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Research Article

Spectral properties of non-selfadjoint Sturm-Liouville operator equation on the real axis

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Abstract

In this paper, we analyze the non-selfadjoint Sturm-Liouville operator L defined in the Hilbert space $L_2(\mathbb{R}, H)$ of vector-valued functions which are strongly-measurable and square-integrable in $\mathbb{R}$. L is defined

$$L(y) = -y'' + Q(x)y, \quad x \in \mathbb{R},$$

for every $y \in L_2(\mathbb{R}, H)$ where the potential $Q(x)$ is a non-selfadjoint, completely continuous operator in a separable Hilbert space H for each $x \in \mathbb{R}$. We obtain the Jost solutions of this operator and examine the analytic and asymptotic properties. Moreover, we find the point spectrum and the spectral singularities of L and also obtain the sufficient condition which assures the finiteness of the eigenvalues and spectral singularities of L.

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1. Introduction

Non-selfadjoint operators are seen in physical systems which do not involve the conservation of energy law. Some selfadjoint problems also give us non-selfadjoint operators after separation of variables. The theory of non-selfadjoint operators has initially begun to analyze ordinary differential equations. M.V. Keldysh played a significant role to develop a general theory for non-selfadjoint operators by inventing a new method for establishing the resolvent of an arbitrary completely continuous, non-selfadjoint operator of finite order [16, 17].

Spectral analysis of non-selfadjoint differential operators has been studied by M.A. Naimark [24, 25]. In particular, he analyzed the non-selfadjoint Sturm-Liouville operator defined by

$$l(y) = -y'' + p(x)y, \quad 0 < x < \infty, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

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where $p(x)$ is a complex-valued function satisfying

$$\int_0^\infty (1+t^2)|p(t)|dt < \infty, \tag{1.3}$$

and $h \in \mathbb{C}$. Several authors investigated the non-selfadjoint Sturm-Liouville operator defined by Equations (1.1) and (1.2) in detail [22–25, 28]. The results of Naimark [24, 25] have been generalized in [7, 8] to the operator l_0 generated in $L_2(\mathbb{R})$ which is defined by

$$l_o(y) = -y'' + q(x)y, \quad x \in \mathbb{R},$$

where the potential q is a complex-valued function. The authors generalized the results of [24] and applied to the non-selfadjoint Schrödinger operator in the three-dimensional space [13].

Non-selfadjoint Hamiltonians and complex extensions of Quantum Mechanics have been studied by many mathematicans, recently. Moreover, spectral properties of the selfadjoint matrix differential and difference equations have been examined [9, 10, 15]. For the non-selfadjoint case, discrete spectrum and the spectral singularities of the matrix Sturm-Liouville operator were investigated [4, 11, 26, 27]. Further, the authors examined a system of non-selfadjoint Sturm-Liouville equations [2, 5, 6].

B. M. Levitan et al. have studied the point spectrum of the following Sturm-Liouville operator equation in detail [14, 19–21]. Let H be a separable Hilbert space and $L_2(\mathbb{R}_+, H)$ denote the space of vector-valued functions $f(x)$ defined on $(0,\infty)$ which are strongly-integrable and also square-integrable on $(0,\infty)$ i.e.,

$$\int_0^\infty \|f(x)\|^2 dx < \infty.$$

Consider the operator l_1 defined on $L_2(\mathbb{R}_+, H)$ by

$$l_1(Y) = -Y'' + Q(x)Y, \quad 0 < x < \infty, \tag{1.4}$$

and the boundary condition $Y(0) = 0$ where $Q(x)$ is a completely continuous, selfadjoint operator defined on H for every $x \in (0,\infty)$. Equation (1.4) is called Sturm-Liouville operator equation.

In our previous paper [3], we considered the non-selfadjoint analogue of the above problem and investigated the spectral properties of the non-selfadjoint Sturm-Liouville operator equation on the half line on the contrary to [14, 19–21]. We also generalized the results in [2, 4, 11, 26, 27] by considering the coefficients as operators not only finite dimensional matrices. In this study, we extend these results to the whole real axis. Explicitly, we focus on the following non-selfadjoint operator.

Assume H is a separable Hilbert space and $H_1 := L_2(\mathbb{R}, H)$ denotes the space of vector-valued funtions $f(x)$ defined on $\mathbb{R}$ which are strongly-integrable and square-integrable. Note that H_1 is a Hilbert space with the inner product (see [29]);

$$(f,g)_1 = \int_{-\infty}^\infty (f(x), g(x))_H dx.$$

Let us consider the non-selfadjoint operator L defined in H_1;

$$L(y) = -y'' + Q(x)y, \quad x \in \mathbb{R}, \tag{1.5}$$

where the potential $Q(x)$ is a non-selfadjoint, completely continuous operator in H for each $x \in \mathbb{R}$. In this paper, we specify the domain of L and express the Jost solutions. Then, we find the discrete spectrum and the set of spectral singularities of L by using the properties of the Jost solutions. Finally, we prove that L has a finite number of eigenvalues and spectral singularities.

The domain $D(L)$ of L is the subspace consisting of all $y \in H_1$ which satisfies the following conditions;

(i) y is twice strongly-differentiable,
(ii) $L(y) \in H_1$.

Let us consider the eigenvalue equation;

$$-y'' + Q(x)y = \lambda^2 y, \quad x \in \mathbb{R}. \tag{1.6}$$

2. The Jost solutions of (1.6)

We shall also focus on the equation

$$-Y'' + Q(x)Y = \lambda^2 Y, \quad x \in \mathbb{R}, \tag{2.1}$$

where $Y(x)$ is an operator-valued function i.e, $Y(x)$ is an operator in H for each $x \in \mathbb{R}$.

Lemma 2.1. *Every sequence of solutions of Equation* (1.6) *can be represented as an operator-valued function which satisfies Equation* (2.1)*. Conversely, one can construct a sequence of vector-valued functions which satisfy Equation* (1.6) *for a given operator-valued solution of Equation* (2.1)*.*

Proof. Since H is a separable Hilbert space, there exists an orthonormal basis $(u_n)_{n\in\mathbb{N}}$. Suppose vector-valued functions $(y_n(x))_{n\in\mathbb{N}}$ satisfy Equation (1.6). We can construct an operator-valued function $Y(x)$ such that $Y(x)u_n = y_n(x)$ for every $n \in \mathbb{N}$. It is clear that $Y(x)$ satisfies Equation (2.1).
Conversely, suppose operator-valued function $Y(x)$ satisfies Equation (2.1). Let $y_n(x) = Y(x)u_n$ for every $n \in \mathbb{N}$. Then, it is clear that $(y_n(x))$ satisfies Equation (1.6) for every $n \in \mathbb{N}$. ☐

As a result of this one to one correspondence, we can focus on the solutions of only one of the Equations (1.6)-(2.1).

We shall use the notations;

$$\sigma^+(x) = \int_x^\infty \|Q(t)\|\, dt, \quad \sigma_1^+(x) = \int_x^\infty \sigma^+(t)dt,$$
$$\sigma^-(x) = \int_{-\infty}^x \|Q(t)\|\, dt, \quad \sigma_1^-(x) = \int_{-\infty}^x \sigma^-(t)dt.$$

Suppose that the condition

$$\int_{-\infty}^{\infty} (1+|t|)\, \|Q(t)\|\, dt < \infty, \tag{2.2}$$

holds. Then, Equation (2.1) has operator solutions $E^+(x,\lambda)$ and $F^-(x,\lambda)$ satisfying the initial conditions;

$$\lim_{x\to\infty} e^{-i\lambda x} E^+(x,\lambda) = I, \quad \mathrm{Im}\lambda \geq 0, \tag{2.3}$$

and

$$\lim_{x\to-\infty} e^{i\lambda x} F^-(x,\lambda) = I, \quad \mathrm{Im}\lambda \geq 0, \tag{2.4}$$

respectively. Indeed, consider the integral equation

$$E^+(x,\lambda) = e^{i\lambda x} I + \frac{1}{\lambda}\int_x^\infty sin\left(\lambda\left(t-x\right)\right) Q(t) E^+(t,\lambda)dt, \quad \mathrm{Im}\lambda \geq 0,$$

which is easily seen to be a solution of Equation (2.1) satisfying (2.3). Similarly, if we define

$$F^-(x,\lambda) = E^+(-x,\lambda), \quad \mathrm{Im}\lambda \geq 0,$$

it easily follows that $F^{-}(x,\lambda)$ satisfies (2.4). Under the condition (2.2), the solution $E^{+}(x,\lambda)$ can be represented (see [1]);

$$E^{+}(x,\lambda) = e^{i\lambda x}I + \int_x^{\infty} e^{i\lambda t}K^{+}(x,t)dt, \quad \mathrm{Im}\lambda \geq 0. \tag{2.5}$$

Let us consider the equation;

$$-Z'' + ZQ(x) = \lambda^2 Z, \quad x \in \mathbb{R}, \tag{2.6}$$

where $Z(x)$ is an operator-valued function. Similarly, Equation (2.6) has an operator solution $E^{-}(x,\lambda)$ which satisfies the initial condition;

$$\lim_{x\to-\infty} e^{i\lambda x}E^{-}(x,\lambda) = I, \quad \mathrm{Im}\lambda \geq 0,$$

and has the representation

$$E^{-}(x,\lambda) = e^{-i\lambda x}I + \int_{-\infty}^{x} e^{-i\lambda t}K^{-}(x,t)dt, \quad \mathrm{Im}\lambda \geq 0.$$

Further, the operator-valued kernels $\overset{+}{K^{-}}(x,t)$ are differentiable with respect to x and t and satisfy

$$\left\| \overset{+}{K^{-}}(x,t) \right\| \leq \frac{1}{2}\overset{+}{\sigma^{-}}\left(\frac{x+t}{2}\right)\exp\left[\overset{+}{\sigma_1^{-}}(x) - \overset{+}{\sigma_1^{-}}\left(\frac{x+t}{2}\right)\right], \tag{2.7}$$

$$\left\| \overset{+}{K_x^{-}}(x,t) \pm \frac{1}{4}Q\left(\frac{x+t}{2}\right) \right\| \leq \frac{1}{2}\overset{+}{\sigma_1^{-}}(x)\overset{+}{\sigma^{-}}\left(\frac{x+t}{2}\right)\exp \overset{+}{\sigma_1^{-}}(x), \tag{2.8}$$

$$\left\| \overset{+}{K_t^{-}}(x,t) \pm \frac{1}{4}Q\left(\frac{x+t}{2}\right) \right\| \leq \frac{1}{2}\overset{+}{\sigma_1^{-}}(t)\overset{+}{\sigma^{-}}\left(\frac{x+t}{2}\right)\exp \overset{+}{\sigma_1^{-}}(t), \tag{2.9}$$

As a result, the solutions $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$ are analytic for $\mathrm{Im}\lambda > 0$ and continuous for $\mathrm{Im}\lambda \geq 0$. $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$ are called the Jost solutions of Equation (1.6). The proofs of above results are very similar to the matrix coefficient case which have been obtained in [1,4]. In addition, we obtained analogous properties in our previous paper [3]. Hence, we omitted the proofs.

Lemma 2.2. *Let $Y(x)$ be a solution of Equation* (2.1) *and $Z(x)$ be a solution of Equation* (2.6)*. Then, the Wronskian $W[Y,Z](x) := Z'(x)Y(x) - Z(x)Y'(x)$ is independent of x.*

Proof. We have

$$\begin{aligned} -Y'' + Q(x)Y &= \lambda^2 Y, \\ -Z'' + ZQ(x) &= \lambda^2 Z. \end{aligned}$$

If we multiply the first equality from the left with Z and the second equality from the right with Y and subtract them, we have

$$Z''(x)Y(x) - Z(x)Y''(x) = 0,$$

which implies $W[Y,Z](x)$ is constant and hence independent of the variable x. □

Let us define the function

$$D(\lambda) := W\left[E^{-}(x,\lambda), E^{+}(x,\lambda)\right], \quad \mathrm{Im}\lambda \geq 0.$$

Since the Wronskian of $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$ is independent of x, $D(\lambda)$ is a function of λ which is also analytic for $\mathrm{Im}\lambda > 0$ and continuous for $\mathrm{Im}\lambda \geq 0$. The function $D(\lambda)$ is called the Jost function of Equation (1.6).

Theorem 2.3. *The function $D(\lambda)$ satisfies*

$$D(\lambda) = 2i\lambda I - 2K^{+}(0,0) - 2K^{-}(0,0) + \int_0^\infty e^{i\lambda t} F(t)dt, \tag{2.10}$$

where

$$\begin{aligned} F(t) = {} & K_x^{+}(0,t) - K_x^{-}(0,-t) - K^{-}(0,0)K^{+}(0,t) - K^{+}(0,0)K^{-}(0,-t) \\ & + K^{-}(0,-t) * K_x^{+}(0,t) - K_x^{-}(0,-t) * K^{+}(0,t) + K_t^{-}(0,-t) \\ & - K_t^{+}(0,t), \end{aligned} \tag{2.11}$$

and $F \in L_1(\mathbb{R}, H)$ where "$*$" is the convolution operation.

Proof. Since the Wronskian of $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$ is independent of x, we put $x = 0$ and obtain

$$D(\lambda) = W\left[E^{-}(x,\lambda), E^{+}(x,\lambda)\right] = E_x^{+}(\lambda)E^{-}(\lambda) - E^{+}(\lambda)E_x^{-}(\lambda).$$

By using the integral representations of $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$ we get (2.10) and (2.11). From (2.7)-(2.9) we have $F \in L_1(\mathbb{R}, H)$. □

Theorem 2.4. *The following asymptotic relations hold;*

$$D(\lambda) = 2i\lambda I - 2K^{+}(0,0) - 2K^{-}(0,0) + o(1), \quad \mathrm{Im}\lambda \geq 0, \quad |\lambda| \to \infty, \tag{2.12}$$

$$D(\lambda) = 2i\lambda I + O(1), \quad \mathrm{Im}\lambda \geq 0, \quad |\lambda| \to \infty. \tag{2.13}$$

Proof. Let $\lambda \in \mathbb{R}$. By Riemann-Lebesgue Lemma for Fourier transforms [18] we have

$$\int_0^\infty e^{i\lambda t} F(t)dt = o(1), \quad \lambda \in \mathbb{R}, \quad |\lambda| \to \infty. \tag{2.14}$$

Now, let $\mathrm{Im}\lambda > 0$. Lebesgue Theorem [18] implies

$$\int_0^\infty e^{i\lambda t} F(t)dt = o(1), \quad \mathrm{Im}\lambda > 0, \quad |\lambda| \to \infty. \tag{2.15}$$

If we use (2.14), (2.15) we get (2.12). The proof is similar for (2.13). □

3. Point spectrum and spectral singularities of L

Now, we introduce the point spectrum and the set of spectral singularities of L according to the definitions given in [22–24]

$$\begin{aligned} \sigma_d(L) &= \left\{\lambda^2 : \ \mathrm{Im}\lambda > 0, \ D(\lambda) \text{ is not invertible}\right\}, \\ \sigma_{ss}(L) &= \left\{\lambda^2 : \ \lambda \in \mathbb{R} \setminus \{0\}, \ D(\lambda) \text{ is not invertible}\right\}. \end{aligned}$$

Now, we try to examine the eigenvalues of L by employing the results in [17]. Let us recall;

$$D(\lambda) = 2i\lambda I - 2K^{+}(0,0) - 2K^{-}(0,0) + \int_0^\infty e^{i\lambda t} F(t)dt, \quad \mathrm{Im}\lambda \geq 0. \tag{3.1}$$

Let

$$\begin{aligned} A(\lambda) &:= \frac{1}{2i\lambda}\left[-2K^{+}(0,0) - 2K^{-}(0,0) + \int_0^\infty e^{i\lambda t} F(t)dt\right], \\ G(\lambda) &:= \frac{1}{2i\lambda} D(\lambda). \end{aligned}$$

Then,

$$G(\lambda) = I + A(\lambda), \quad \mathrm{Im}\lambda \geq 0,$$

and for $\lambda \neq 0$, it follows $D(\lambda)$ is invertible iff $G(\lambda)$ is invertible. Hence

$$\begin{aligned} \sigma_d(L) &= \left\{\lambda^2 : \ \mathrm{Im}\lambda > 0, \ G(\lambda) \text{ is not invertible}\right\}, \\ \sigma_{ss}(L) &= \left\{\lambda^2 : \ \lambda \in \mathbb{R} \setminus \{0\}, \ G(\lambda) \text{ is not invertible}\right\}. \end{aligned}$$

Let us define $M_1 := \{\lambda : \mathrm{Im}\lambda > 0,\ G(\lambda) \text{ is not invertible}\}$. It follows $\sigma_d(L) = \{\lambda^2 : \lambda \in M_1\}$.

Since $\int_{-\infty}^{\infty} (1+|t|)\,\|Q(t)\|\,dt < \infty$ and $Q(x)$ is completely continuous operator for each $x \in \mathbb{R}$, it follows $F(t)$ is completely continuous operator for each $0 < t < \infty$ and as a result $A(\lambda)$ is completely continuous for $\mathrm{Im}\lambda > 0$. Also, since $D(\lambda)$ is analytic for $\mathrm{Im}\lambda > 0$, $A(\lambda)$ is also analytic in the same domain. Now, we can use the results in [17].

Definition 3.1. If

$$(I-R)\,(I+A) = I,$$

holds, then the operator R is called the resolvent of the operator A, [17].

Let us denote the resolvent of $A(\lambda)$ by $R(\lambda)$. It follows

$$I - R(\lambda) = (I + A(\lambda))^{-1} = (G(\lambda))^{-1}.$$

If $I - R(\lambda)$ exists at least for one λ which means $G(\lambda)$ is invertible, this implies $I - R(\lambda)$ exists on the domain $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$ except for a set of isolated points, and also $I - R(\lambda)$ is a meromorphic operator function in the same domain [17]. It is obvious that $M_1 \neq \mathbb{C}_+$. This implies there exists at least one λ such that $I - R(\lambda)$ is defined. [17] implies that $I - R(\lambda)$ should exist on the domain $\mathbb{C}_+$ except for a set of isolated points. These isolated points are obviously the eigenvalues of L. Moreover, $I - R(\lambda)$ is a meromorphic operator function on $\mathbb{C}_+$. Therefore, we can express $I - R(\lambda)$ as a ratio of two analytical functions in the domain $\mathbb{C}_+$ as;

$$(G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \tag{3.2}$$

where $S(\lambda)$ is an operator function and $d(\lambda)$ is a scalar function on $\mathbb{C}_+$. Moreover, the above isolated points are poles of the function $I - R(\lambda)$ and they are the zeros of the function $d(\lambda)$. As a result, it follows

$$M_1 = \{\lambda : \mathrm{Im}\lambda > 0,\ d(\lambda) = 0\}. \tag{3.3}$$

Theorem 3.2. *If the condition* (2.2) *holds, then $\sigma_d(L)$ is a bounded and countable set. Further, the limit points of $\sigma_d(L)$ should lie in a bounded interval of the real axis.*

Proof. The relation (2.12) implies

$$G(\lambda) = I + o(1), \quad \mathrm{Im}\lambda \geq 0, \quad |\lambda| \to \infty,$$

which means for sufficiently large $\lambda \in \overline{\mathbb{C}_+}$, $G(\lambda) \to I$ and thus $G(\lambda)$ is invertible. Therefore, M_1 is bounded. Since the function $d(\lambda)$ is analytic, its zeros are isolated. This implies M_1 is countable. Further, the limit points of the zeros of $d(\lambda)$ should lie in an interval of the real line [12]. The proof is complete since

$$\sigma_d(L) = \left\{\lambda^2 : \lambda \in M_1\right\}.$$

□

Now, let us assume that the condition

$$\int_{-\infty}^{\infty} e^{\epsilon|t|}\,\|Q(t)\|\,dt < \infty,\ \ \epsilon > 0, \tag{3.4}$$

holds.

Theorem 3.3. *L has a finite number of eigenvalues.*

Proof. From the equalities (2.7)-(2.9) and (3.4) we have

$$\left\|K^{\pm}(x,t)\right\|,\ \left\|K_x^{\pm}(x,t)\right\|,\ \left\|K_t^{\pm}(x,t)\right\| \leq c\exp\left(-\epsilon(\frac{x+t}{2})\right),$$

and hence

$$\|F(t)\| \leq c \exp\left(-\epsilon(\frac{t}{2})\right), \ \forall t \in [0, \infty),$$

where c is a positive constant. Further,

$$\|F(t)\| \left|e^{i\lambda t}\right| \leq c \exp^{-t\left(\frac{\epsilon}{2} + \mathrm{Im}\lambda\right)}, \quad \forall t \in [0, \infty),$$

and thus

$$\begin{aligned} \left\|\int_0^\infty e^{i\lambda t} F(t) dt\right\| &\leq \int_0^\infty \left|e^{i\lambda t}\right| \|F(t)\| \, dt \\ &\leq \int_0^\infty c \exp^{-t\left(\frac{\epsilon}{2} + \mathrm{Im}\lambda\right)} dt, \end{aligned}$$

$$\int_0^\infty c \exp^{-t\left(\frac{\epsilon}{2} + \mathrm{Im}\lambda\right)} dt < \infty \Leftrightarrow \mathrm{Im}\lambda + \frac{\epsilon}{2} > 0.$$

The Uniform Convergence Test implies that the integral $\int_0^\infty e^{i\lambda t} F(t) dt$ is uniformly convergent in the domain $\mathrm{Im}\lambda > -\frac{\epsilon}{2}$. This implies $D(\lambda)$ and also $G(\lambda)$ have analytic continuations to the domain $\mathrm{Im}\lambda > -\frac{\epsilon}{2}$. Since the analytic continuation is unique, it follows

$$(G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \ \mathrm{Im}\lambda > -\frac{\epsilon}{2}.$$

Let us recall $M_1 = \{\lambda : \ \mathrm{Im}\lambda > 0, \ d(\lambda) = 0\}$, $\sigma_d(L) = \{\lambda^2 : \ \lambda \in M_1\}$ and M_1 is bounded. Suppose that M_1 is not finite. Let us recall Bolzano-Weierstrass Theorem which states that each bounded sequence in $\mathbb{R}$ has a convergent subsequence. Bolzano-Weierstrass Theorem implies that M_1 must have one limit point. Also, Theorem 3.2 states that the limit points of M_1 can only lie on the real axis. However, since $d(\lambda)$ is analytic in the domain $\mathrm{Im}\lambda > -\frac{\epsilon}{2}$, the limit points of M_1 should be on the boundary of this domain [12]. This contradicts with the assumption $\epsilon > 0$. Thus, M_1 and $\sigma_d(L)$ are finite. □

Let $M_2 := \{\lambda : \ \lambda \in \mathbb{R}, \ G(\lambda) \text{ is not invertible}\}$. It is obvious that

$$\sigma_{ss}(L) = \left\{\lambda^2 : \ \lambda \in M_2\right\} \setminus \{0\}.$$

From the representation (3.2), we have

$$\frac{G(\lambda)}{d(\lambda)} S(\lambda) = I, \quad \lambda \in \mathbb{C}_+,$$

and this implies $S(\lambda)$ is invertible iff $G(\lambda)$ is invertible or equivalently $d(\lambda) \neq 0$ for $\lambda \in \mathbb{C}_+$. If $d(\lambda) \neq 0$ it follows

$$(S(\lambda))^{-1} = \frac{G(\lambda)}{d(\lambda)}, \ \lambda \in \mathbb{C}_+,$$

and also

$$G(\lambda) = d(\lambda) \left(S(\lambda)\right)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{3.5}$$

Since $G(\lambda)$ is continuous on the real line, Equation (3.5) suggests that the functions $S(\lambda)$ and $d(\lambda)$ are continuous on the real line. Hence, we can extend the representation (3.2) continuously to the real line and obtain

$$(G(\lambda))^{-1} = \frac{S(\lambda)}{d(\lambda)}, \ \lambda \in \overline{\mathbb{C}_+}. \tag{3.6}$$

(3.6) implies that $G(\lambda)$ is invertible iff $d(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$. Thus, we have

$$M_2 = \{\lambda \in \mathbb{R} : \ d(\lambda) = 0\}.$$

Theorem 3.4. *M_2 is compact and has zero Lebesgue measure under the condition (2.2).*

Proof. Theorem 3.2 implies M_2 is bounded. We only have to show that M_2 is closed. Let $\{\lambda_n\} \subset M_2$ such that $\lambda_n \to \lambda_0$. $\{\lambda_n\} \subset M_2$ implies $\lambda_n \in \mathbb{R}$ and $G(\lambda_n)^{-1}$ doesn't exist for $n \in \mathbb{N}$. Further, $\lambda_n \to \lambda_0$ implies $\lambda_0 \in \mathbb{R}$. We have $G(\lambda)$ is a continuous operator function on the real line. Now, $\lambda_n \to \lambda_0$ suggests that $G(\lambda_n) \to G(\lambda_0)$ where the latter convergence is strong.

Let $GL(H) := \{A : A$ is invertible, bounded, linear operator on $H\}$. It is well known that $GL(H)$ is an open subset of the space $B(H)$ of bounded, linear operators on H. It follows $B(H) \setminus GL(H)$ is a closed set. This implies $G(\lambda_0) \in B(H) \setminus GL(H)$ and $\lambda_0 \in M_2$. Finally, Privalov's Theorem states that M_2 has zero Lebesgue measure [12] . □

Corollary 3.5. *$\sigma_{ss}(L)$ is bounded and has zero Lebesgue measure, under the condition (2.2).*

Theorem 3.6. *L has a finite number of spectral singularities, under the condition (3.4).*

Proof. It can be shown similarly (see the proof of Theorem 3.3) that $G(\lambda)$ has an analytic continuation to the domain $\text{Im}\lambda > -\frac{\epsilon}{2}$ for arbitrary $\epsilon > 0$. Since this analytic continuation is unique, it follows

$$(G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \ \text{Im}\lambda > -\frac{\epsilon}{2}.$$

Suppose that M_2 is not finite. Since M_2 is bounded (see Theorem 3.2), Bolzano-Weierstrass Theorem implies that M_2 has a limit point. This limit point as a zero of the analytic function $d(\lambda)$ should lie on the boundary of the domain $\text{Im}\lambda > -\frac{\epsilon}{2}$ [12]. It contradicts with $\epsilon > 0$. □

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