will denote a power series ring over a perfect field $\Bbbk$ of prime characteristic p, and we will identify E with the module of inverse polynomials. Since F_*E is the injective hull of the residue field of F_*R, by [4, Theorem A25], we know that $\operatorname{Hom}_R(F_*R, E) \cong F_*E$ as F_*R-modules. In this section, we define an explicit isomorphism between $\operatorname{Hom}_R(F_*R, E)$ and F_*E.

Notation 4.1. Let k, k_1, k_2 be integers. We will write

(1) $\bar{x}$ to denote $x_1 \ldots x_n$,
(2) $\bar{\alpha}$, $\bar{\alpha} + \bar{\beta}$ and $k\bar{\alpha}$ to denote the n-tuples $(\alpha_1, \ldots, \alpha_n)$, $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$ and $(k\alpha_1, \ldots, k\alpha_n)$, respectively,
(3) $\bar{x}^{\bar{\alpha}}$ to denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and $\bar{x}^k$ to denote $x_1^k \ldots x_n^k$,
(4) $k_1 < \bar{\alpha} < k_2$ to mean that $k_1 < \alpha_i < k_2$ for each i,
(5) $r_{\bar{\alpha}}$ and $r_{\bar{k}}$ to denote the elements of R indexed with the n-tuples $\bar{\alpha}$ and $(k, \ldots, k)$, respectively,
(6) $-\nu$ to denote n-tuples $(\nu_1, \ldots, \nu_n)$.

By Remark 2.1, $F_*^e R$ is a free R-module with the basis set $\mathcal{B} = \{F_*^e x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid 0 \leq \alpha_1, \ldots, \alpha_n < p^e\}$. Therefore, an R-linear map from F_*R to any other R-module is simply a choice of where to send these basis elements.

Lemma 4.2. *The map $\Phi : \operatorname{Hom}_R(F_*R, E) \to F_*E$ given by*

$$\Phi(g) = \sum_{0 \leq \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(g(F_*\bar{x}^{\bar{\alpha}}))] \tag{4.1}$$

for all $g \in \operatorname{Hom}_R(F_*R, E)$, where T is the natural Frobenius map on E, is an F_*R-isomorphism.

Proof. By the definitions of T and g, it can easily be seen that Φ is well-defined and additive. For any $r \in R$, we further have the following

$$\begin{aligned} r\Phi(g) &= \sum_{0 \leq \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} r^p T(g(F_*\bar{x}^{\bar{\alpha}}))] \\ &= \sum_{0 \leq \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(rg(F_*\bar{x}^{\bar{\alpha}}))] = \Phi(rg) \end{aligned}$$

which means that Φ is R-linear. Thus, for F_*R-linearity of Φ, since F_*R is a free R-module, it is enough to show that $F_*\bar{x}^{\bar{\beta}}\Phi(g) = \Phi(F_*\bar{x}^{\bar{\beta}}g)$ for any basis element $F_*\bar{x}^{\bar{\beta}} \in \mathcal{B}$, and so we will show that the right hand sides of following equations are equal.

$$F_*\bar{x}^{\bar{\beta}}\Phi(g) = \sum_{0\le\bar{\alpha}<p} F_*[\bar{x}^{p-1-\bar{\alpha}+\bar{\beta}}T(g(F_*\bar{x}^{\bar{\alpha}}))], \tag{4.2}$$

$$\Phi(F_*\bar{x}^{\bar{\beta}}g) = \sum_{0\le\bar{\alpha}<p} F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*\bar{x}^{\bar{\alpha}+\bar{\beta}}))]. \tag{4.3}$$

Moreover, since $F_*\bar{x}^{\bar{\beta}} = F_*x_1^{\beta_1}\cdots F_*x_n^{\beta_n}$, it is enough to show that

$$\begin{aligned} F_*x_i^{\beta_i}\Phi(g) &= \sum_{0\le\bar{\alpha}<p} F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \\ &= \sum_{0\le\bar{\alpha}<p} F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] = \Phi(F_*x_i^{\beta_i}g) \end{aligned}$$

for each $F_*x_i^{\beta_i}$. To do that we will show the following sets are the same

$$S_1 = \{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid 0\le\alpha_i<p\},$$
$$S_2 = \{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid 0\le\alpha_i<p\}.$$

In the case that $\alpha_i+\beta_i<p$,

$$\{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid 0\le\alpha_i<p-\beta_i\} =$$
$$\{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid \beta_i\le\alpha_i<p\}$$

since substituting α_i with $\alpha_i+\beta_i$ in the latter set gives us the former set. On the other hand, in the case that $\alpha_i+\beta_i\ge p$,

$$\{F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] \mid p-\beta_i\le\alpha_i<p\} =$$
$$\{F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*\bar{x}^{\bar{\alpha}}))] \mid 0\le\alpha_i<\beta_i\}$$

since for each $k\in\{0,\dots,\beta_i-1\}$, where $\alpha_i+\beta_i=p+k$ (i.e. $p-\alpha_i=\beta_i-k$),

$$F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*x_1^{\alpha_1}\dots x_i^{\alpha_i+\beta_i}\dots x_n^{\alpha_n}))] =$$
$$F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-\alpha_i+p}\dots x_n^{p-1-\alpha_n}T(g(F_*x_1^{\alpha_1}\dots x_i^{k}\dots x_n^{\alpha_n}))] =$$
$$F_*[x_1^{p-1-\alpha_1}\dots x_i^{p-1-k+\beta_i}\dots x_n^{p-1-\alpha_n}T(g(F_*x_1^{\alpha_1}\dots x_i^{k}\dots x_n^{\alpha_n}))].$$

Therefore, $S_1=S_2$, and so the right hand sides of (4.2) and (4.3) are equal.

For injectivity of Φ, we first need the following. For any $g\in\mathrm{Hom}_R(F_*R,E)$, we have $g(F_*\bar{x}^{\bar{\alpha}})\in\Bbbk[x_1^-,\dots,x_n^-]$, and so $g(F_*\bar{x}^{\bar{\alpha}})$ is a finite $\Bbbk$-linear combination of monomials $x_1^{-\nu_1}\dots x_n^{-\nu_n}$, where ν_i's are positive integers. Therefore, for each $F_*\bar{x}^{\bar{\alpha}}\in\mathcal{B}$, $F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*\bar{x}^{\bar{\alpha}}))]$ is a finite $\Bbbk$-linear combination of monomials

$$F_*x_1^{p-1-\alpha_1-p\nu_1}\dots x_n^{p-1-\alpha_n-p\nu_n}.$$

This means that

$$\Phi(g) = \sum_{0\le\bar{\alpha}<p} F_*[\bar{x}^{p-1-\bar{\alpha}}T(g(F_*\bar{x}^{\bar{\alpha}}))] = \sum_{0\le\bar{\alpha}<p}\Bigg(\sum_{0<\bar{\nu}<\infty}\lambda_\nu F_*\bar{x}^{p-1-\bar{\alpha}-p\bar{\nu}}\Bigg).$$

Therefore, $\Phi(g)=0$ if and only if $\lambda_\nu=0$ for all $\bar{\nu}>0$ since $p-1-\bar{\alpha}<p$ implies that $p-1-\bar{\alpha}-p\bar{\nu}<0$ and the terms $F_*\bar{x}^{p-1-\bar{\alpha}-p\bar{\nu}}\ne F_*\bar{x}^{p-1-\bar{\beta}-p\bar{\mu}}$ unless $\bar{\alpha}=\bar{\beta}$ and $\bar{\mu}=\bar{\nu}$ at the same time. Hence,

$$\begin{aligned} \Phi(g)=0 &\Leftrightarrow T(g(F_*\bar{x}^{\bar{\alpha}}))=0 \text{ for all } 0\le\bar{\alpha}<p \\ &\Leftrightarrow g(F_*\bar{x}^{\bar{\alpha}})=0 \text{ for all } 0\le\bar{\alpha}<p \\ &\Leftrightarrow g=0. \end{aligned}$$

For surjectivity of Φ, we take an element F_*e of $F_*\Bbbk[x_1^-, \dots, x_n^-]$. We know that it can be written as a finite sum of terms $F_*\mu x_1^{-\nu_1} \dots x_n^{-\nu_n}$, where $\mu \in \Bbbk$, and these terms can be written as

$$\begin{aligned} F_*\mu x_1^{-\nu_1} \dots x_n^{-\nu_n} &= F_* x_1^{k_1} \dots x_n^{k_n} F_* \lambda^p x_1^{-p\beta_1} \dots x_n^{-p\beta_n} \\ &= F_* x_1^{k_1} \dots x_n^{k_n} F_* T(\lambda x_1^{-\beta_1} \dots x_n^{-\beta_n}), \end{aligned}$$

where $\mu = \lambda^p$, and for each i, $k_i = p\beta_i - \nu_i$ and $0 \le k_i < p$. Now we rewrite F_*e as a finite sum

$$\sum_{0 \le k_1, \dots, k_n < p} F_* x_1^{k_1} \dots x_n^{k_n} F_* T(e_{k_1, \dots, k_n}),$$

where $e_{k_1,\dots,k_n} \in \Bbbk[x_1^-, \dots, x_n^-]$, and we choose a map $g \in \mathrm{Hom}_R(F_*R, E)$ which sends $F_* x_1^{p-1-k_1} \dots x_n^{p-1-k_n}$ to $e_{k_1,\dots,k_n}$. This means that $\Phi(g) = F_*e$, i.e. Φ is surjective. Hence, it is an isomorphism of F_*R-modules. $\square$

5. The correspondence

In this section, we introduce our explicit correspondence between finitely generated Cartier modules and Artinian modules equipped with a Frobenius map via the Matlis duality. We use Lemma 4.2 repeatedly to give our explicit proofs.

Proposition 5.1. *Let α be a non-negative integer. There is a bijective correspondence between $\mathrm{Hom}_R(F_*R^\alpha, R^\alpha)$ and $\mathrm{Hom}_R(E^\alpha, F_*E^\alpha)$ such that the trace map Π on F_*R^α corresponds to the natural Frobenius map T on E^α and $\Pi(F_*U-)$ corresponds to U^tT for any $\alpha \times \alpha$ matrix U with entries in R.*

Proof. We start by identifying $\mathrm{Hom}_R(F_*R, E)$ with F_*E using the isomorphism Φ defined in Lemma 4.2. Then we first assume that $\alpha = 1$ and let $\phi : F_*R \to R$ be a Cartier map. We know that there is an element $u \in R$ such that $\phi(-) = \pi(F_*u-)$. Applying the Matlis duality functor to this map gives us $\mathrm{Hom}_R(R, E) \xrightarrow{f \mapsto f \circ \phi} \mathrm{Hom}_R(F_*R, E)$. Next we use the isomorphism $E \xrightarrow{e \mapsto f_e} \mathrm{Hom}_R(R, E)$, where $f_e(1) = e$, to get the following composition:

$$\begin{array}{cccc} E \to & \mathrm{Hom}_R(R,E) \to & \mathrm{Hom}_R(F_*R,E) \to & F_*E \\ e \mapsto & f_e \mapsto & f_e \circ \phi \mapsto & \Phi(f_e \circ \phi) \end{array}$$

$$\begin{aligned} \Phi(f_e \circ \phi) &= \Phi(f_e \circ \pi(F_*u-)) \text{ (by } F_*R\text{-linearity of } \Phi) \\ &= F_*u\Phi(f_e \circ \pi) = F_*u \sum_{0 \le \bar{\alpha} < p} F_*[\bar{x}^{p-1-\bar{\alpha}} T(f_e \circ \pi(F_*\bar{x}^{\bar{\alpha}}))] \\ &= F_*u[F_*T(f_e \circ \pi(F_*\bar{x}^{p-1}))] = F_*u[F_*T(f_e(1))] \\ &= F_*uF_*T(e) = F_*(uT(e)). \end{aligned}$$

Therefore, the composition above gives us the Frobenius map uT on E. In particular, if $u = 1$ we get the natural Frobenius T on E.

We now assume that $\alpha > 1$ and let $\phi : F_*R^\alpha \to R^\alpha$ be a Cartier map. We know that there is an $\alpha \times \alpha$ matrix U with entries $(u_{ij})_{1 \le i,j \le \alpha}$ in R such that $\phi(-) = \Pi(F_*U-)$. Applying the Matlis duality functor to this map gives $\mathrm{Hom}_R(R^\alpha, E) \xrightarrow{f \mapsto f \circ \phi} \mathrm{Hom}_R(F_*R^\alpha, E)$.

Then we get the following composition

$$\begin{array}{ccc}
E^\alpha & \dashrightarrow & F_*E^\alpha \\
a=(a_1,\dots,a_\alpha)^t & & (\Phi(f_a\circ\phi\circ\epsilon_1),\dots,\Phi(f_a\circ\phi\circ\epsilon_\alpha))^t \\
\downarrow & & \uparrow \\
\mathrm{Hom}_R(R,E)^\alpha & & \mathrm{Hom}_R(R^\alpha,E) \\
(f_{a_1},\dots,f_{a_\alpha})^t & & (f_a\circ\phi\circ\epsilon_1,\dots,f_a\circ\phi\circ\epsilon_\alpha)^t \\
\downarrow & & \uparrow \\
\mathrm{Hom}_R(F_*R^\alpha,E) & \longmapsto & \mathrm{Hom}_R(F_*R,E)^\alpha \\
f_a & & f_a\circ\phi
\end{array}$$

where $a\in E^\alpha$ and we use the following obvious isomorphisms

$$E^\alpha\to\mathrm{Hom}_R(R,E)^\alpha \text{ given by } (a_1,\dots,a_\alpha)^t\mapsto(f_{a_1},\dots,f_{a_\alpha})^t$$

such that $f_{a_i}(1)=a_i$ for each i,

$$\mathrm{Hom}_R(R,E)^\alpha\to\mathrm{Hom}_R(R^\alpha,E) \text{ given by } (g_1,\dots,g_\alpha)^t\mapsto g$$

such that $g(e_i)=g_i(1)$ for each elementary vector e_i, and

$$\mathrm{Hom}_R(F_*R^\alpha,E)\to\mathrm{Hom}_R(F_*R,E)^\alpha \text{ given by } h\mapsto(h\circ\epsilon_1,\dots,h\circ\epsilon_\alpha)^t$$

such that the map $\epsilon_i:F_*R\to F_*R^\alpha$ given by $F_*r\mapsto F_*rF_*e_i$ is the canonical injection for each i. Then for a fixed i where $1\le i\le\alpha$, we have

$$\Phi(f_a\circ\phi\circ\epsilon_i)=\sum_{0\le\bar\alpha<p}F_*\Big[\bar x^{p-1-\bar\alpha}T\Big(f_a\Big(\Pi(F_*U\epsilon_i(F_*\bar x^{\bar\alpha}))\Big)\Big)\Big]$$

Since

$$\begin{aligned}
f_a\Big(\Pi(F_*U\epsilon_i(F_*\bar x^{\bar\alpha}))\Big)&=f_a\Big(\Pi((F_*u_{1i}\bar x^{\bar\alpha},\dots,F_*u_{\alpha i}\bar x^{\bar\alpha})^t)\Big)\\
&=f_a\Big((\pi(F_*u_{1i}\bar x^{\bar\alpha}),\dots,\pi(F_*u_{\alpha i}\bar x^{\bar\alpha}))^t\Big)\\
&=\sum_{1\le j\le\alpha}f_{a_j}(\pi(F_*u_{ji}\bar x^{\bar\alpha})),
\end{aligned}$$

we have

$$\Phi(f_a\circ\phi\circ\epsilon_i)=\sum_{1\le j\le\alpha}\Big(\sum_{0\le\bar\alpha<p}F_*\Big[\bar x^{p-1-\bar\alpha}T\Big(f_{a_j}(\pi(F_*u_{ji}\bar x^{\bar\alpha}))\Big)\Big]\Big).$$

Then by definition of Φ,

$$\begin{aligned}
\Phi(f_a\circ\phi\circ\epsilon_i)&=\sum_{1\le j\le\alpha}F_*u_{ji}\Phi(f_{a_j}\circ\pi)\\
&=\sum_{1\le j\le\alpha}F_*u_{ji}F_*T\Big(f_{a_j}(\pi(\bar x^{p-1}))\Big)\\
&=\sum_{1\le j\le\alpha}F_*u_{ji}T(a_j)=F_*[\sum_{1\le j\le\alpha}u_{ji}T(a_j)].
\end{aligned}$$

Therefore, for $1\le i\le\alpha$

$$\begin{pmatrix}\Phi(f_a\circ\phi\circ\epsilon_1)\\ \vdots\\ \Phi(f_a\circ\phi\circ\epsilon_\alpha)\end{pmatrix}=\begin{pmatrix}F_*[\sum_{1\le j\le\alpha}u_{j1}T(a_j)]\\ \vdots\\ F_*[\sum_{1\le j\le\alpha}u_{j\alpha}T(a_j)]\end{pmatrix}$$

which is equal to

$$F_*[U^t(T((a_1,\dots,a_\alpha)^t)]=F_*[U^tT(a)].$$

Hence, the composition above gives us the Frobenius map U^tT on E^α. In particular, if U is the identity matrix we get the natural Froebenius T on E^α.

The construction above gives us a map $\Omega : \mathrm{Hom}_R(F_*R^\alpha, R^\alpha) \to \mathrm{Hom}_R(E^\alpha, F_*E^\alpha)$ defined by $\Omega(\phi) = \Theta$ such that $\phi(-) = \Pi(F_*U-)$ and $\Theta(-) = F_*U^tT(-)$. We claim that this map actually is an F_*R-linear isomorphism. Let $\Omega(F_*r.\phi) = \Theta'$ for any $r \in R$. Then since $(F_*r.\phi)(-) = \phi(F_*r-) = \Pi(F_*Ur-)$, we have $\Theta'(-) = F_*(Ur)^tT(-) = F_*rF_*U^tT(-) = F_*r\Theta(-)$, i.e. $\Omega(F_*r.\phi) = \Theta' = F_*r\Theta = F_*r\Omega(\phi)$, and so Ω is F_*r-linear. Surjectivity of Ω is clear since for any Frobenius map $\Theta(-) = F_*U^tT(-)$ we can define a Cartier map $\phi(-) = \Pi(F_*U-)$. We also have $\Omega(\phi) = 0 \Rightarrow U^tT = 0 \Rightarrow U = 0 \Rightarrow \phi = 0$, because if any entry of U was non zero there would be non zero elements in the image of U^tT, i.e. Ω is injective. This means that we get the promised bijective correspondence. $\square$

Next we see that the Matlis duality functor $(-)^\vee = \mathrm{Hom}_R(-, E)$ commutes with $F_*(-)$ (cf. Lemma 5.1 in [1]).

Lemma 5.2. *Let M be a finitely generated or an Artinian R-module. Then $F_*M^\vee \cong (F_*M)^\vee$.*

Proof. We first assume that M is finitely generated. Then M has a presentation $\cdots \to R^\beta \xrightarrow{A} R^\alpha \twoheadrightarrow M \to 0$ where A is an $\alpha \times \beta$ matrix with entries in R. If we apply the Matlis dual to this presentation we get $0 \to M^\vee \hookrightarrow E^\alpha \xrightarrow{A^t} E^\beta \to \dots$. So $M^\vee = \mathrm{Ker}\, A^t = \mathrm{Ann}_{E^\alpha} A^t$. On the other hand, F_*M has a presentation $\cdots \to F_*R^\beta \xrightarrow{F_*A} F_*R^\alpha \twoheadrightarrow F_*M \to 0$. Then if we apply the Matlis dual again and identify $\mathrm{Hom}_R(F_*R, E)$ with F_*E using the isomorphism Φ defined in Lemma 4.2, we get $0 \to (F_*M)^\vee \hookrightarrow F_*E^\alpha \xrightarrow{F_*A^t} F_*E^\beta \to \dots$, and so $(F_*M)^\vee = \mathrm{Ker}\, F_*A^t = \mathrm{Ann}_{F_*E^\alpha} F_*A^t = F_*(\mathrm{Ann}_{E^\alpha} A^t) = F_*M^\vee$.

If now M is Artinian, we know that $M^\vee$ is Noetherian and $M \cong M^{\vee\vee}$. Then it follows from the first assumption, $F_*M^\vee \cong (F_*M^\vee)^{\vee\vee} \cong (F_*M^{\vee\vee})^\vee \cong (F_*M)^\vee$. $\square$

Next theorem extends Proposition 5.2 in [1] to a computational level.

Theorem 5.3. *The Matlis duality functor induces a bijective correspondence between finitely generated Cartier modules and Artinian modules equipped with Frobenius maps given as follows: if M is a finitely generated Cartier module with a square matrix U defining the Cartier module structure on M, then the corresponding Artinian module is $M^\vee$ with the corresponding Frobenius map U^tT.*

Proof. Let (M, C) be a finitely generated Cartier module with a square matrix U defining Cartier module structure on M. Then we have a presentation of M as follows $\cdots \to R^\beta \xrightarrow{A} R^\alpha \twoheadrightarrow M \to 0$ and the following commutative diagram with exact rows

$$\begin{tikzcd} F_*R^\alpha \arrow[r] \arrow[d, "\Pi(F_*U-)"'] & F_*M \arrow[r] \arrow[d, "C"] & 0 \\ R^\alpha \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where C is induced by $\Pi(F_*U-)$ on M. If we apply the Matlis dual to the diagram above and if we use Lemma 4.2, Proposition 5.1, and Lemma 5.2 we get

$$\begin{tikzcd} 0 \arrow[r] & M^\vee \arrow[r] \arrow[d, "\theta"'] & E^\alpha \arrow[d, "F_*U^tT"] \\ 0 \arrow[r] & F_*M^\vee \arrow[r] & F_*E \end{tikzcd}$$

where θ is the restriction of F_*U^tT on $M^\vee$. The same construction follows the converse. $\square$

Question: The theory behind explicit results in this paper works for the case that $R = \Bbbk[\![x_1, ..., x_n]\!]$ is a power series ring over an F-finite field $\Bbbk$ of prime characteristic p. Is it possible to write an explicit isomorphism between $\mathrm{Hom}_R(F_*R, E)$ and F_*E for the F-finite case as in Lemma 4.2 ?

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