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Harmonic Aspects in an η-Ricci Soliton

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ABSTRACT

We characterize η-Ricci solitons (g, ξ, λ, μ) in some special cases when the 1-form η, which is the g-dual of ξ, is a harmonic or a Schrödinger-Ricci harmonic form. We also provide necessary and sufficient conditions for η to be a solution of the Schrödinger-Ricci equation and point out the relation between the three notions in our context. In particular, we apply these results to a perfect fluid spacetime and using Bochner-Weitzenböck techniques, we formulate some more conclusions for gradient solitons and deduce topological properties of the manifold and its universal covering.

Keywords: *gradient Ricci solitons; Schrödinger-Ricci equation; harmonic form.*

AMS Subject Classification (2020): *Primary: 35C08; 53C25.*

1. Introduction

Self-similar solutions to the Ricci flow, the *Ricci solitons* [31] have been studied in different geometrical contexts on complex, contact and paracontact manifolds. The more general notion of *η-Ricci soliton* was introduced by J. T. Cho and M. Kimura [22] on real hypersurfaces in a Kähler manifold and treated in complex space forms [21], Euclidean hypersurfaces [1], paracontact geometries [4], [5], [17], [18], [19], [26]. Different geometrical aspects of Ricci and η-Ricci solitons have been studied by author in [6], [13], [15]. Further generalizations of this notion and properties of other geometrical solitons can be found in [9], [11] and [2], [14].

A particular case of solitons arise when they evolve by diffeomorphism generated by a gradient vector field, namely when the potential vector field is the gradient of a smooth function. The gradient vector fields play a central rôle in Morse-Smale theory [37] and some aspects of gradient η-Ricci solitons were discusses by author in [3], [7], [8], [10], [12], [16].

In Section 2, after we point out the basic properties of an η-Ricci soliton (g, ξ, λ, μ), we provide necessary and sufficient conditions for the g-dual 1-form of the potential vector field ξ to be a solution of the Schrödinger-Ricci equation, a harmonic or a Schrödinger-Ricci harmonic form and characterize the 1-forms orthogonal to η. We end these considerations by discussing the case of a perfect fluid spacetime. In Section 3 we formulate the results for the special case of gradient solitons and deduce topological properties of the manifold and its universal covering [33].

2. Geometrical aspects of η

Let (M, g) be an n-dimensional Riemannian manifold, $n > 2$, and denote by $\flat : TM \to T^*M$, $\flat(X) := i_X g$, $\sharp : T^*M \to TM$, $\sharp := \flat^{-1}$ the musical isomorphism. Consider the set $\mathcal{T}^0_{2,s}(M)$ of symmetric $(0,2)$-tensor fields on M and for $Z \in \mathcal{T}^0_{2,s}(M)$, denote by $Z^\sharp : TM \to TM$ and by $Z_\sharp : T^*M \to T^*M$ the maps defined as follows:

$$g(Z^\sharp(X), Y) := Z(X, Y), \quad Z_\sharp(\alpha)(X) := Z(\sharp(\alpha), X).$$

We also denote by $Z^\sharp$ the map $Z^\sharp : T^*M \times T^*M \to C^\infty(M)$:

$$Z^\sharp(\alpha, \beta) := Z(\sharp(\alpha), \sharp(\beta))$$

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and can identify $Z_\sharp$ with the map also denoted by $Z_\sharp : T^*M \times TM \to C^\infty(M)$:

$$Z_\sharp(\alpha, X) := Z_\sharp(\alpha)(X).$$

Given a vector field X, its g-dual 1-form $X^\flat =: \flat(X)$ is said to be a *solution of the Schrödinger-Ricci equation* if it satisfies:

$$div(L_X g) = 0, \tag{2.1}$$

where $L_X g$ denotes the Lie derivative along the vector field X.

It is known that [24]:

$$div(L_X g) = (\Delta + S_\sharp)(X^\flat) + d(div(X)), \tag{2.2}$$

where Δ denotes the Laplace-Hodge operator on forms w.r.t. the metric g and S the Ricci curvature tensor field. Denoting by Q the Ricci operator defined by $g(QX, Y) := S(X, Y)$, for any vector fields X and Y, by a direct computation we deduce that $S_\sharp(\gamma) = i_{Q\gamma^\sharp} g$, for any 1-form γ.

η-Ricci solitons. We are interested to find the necessary and sufficient conditions for the g-dual 1-form η of the potential vector field ξ in an η-Ricci soliton to be a solution of the Schrödinger-Ricci equation, a harmonic or Schrödinger-Ricci harmonic form.

Consider the equation:

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2.3}$$

where g is a Riemannian metric, S its Ricci curvature tensor field, ξ a vector field, η a 1-form and λ and μ are real constants. The data (g, ξ, λ, μ) which satisfy the equation (2.3) is said to be an *η-Ricci soliton* on M [22]; in particular, if $\mu = 0$, (g, ξ, λ) is a *Ricci soliton* [31] and it is called *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive, respectively [25]. If the potential vector field ξ is of gradient-type, $\xi = grad(f)$, for f a smooth function on M, then (g, ξ, λ, μ) is called a *gradient η-Ricci soliton*.

Taking the trace of the equation (2.3) we obtain:

$$div(\xi) + scal + \lambda n + \mu|\xi|^2 = 0. \tag{2.4}$$

From a direct computation we get:

$$div(\eta \otimes \eta) = div(\xi)\eta + \nabla_\xi \eta.$$

Now taking the divergence of (2.3) and using (2.2) we obtain:

$$div(L_\xi g) + d(scal) + 2\mu[div(\xi)\eta + \nabla_\xi \eta] = 0. \tag{2.5}$$

Schrödinger-Ricci solutions. We say that a 1-form γ is a *solution of the Schrödinger-Ricci equation* if

$$(\Delta + S_\sharp)(\gamma) + d(div(\gamma^\sharp)) = 0. \tag{2.6}$$

Theorem 2.1. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ. Then η is a solution of the Schrödinger-Ricci equation if and only if*

$$d(scal) = 2\mu[(scal + \lambda n + \mu|\xi|^2)\eta - \nabla_\xi \eta]. \tag{2.7}$$

Moreover, in this case, $scal$ is constant if and only if $\mu = 0$ (which yields a Ricci soliton) or $(scal + \lambda n + \mu|\xi|^2)\eta = \nabla_\xi \eta$.

Proof. From (2.3), (2.4), (2.5) and

$$2div(S) = d(scal)$$

it follows that η is a solution of the Schrödinger-Ricci equation if and only if (2.7) holds. □

Remark 2.1. Under the hypotheses of Theorem 2.1, if the potential vector field is of constant length k, then from (2.7) we deduce that the scalar curvature is constant if either the soliton is a Ricci soliton or, $(scal + \lambda n + \mu k^2)\eta = \nabla_\xi \eta$ which implies $scal = -\lambda n - \mu k^2$.

Corollary 2.1. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ and assume that η is a nontrivial solution of the Schrödinger-Ricci equation. If $scal$ is constant and $\mu \neq 0$, then $\frac{1}{2|\xi|^2}\xi(|\xi|^2) - \mu|\xi|^2 = scal + \lambda n$ (constant).*

Proof. Under the hypotheses, from (2.7) we obtain:

$$(scal + \lambda n + \mu|\xi|^2)\eta = \nabla_\xi \eta,$$

applying ξ and taking into account that

$$(\nabla_\xi \eta)\xi = \frac{1}{2}\xi(|\xi|^2),$$

we deduce that $(scal + \lambda n + \mu|\xi|^2)|\xi|^2 = \frac{1}{2}\xi(|\xi|^2)$. □

For the case of Ricci solitons, from Theorem 2.1 we have:

Corollary 2.2. *If (g, ξ, λ) is a Ricci soliton on the n-dimensional manifold M and η is the g-dual of ξ, then η is a solution of the Schrödinger-Ricci equation if and only if the scalar curvature of the manifold is constant.*

Schrödinger-Ricci harmonic forms. We say that a 1-form γ is *Schrödinger-Ricci harmonic* if

$$(\Delta + S_\sharp)(\gamma) = 0.$$

From (2.6), (2.4) and (2.5) we deduce:

Theorem 2.2. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ. Then η is a Schrödinger-Ricci harmonic form if and only if $\mu = 0$ (which yields a Ricci soliton) or*

$$(scal + \lambda n + \mu|\xi|^2)\eta = \nabla_\xi \eta - \frac{1}{2}d(|\xi|^2). \tag{2.8}$$

Remark 2.2. Under the hypotheses of Theorem 2.2, if $\mu \neq 0$, then from (2.8) we deduce that the scalar curvature is constant if and only if the potential vector field is of constant length.

Harmonic forms. We know that on a Riemannian manifold (M, g), a 1-form γ is *harmonic* (i.e. $\Delta(\gamma) = 0$) if and only if it is closed and divergence free.

Remark that on an η-Ricci soliton, a harmonic 1-form γ is Schrödinger-Ricci harmonic if and only if

$$\gamma \circ \nabla\xi + \lambda\gamma + \mu\gamma(\xi)\eta = 0$$

which implies (using the fact that $(\nabla_X \gamma)^\sharp = \nabla_X \gamma^\sharp$, for any vector field X and any 1-form γ):

$$\gamma^\sharp \in \ker[\nabla_\xi \eta + (\lambda + \mu|\xi|^2)\eta].$$

From (2.2) and (2.5) we deduce:

Theorem 2.3. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ. Then η is a harmonic form if and only if*

$$i_{Q\xi}g = \mu\{2[(scal + \lambda n + \mu|\xi|^2)\eta - \nabla_\xi \eta] + d(|\xi|^2)\}. \tag{2.9}$$

For the case of Ricci solitons, from Theorem 2.3 we have:

Corollary 2.3. *If (g, ξ, λ) is a Ricci soliton on the n-dimensional manifold M and η is the g-dual of ξ, then η is a harmonic form if and only if $\xi \in \ker Q$.*

From (2.4), (2.8) and (2.9) we deduce:

Corollary 2.4. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ. If η is a harmonic form, then i) $\xi \in \ker Q$ and ii) the scalar curvature is constant if and only if the potential vector field ξ is of constant length.*

The relation between the cases when η is a solution of the Schrödinger-Ricci equation, harmonic or the Schrödinger-Ricci harmonic form is stated in the following result:

Lemma 2.1. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ.*
i) If η is a solution of the Schrödinger-Ricci equation, then η is:
a) Schrödinger-Ricci harmonic form if and only if $scal + \mu|\xi|^2$ is constant;
b) harmonic form if and only if $i_{Q\xi}g = d(scal + \mu|\xi|^2)$; also η harmonic implies $\xi \in \ker Q$.
ii) If η is Schrödinger-Ricci harmonic form, then η is:
a) a solution of the Schrödinger-Ricci equation if and only if $scal + \mu|\xi|^2$ is constant;
b) harmonic form if and only if $\xi \in \ker Q$.
iii) If η is a harmonic form, then η is:
a) a solution of the Schrödinger-Ricci equation if and only if $\xi \in \ker Q$;
b) Schrödinger-Ricci harmonic form if and only if $\xi \in \ker Q$.

We can synthesize:
i) if $scal + \mu|\xi|^2$ is constant, then η is Schrödinger-Ricci harmonic if and only if it is a solution of the Schrödinger-Ricci equation;
ii) if $\xi \in \ker Q$, then η is Schrödinger-Ricci harmonic if and only if it is harmonic.

1-***forms orthogonal to*** η**.** We say that two 1-forms γ_1 and γ_2 are *orthogonal* if $g(\gamma_1^\sharp, \gamma_2^\sharp) = 0$ (i.e. $\langle\gamma_1, \gamma_2\rangle = 0$, where $\langle\gamma_1, \gamma_2\rangle := \sum_{i=1}^{n} \gamma_1(E_i)\gamma_2(E_i)$, for $\{E_i\}_{1\leq i\leq n}$ a local orthonormal frame field).
Remark that γ_1 and γ_2 are orthogonal if and only if

$$\gamma_1^\sharp \in \ker \gamma_2 \ \ or \ \ \gamma_2^\sharp \in \ker \gamma_1.$$

Theorem 2.4. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ and $\mu \neq 0$. If γ is a 1-form, then γ is orthogonal to η if and only if*

$$\nabla_{\gamma^\sharp}\xi + Q\gamma^\sharp + \lambda\gamma^\sharp \in \ker \gamma. \tag{2.10}$$

Proof. Observe that computing the soliton equation in $(\gamma^\sharp, \gamma^\sharp)$ and using the orthogonality condition we obtain:

$$g(\nabla_{\gamma^\sharp}\xi, \gamma^\sharp) + g(Q\gamma^\sharp, \gamma^\sharp) + \lambda|\gamma^\sharp|^2 = 0 \tag{2.11}$$

which is equivalent to the condition (2.10). □

Example We end these considerations by discussing the case of a perfect fluid spacetime (M, g, ξ) [12]. If we denote by p the isotropic pressure, σ the energy-density, λ the cosmological constant, k the gravitational constant, S the Ricci curvature tensor field and $scal$ the scalar curvature of g, then [12]:

$$S = -(\lambda - \frac{scal}{2} - kp)g + k(\sigma + p)\eta \otimes \eta \tag{2.12}$$

and the scalar curvature of M is:

$$scal = 4\lambda + k(\sigma - 3p). \tag{2.13}$$

From Theorem 2.1, we deduce that if (g, ξ, a, b) is an η-Ricci soliton on (M, g, ξ), then η is a solution of the Schrödinger-Ricci equation if and only if

$$kd(\sigma - 3p) = 2b\{[4(a + \lambda) - b + k(\sigma - 3p)]\eta - \nabla_\xi\eta\}.$$

Moreover, the fluid is a radiation fluid (i.e. $\sigma = 3p$) if and only if $b = 0$ (which yields the Ricci soliton) or $[4(a + \lambda) - b]\eta = \nabla_\xi\eta$ which implies $b = 4(a + \lambda)$.
From Theorem 2.2, we deduce that if (g, ξ, a, b) is an η-Ricci soliton on (M, g, ξ), then η is a Schrödinger-Ricci harmonic form if and only if $b = 0$ (which yields a Ricci soliton) or

$$[4(a + \lambda) - b + k(\sigma - 3p)]\eta = \nabla_\xi\eta$$

which implies $b = 4(a + \lambda) + k(\sigma - 3p)$.
From Theorem 2.3, we deduce that if (g, ξ, a, b) is an η-Ricci soliton on (M, g, ξ), then η is a harmonic form if and only if

$$\{4b[4(a + \lambda) - b + k(\sigma - 3p)] - 2\lambda + k(\sigma + 3p)\}\eta = 4b\nabla_\xi\eta.$$

For the case of Ricci soliton (g, ξ, a) in a radiation fluid we obtain the constant pressure $p = \frac{\lambda}{3k}$.

3. Applications to gradient solitons

Let $f \in C^\infty(M)$, $\xi := grad(f)$, $\eta := \xi^\flat$ and λ and μ real constants. Then $\eta = df$ and

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{3.1}$$

for any $X, Y \in \mathfrak{X}(M)$. Also [5]:

$$trace(\eta \otimes \eta) = |\xi|^2, \tag{3.2}$$

$$div(\eta \otimes \eta) = div(\xi)\eta + \frac{1}{2}d(|\xi|^2) \tag{3.3}$$

and

$$\nabla_\xi \eta = \frac{1}{2}d(|\xi|^2). \tag{3.4}$$

For the gradient η-Ricci solitons we have:

Proposition 3.1. *If $(g, \xi := grad(f), \lambda, \mu)$ is a gradient η-Ricci soliton on the n-dimensional manifold M and $\eta = df$ is the g-dual of ξ, then η is a solution of the Schrödinger-Ricci equation if and only if*

$$d(scal) = 2\mu[(scal + \lambda n + \mu|\xi|^2)df - \frac{1}{2}d(|\xi|^2)]. \tag{3.5}$$

Moreover, in this case, $scal$ is constant if and only if $\mu = 0$ (which yields a gradient Ricci soliton) or $(scal + \lambda n + \mu|\xi|^2)df = \frac{1}{2}d(|\xi|^2)$.

Remark 3.1. Under the hypotheses of Proposition 3.1, if the potential vector field is of constant length k, then (3.5) becomes:

$$d(scal) = 2\mu(scal + \lambda n + \mu k^2)df, \tag{3.6}$$

so the scalar curvature is constant if either the soliton is a gradient Ricci soliton or $scal = -\lambda n - \mu k^2$.

Remark 3.2. i) Taking into account that for a gradient vector field ξ [10]:

$$div(L_\xi g) = 2d(div(\xi)) + 2i_{Q\xi}g, \tag{3.7}$$

the condition for the g-dual $\eta = df$ of the potential vector field $\xi := grad(f)$ of a gradient η-Ricci soliton (g, ξ, λ, μ) to be a solution of the Schrödinger-Ricci equation is:

$$d(scal + \mu|\xi|^2) = i_{Q\xi}g. \tag{3.8}$$

In this case, $scal + \mu|\xi|^2$ is constant if and only if $\xi \in \ker Q$ and from the η-Ricci soliton equation we obtain $\nabla_\xi \xi = -(\lambda + \mu|\xi|^2)\xi$. Applying η we get $\lambda + \mu|\xi|^2 = -\frac{1}{2|\xi|^2}\xi(|\xi|^2)$, therefore, if the length of ξ is constant (also, the scalar curvature will be constant), then $|\xi|^2 = -\frac{\lambda}{\mu}$, hence ξ is a geodesic vector field.

ii) If ξ is an eigenvector of Q (i.e. $Q\xi = a\xi$, with a a smooth function), then η is a solution of the Schrödinger-Ricci equation if and only if $scal + \mu|\xi|^2 - af$ is constant. In particular, if $\xi \in \ker Q$, then η is a solution of the Schrödinger-Ricci equation if and only if η is a harmonic form.

iii) If η is a Schrödinger-Ricci harmonic form, then $d(scal + \mu|\xi|^2) = 2i_{Q\xi}g$. In this case, $scal + \mu|\xi|^2$ is constant if and only if $\xi \in \ker Q$ and using the same arguments as in i) we deduce that ξ is a geodesic vector field.

Also, an exact 1-form df is harmonic if and only if the function f is harmonic. In the case of a gradient η-Ricci soliton, for η harmonic form, denoting by $\Delta_f := \Delta - \nabla_{grad(f)}$ the f-Laplace-Hodge operator, the result stated in Theorem 3.2 from [10] becomes:

Theorem 3.1. *Let $(g, \xi := grad(f), \lambda, \mu)$ be a gradient η-Ricci soliton on the n-dimensional manifold M with $\eta = df$ the g-dual of ξ. If η is a harmonic form, then:*

$$\frac{1}{2}\Delta_f(|\xi|^2) = |Hess(f)|^2 + \lambda|\xi|^2 + \mu|\xi|^4. \tag{3.9}$$

Using Corollary 2.4 we get:

Corollary 3.1. *Under the hypotheses of Theorem 3.1, if M is of constant scalar curvature, then at least one of λ and μ is non positive.*

As a consequence for the case of gradient Ricci soliton, we have:

Proposition 3.2. *Let $(g, \xi := grad(f), \lambda)$ be a gradient Ricci soliton on the n-dimensional manifold M of constant scalar curvature, with $\eta = df$ the g-dual of ξ. If η is a harmonic form, then the soliton is shrinking.*

Proof. From Theorem 2.4 and Theorem 3.1 we obtain $|Hess(f)|^2 + \lambda|\xi|^2 = 0$, hence $\lambda < 0$. □

Remark 3.3. i) Assume that $\mu \neq 0$. If $\lambda \geq -\mu|\xi|^2$, then $\Delta_f(|\xi|^2) \geq 0$ and from the maximum principle follows that $|\xi|^2$ is constant in a neighborhood of any local maximum. If $|\xi|$ achieve its maximum, then M is quasi-Einstein. Indeed, since $Hess(f) = 0$, from the soliton equation we have $S = -\lambda g - \mu df \otimes df$. Moreover, in this case, $|\xi|^2(\lambda + \mu|\xi|^2) = 0$, which implies either $\xi = 0$ or $|\xi|^2 = -\frac{\lambda}{\mu} \geq 0$. Since $scal + \lambda n + \mu|\xi|^2 = 0$ we get $scal = \lambda(1 - n)$.

ii) For $\mu = 0$, we get the Ricci soliton case [35].

Computing the gradient soliton equation in $(\gamma^\sharp, X)$, $X \in \mathfrak{X}(M)$, we obtain:

$$g(\nabla_{\gamma^\sharp}\xi, X) + g(Q\gamma^\sharp, X) + \lambda g(\gamma^\sharp, X) + \mu\eta(\gamma^\sharp)\eta(X) = 0$$

and taking $X := \xi$ we get:

$$\frac{1}{2}\gamma^\sharp(|\xi|^2) + \gamma(Q\xi) + (\lambda + \mu|\xi|^2)\eta(\gamma^\sharp) = 0.$$

Therefore:

Proposition 3.3. *Let (g, ξ, λ, μ) be an η-Ricci soliton on the n-dimensional manifold M with η the g-dual of ξ and $\mu \neq 0$. If γ is a 1-form, then γ is orthogonal to η if and only if*

$$\nabla_{\gamma^\sharp}\xi + Q\gamma^\sharp + \lambda\gamma^\sharp = 0, \tag{3.10}$$

hence:

$$\frac{1}{2}\gamma^\sharp(|\xi|^2) = -\gamma(Q\xi). \tag{3.11}$$

Some results concerning the harmonic 1-forms on gradient η-Ricci solitons are further presented.

For two $(0,2)$-tensor fields T_1 and T_2, denote by $\langle T_1, T_2\rangle := \sum_{1\leq i,j\leq n} T_1(E_i, E_j)T_2(E_i, E_j)$, for $\{E_i\}_{1\leq i\leq n}$ a local orthonormal frame field.

Theorem 3.2. *Let M be a compact and oriented n-dimensional manifold M, $(g, \xi := grad(f), \lambda, \mu)$ a gradient η-Ricci soliton with $\eta = df$ the g-dual of ξ and γ a 1-form.*

1. *If γ is orthogonal to η and $\mu \neq 0$, then $\gamma^\sharp \in \ker(\nabla_\xi\eta + \eta \circ Q)$.*
2. *If γ is harmonic, then either we have a Ricci soliton or $\nabla_\xi\gamma^\sharp \in \ker\eta$.*
3. *If γ is exact with $\gamma = du$, then:*

$$\int_M \langle S, div(du)\rangle = -\int_M \langle Hess(f), Hess(u)\rangle - \mu(df|\nabla_{grad(f)}grad(u)). \tag{3.12}$$

Moreover, if γ is harmonic, the relation (3.12) becomes:

$$\int_M \langle Hess(f), Hess(u)\rangle = -\mu(df|\nabla_{grad(f)}grad(u)). \tag{3.13}$$

Proof. From (3.11) and using (3.1) we get:

$$0 = g(\nabla_{\gamma^\sharp}\xi, \xi) + g(Q\xi, \gamma^\sharp) = \xi(\eta(\gamma^\sharp)) - \eta(\nabla_\xi\gamma^\sharp) + g(\xi, Q\gamma^\sharp) = (\nabla_\xi\eta)\gamma^\sharp + \eta(Q\gamma^\sharp)$$

and hence 1.

Let $\{E_i\}_{1\leq i\leq n}$ be a local orthonormal frame field with $\nabla_{E_i}E_j = 0$ in a point. For any symmetric $(0,2)$-tensor field Z and any 1-form γ:

$$\langle Z, L_{\gamma^\sharp}g\rangle = \sum_{1\leq i,j\leq n} Z(E_i,E_j)(L_{\gamma^\sharp}g)(E_i,E_j) = 2\sum_{1\leq i,j\leq n} Z(E_i,E_j)g(\nabla_{E_i}\gamma^\sharp, E_j) =$$

$$= 2\sum_{1\leq i,j\leq n} Z(E_i,E_j)E_i(\gamma(E_j)) = 2\langle Z, div(\gamma)\rangle.$$

Also:

$$\langle g, L_{\gamma^\sharp}g\rangle = \sum_{i=1}^{n}(L_{\gamma^\sharp}g)(E_i,E_i) = 2\sum_{i=1}^{n} g(\nabla_{E_i}\gamma^\sharp, E_i) = 2div(\gamma^\sharp)$$

and

$$\langle df\otimes df, L_{\gamma^\sharp}g\rangle = \sum_{1\leq i,j\leq n} df(E_i)df(E_j)(L_{\gamma^\sharp}g)(E_i,E_j) = 2\sum_{1\leq i,j\leq n} df(E_i)df(E_j)g(\nabla_{E_i}\gamma^\sharp, E_j) =$$

$$= 2g(\nabla_{grad(f)}\gamma^\sharp, grad(f)) = 2g((\nabla_{grad(f)}\gamma)^\sharp, (df)^\sharp).$$

Computing $\langle S, div(\gamma)\rangle$ by replacing S from the η-Ricci soliton equation, we obtain:

$$\langle S, div(\gamma)\rangle = -\frac{1}{2}\langle Hess(f), L_{\gamma^\sharp}g\rangle - \lambda div(\gamma^\sharp) - \mu g((\nabla_{grad(f)}\gamma)^\sharp, (df)^\sharp).$$

For 2. we use $div(\gamma) = 0 = div(\gamma^\sharp)$ and for 3. we use the fact that $\gamma^\sharp = grad(u)$, hence $L_{\gamma^\sharp}g = 2Hess(u)$ and apply the divergence theorem. □

Since

$$\eta(\nabla_\xi\xi) = \frac{1}{2}\xi(|\xi|^2)$$

and for η harmonic:

$$\int_M |Hess(f)|^2 = -\mu\int_M df(\nabla_\xi\xi),$$

we get:

Corollary 3.2. *Under the hypotheses of Theorem 3.2, if η is a harmonic form, then either we have a Ricci soliton or the potential vector field ξ is of constant length. In the second case, η is a solution of the Schrödinger-Ricci equation and M is a quasi-Einstein manifold.*

We know that the Bochner formula, for an arbitrary vector field ξ [10], states:

$$\frac{1}{2}\Delta(|\xi|^2) = |\nabla\xi|^2 + S(\xi,\xi) + \xi(div(\xi))$$

and taking into account that the g-dual 1-form η of ξ satisfies

$$|\xi| = |\eta|, \ \ |\nabla\xi| = |\nabla\eta|, \ \ S(\xi,\xi) = S^\sharp(\eta,\eta), \ \ \xi(div(\xi)) = \langle\Delta(\eta),\eta\rangle,$$

we have the corresponding relation for η:

$$\frac{1}{2}\Delta(|\eta|^2) = |\nabla\eta|^2 + S^\sharp(\eta,\eta) + \langle\Delta(\eta),\eta\rangle. \tag{3.14}$$

Let γ be a 1-form and writing the previous relation for $\eta+\gamma$ we obtain:

$$\frac{1}{2}\Delta(\langle\eta,\gamma\rangle) = \langle\nabla\eta,\nabla\gamma\rangle + S^\sharp(\eta,\gamma) + \frac{1}{2}(\langle\Delta(\eta),\gamma\rangle + \langle\Delta(\gamma),\eta\rangle).$$

Theorem 3.3. *Let M be an n-dimensional manifold, $(g, \xi := grad(f), \lambda, \mu)$ a gradient η-Ricci soliton with $\eta = df$ the g-dual of ξ and γ a 1-form. Then:*

$$\frac{1}{2}\Delta(\langle df,\gamma\rangle) = \langle Hess(f),\nabla\gamma\rangle - \mu\Delta(f)\langle df,\gamma\rangle + \frac{1}{2}\langle df,\Delta(\gamma)\rangle. \tag{3.15}$$

Proof. From (2.4), (3.3), (3.7) and $2div(S) = d(scal)$, we get:

$$S^{\sharp}(\eta,\gamma) = S(\xi,\gamma^{\sharp}) = -\frac{1}{2}d(\Delta(f))(\gamma^{\sharp}) - \mu\Delta(f)df(\gamma^{\sharp}) = -\frac{1}{2}\langle\Delta(df),\gamma\rangle - \mu\Delta(f)\langle df,\gamma\rangle,$$

hence (3.15). □

Proposition 3.4. *Let M be an n-dimensional manifold, $(g, \xi := grad(f), \lambda, \mu)$ a gradient η-Ricci soliton with $\eta = df$ the g-dual of ξ and γ a 1-form.*

1. *If γ is orthogonal to η, then $\langle Hess(f), \nabla\gamma\rangle = -\frac{1}{2}\langle df, \Delta(\gamma)\rangle$.*
2. *If γ is harmonic, then $\frac{1}{2}\Delta(\langle df,\gamma\rangle) = \langle Hess(f),\nabla\gamma\rangle - \mu\Delta(f)\langle df,\gamma\rangle$. In this case, $\langle df,\gamma\rangle$ is harmonic if and only if $\mu\Delta(f)\langle df,\gamma\rangle = \langle Hess(f),\nabla\gamma\rangle$.*

   Moreover, if γ is orthogonal to η, then $\nabla\gamma$ is orthogonal to $\nabla\eta$.

L^2_f harmonic 1-forms. Endow the Riemannian manifold (M, g) with the weighted volume form $e^{-f}dV$ and define *L^2_f forms* those forms γ satisfying $\int_M |\gamma|^2 e^{-f}dV < \infty$.

The most natural operator of Laplacian-type associated to the weighted manifold $(M, g, e^{-f}dV)$ is the f-Laplace-Hodge operator

$$\Delta_f := \Delta - \nabla_{grad(f)}$$

which is self-adjoint with respect to this measure.

We say that a 1-form γ is *f-harmonic* if

$$\Delta_f(\gamma) = 0.$$

Remark that γ is f-harmonic if and only if

$$\Delta(\gamma) = i_{\nabla^{\sharp}_{\gamma}\xi}g.$$

From (2.4) and (3.4) we deduce:

Proposition 3.5. *Let $(g, \xi := grad(f), \lambda, \mu)$ be a gradient η-Ricci soliton on the n-dimensional manifold M with $\eta = df$ the g-dual of ξ. Then η is an f-harmonic form if and only if $scal + (\mu + \frac{1}{2})|\xi|^2$ is constant.*

In terms of Δ_f, the relation (3.14) can be written [34]:

$$\frac{1}{2}\Delta_f(|\gamma|^2) = |\nabla\gamma|^2 + S^{\sharp}_f(\gamma,\gamma) + \langle\Delta_f(\gamma),\gamma\rangle, \tag{3.16}$$

where $S_f := Hess(f) + S$ is the Bakry-Émery Ricci tensor.

Using a Reilly-type formula involving the f-Laplacian, an interesting result was obtained in [29], namely, if the manifold M is the boundary of a compact and connected Riemannian manifold and has non negative m-dimensional Bakry-Émery Ricci curvature and non negative f-mean curvature, then either M is connected or it has only two connected components, in the later case, being totally geodesic.

Another interesting topological property will be stated in the next theorem:

Theorem 3.4. *Let $(M^n, g, e^{-f}dV)$ be a complete, non compact smooth metric measure space and $(g, \xi := grad(f), \lambda, \mu)$ a gradient η-Ricci soliton with $\eta = df$ the g-dual of ξ. If there exists a non trivial L^2_f harmonic 1-form γ_0 such that $\lambda|\gamma_0|^2 + \mu(\gamma_0(\xi))^2 \leq 0$, then M has finite volume and its universal covering splits isometrically into $\mathbb{R} \times N^{n-1}$.*

Proof. The condition $\lambda|\gamma_0|^2 + \mu(\gamma_0(\xi))^2 \leq 0$ is equivalent to $S^{\sharp}_f(\gamma_0,\gamma_0) \geq 0$. From (3.16) and Lemma 3.2 from [38]:

$$|\gamma_0|\Delta_f(|\gamma_0|) \geq 0.$$

Following the same steps as in [38], we obtain the conclusion. □

Remark 3.4. i) Under the hypothesis of Theorem 3.4, in particular, we deduce that γ_0 is ∇-parallel and of constant length. Also, $\lambda \leq 0$ since in [36] was shown that $\lambda > 0$ implies M compact.

ii) In the Ricci soliton case, the hypothesis of Theorem 3.4 requires that the space of L^2_f harmonic 1-forms to be nonempty and the Ricci soliton to be shrinking in order to get the same conclusion.

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