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Internal Categories in Crossed Semimodules and Schreier Internal Categories

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Abstract
In this paper, we characterize internal categories in the category of crossed semimodules and the category of Schreier internal categories within monoids. Then we prove a natural equivalence between their categories. This allows us to produce various examples of double categories.

Keywords: *Crossed module; Crossed semimodule; Schreier internal category; Double category.*

AMS Subject Classification (2020): *Primary: 20L05; Secondary: 18D35; 18B40; 20M50.*

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1. Introduction

Crossed modules, which are defined by Whitehead in [22, 23] as algebraic models for homotopy 2-types, have been used with several branches of mathematics such as homotopy theory [6], homological algebra [10], and algebraic K-theory [12]. The structure of a crossed module (A, B, ∂) has a pair of groups and a homomorphism ∂ with an action $\bullet$ of B on A satisfying $\partial(b \bullet a) = b\partial(a)b^{-1}$ and $\partial(a) \bullet a' = aa'a^{-1}$. The category of crossed modules is naturally equivalent to several algebraic and combinatorial categories such as group-groupoids (or alternatively called in literature as $\mathcal{G}$-groupoids [5], geoup-groupoids [7] or 2-groups [2]). This equivalence has been generalized in different ways: for example, by taking 2-groupoids with a single object [1]. The other one is the generalization to monoids by taking special types of internal categories like Schreier internal categories in the category MON of monoids using crossed semimodules by Patchkoria [15]. For the topological aspect of the results of [15], see [19]. For the 2-categorical approach to Schreier internal categories in MON using Schreier 2-categories with one object, see [20]. Patchkoria also defined the category of Schreier internal groupoids in MON which is equivalent to the category of crossed semimodules $C = (A, B, \partial)$ where A is a group. This natural equivalence is the special case of the main theorem of [16] by Porter.

Double groupoids, which should be thought of as internal groupoids in the category GPD of groupoids, were introduced by Ehresmann in [8, 9] and have been used in mathematics and mathematical physics as an application of categorical methods for some problems. As an example, one can see the reference [6] for the proof of the 2-dimensional Seifert-Van-Kampen Theorem using double groupoids. For an extension of topological quantum field theories via double categories, see [11]. Brown and Spencer proved the categorical equivalence between crossed modules and special double groupoids in the sense that the vertical and horizontal groupoids agree with a single point [4]. Using this equivalence, normal and quotient objects in the category of crossed modules over groupoids and of double groupoids with thin structures were compared and the corresponding structures related double groupoids were characterized in [14] (see also [21]).

In this paper, we characterize internal categories in the category of crossed semimodules which are extensions of internal categories in the category of crossed modules [17] and the category of Schreier internal categories in the category MON of monoids which are general cases of double group-groupoids characterized in [18]. Finally, we prove the categorical equivalence between their structures. This equivalence enables us to obtain more examples of double categories.

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2. Preliminaries

A category $\mathcal{C} = (C_0, C)$ consists of the set C_0 of objects, the set $C = \cup_{x,y \in C_0} C(x,y)$ of morphisms where $C(x,y)$ is the set of morphisms from x to y with the source and the target maps $s, t \colon C \to C_0$, respectively, such that $s(c) = x, t(c) = y$ where $x \xrightarrow{c} y$, the associative composition map $m \colon C(y,z) \times C(x,y) \to C(x,z)$ usually written as $m(d,c) = d \circ c$ and the unit map $\varepsilon \colon C_0 \to C$, $\varepsilon(x) = 1_x \in C(x,x)$ such that $s\varepsilon = t\varepsilon = 1_{C_0}$, $c \circ 1_x = c$ and $1_x \circ c' = c'$, where $s(c) = t(c') = x$. A groupoid $\mathcal{G} = (G_0, G)$ is a category in which all morphisms are invertible. For further details, see [3, 13].

Let $\mathcal{C}$ be a category and D_0, D be two objects of $\mathcal{C}$ with maps s, t, ε, m as morphisms of $\mathcal{C}$ such that satisfy the usual category axioms where $D_s \times_t D$ is the pullback of s, t with π_1, π_2.

$$D_s \times_t D \xrightarrow{\ m\ } D \underset{t}{\overset{s}{\rightrightarrows}} D_0, \qquad D \xleftarrow{\ \varepsilon\ } D_0$$

Then $\mathcal{D} = (D_0, D)$ is called an internal category in $\mathcal{C}$, if the following conditions are satisfied:

IC 1. $s\varepsilon = t\varepsilon = 1_{D_0}$;

IC 2. $sm = s\pi_2, \ \ tm = t\pi_1$;

IC 3. $m(1_D \times m) = m(m \times 1_D)$ and

IC 4. $m(\varepsilon s, 1_D) = m(1_D, \varepsilon t) = 1_D$.

For further details, see [13].

Let $\mathcal{D}$ and $\mathcal{D}'$ be two internal categories in $\mathcal{C}$. Then an internal functor $F = (f_0, f) \colon \mathcal{D} \to \mathcal{D}'$ consists of a pair of morphisms $f_0 \colon D_0 \to D'_0$ and $f \colon D \to D'$ in $\mathcal{C}$ such that

(i) $s'f = f_0 s, t'f = f_0 t,$

(ii) $\varepsilon' f_0 = f_1 \varepsilon,$

(iii) $m'(f \times f) = fm.$

Thus we can construct the category of internal categories in an arbitrary category $\mathcal{C}$ with pullbacks where the morphisms are internal functors (or morphisms of internal categories) as defined above. This category is denoted by CAT($\mathcal{C}$).

Let $\mathcal{M} = (M_0, M)$ be an internal category in the category MON of monoids. If for any $c \in M$ there exists a unique $\widehat{c} \in \ker s$ such that

$$c = \widehat{c} \cdot \varepsilon s(c),$$

then $\mathcal{M}$ is called a *Schreier internal category* in MON and this condition is called *the Schreier condition* [15]. In a Schreier internal category, the monoid multiplication and the composition of morphisms give the following interchange rule

$$(d \circ c) \cdot (d' \circ c') = (d \cdot d') \circ (c \cdot c') \tag{2.1}$$

whenever compositions are defined. Using this rule, the composition of morphisms can be written in terms of the monoid product as

$$d \circ c = \widehat{d} \cdot \widehat{c} \cdot \varepsilon s(c) \tag{2.2}$$

for all $c, d \in M$ with $t(c) = s(d)$. This means that $\widehat{d \circ c} = \widehat{d} \cdot \widehat{c}$.

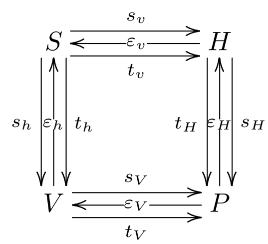
A *Schreier internal groupoid* in MON is a Schreier internal category in which all morphisms are invertible.

Let $\mathcal{M}, \mathcal{N}$ be Schreier internal categories in MON. An internal functor $F = (f_0, f) \colon \mathcal{M} \to \mathcal{N}$ such that $f_0 \colon M_0 \to N_0, f \colon M \to N$ are morphisms of monoids is called a morphism of Schreier internal categories in MON.

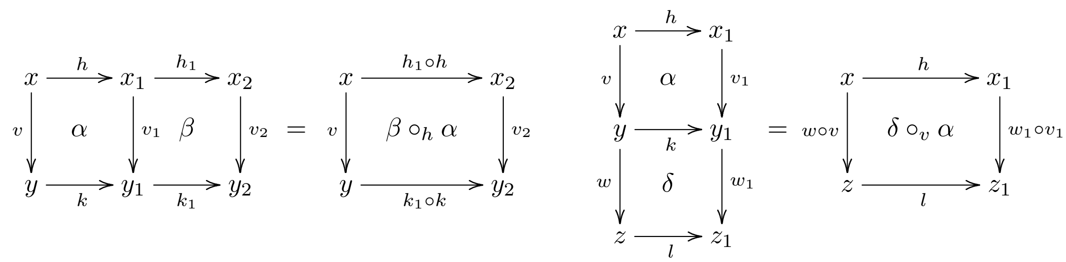
$$\begin{array}{ccccc}
M \,{}_s\times_t M & \xrightarrow{\ m\ } & M & \underset{t}{\overset{s}{\rightrightarrows}} \ (\overset{\varepsilon}{\leftarrow}) & M_0 \\
\downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\
N \,{}_s\times_t N & \xrightarrow{\ m\ } & N & \underset{t}{\overset{s}{\rightrightarrows}} \ (\underset{\varepsilon}{\leftarrow}) & N_0
\end{array}$$

Hence Schreier internal categories in MON form a category which we denote by SIC.

A double category denoted by $\mathfrak{C} = (S, H, V, P)$ consists of the sets S, H, V and P of squares, horizontal and vertical morphisms (or edges) and points, respectively. The structure of a double category contains four compatible category structures as partially shown in the following diagram:



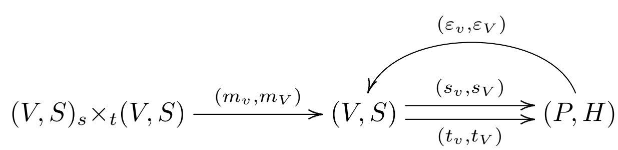
In a double category, vertical and horizontal morphisms can be composed as in an ordinary category, and squares can be composed vertically and horizontally.



Horizontal composition and vertical composition of squares must satisfy the following interchange rule:

$$(\theta \circ_h \delta) \circ_v (\beta \circ_h \alpha) = (\theta \circ_v \alpha) \circ_h (\delta \circ_v \alpha).$$

A double category can be regarded as an internal category in the category CAT of all small categories as shown in the following diagram:



where $s = (s_v, s_V), t = (t_v, t_V)$. A double groupoid is a double category in which four underlying categories are groupoids. For further details, see [4, 8, 9].

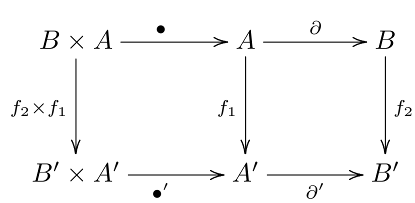
A crossed semimodule $C = (A, B, \partial)$ consists of a pair of monoids A, B and a morphism $\partial\colon A \to B$ of monoids with an action $\bullet\colon B \times A \to A$ of B on A satisfying

CSM 1. $\partial(b \bullet a) \cdot b = b \cdot \partial(a)$

CSM 2. $(\partial(a) \bullet a_1) \cdot a = a \cdot a_1$,

for $a, a_1 \in A$ and $b \in B$ [15].

Let $C = (A, B, \partial)$ and $C' = (A', B', \partial')$ be crossed semimodules. A morphism of crossed semimodules is a mapping $f = \langle f_1, f_2\rangle\colon C \to C'$ where $f_1\colon A \to A'$ and $f_2\colon B \to B'$ are morphisms of monoids such that $f_1(b \bullet a) = f_2(b) \bullet' f_1(a)$ and $f_2\partial = \partial' f_1$.



Hence crossed semimodules and their morphisms form a category which we will denote by CSM.

The following theorem was proved by Patchkoria in [15]. Since we need some details of the proof, we give a sketch proof in terms of our notation.

Theorem 2.1. *The category* SIC *of Schreier internal categories in* MON *is naturally equivalent to the category* CSM *of crossed semimodules.*

Proof: A functor $\delta\colon$ SIC $\to$ CSM is defined as an equivalence of categories. Let $\mathcal{M} = (M_0, M)$ be a Schreier internal category in MON. Then $\delta(\mathcal{M}) = (A, B, \partial)$ is a crossed semimodule where $A = \ker s$, $B = M_0$, $\partial = t|_{\ker s}$ and action of M_0 on $\ker s$ is defined by $(x \bullet c) \cdot \varepsilon s(c) = \varepsilon s(c) \cdot c$ for all $x \in M_0$ and $c \in \ker s$.

A functor $\gamma\colon$ CSM $\to$ SIC is defined as a weak inverse of δ. Let $C = (A, B, \partial)$ be a crossed semimodule. Then $\gamma(A, B, \partial) = (B, B \ltimes A, s, t, \varepsilon, \circ)$ is a Schreier internal category in MON where $B \ltimes A$ is the semi-direct product of monoids with the product $(b, a) \cdot (b', a') = (b \cdot b', a \cdot (b \bullet a'))$. The structure maps are defined by $s(b, a) = b$, $t(b, a) = \partial(a) \cdot b$, $\varepsilon(b) = (b, e_A)$ where e_A is the identity element of A and the composition of morphisms is defined by $(\partial(a) \cdot b, a_1) \circ (b, a) = (b, a_1 \cdot a)$. Since $(b, a) = (e_B, a) \cdot (b, e_A)$, all morphisms satisfy the Schreier condition where e_B is the identity element of B.

In order to define a natural transformation $\eta\colon \mathcal{M} \to \gamma\delta(\mathcal{M})$, a map $\eta_{\mathcal{M}}$ is defined to be identity on objects, and is defined by $\eta_{\mathcal{M}}(c) = (s(c), \widehat{c})$ for $c \in M$.

In order to define a natural transformation $\mu\colon 1_{\text{CSM}} \to \delta\gamma$, a map μ_C is defined to be identity on B and is defined by $a \mapsto (e_B, a)$ on A. □

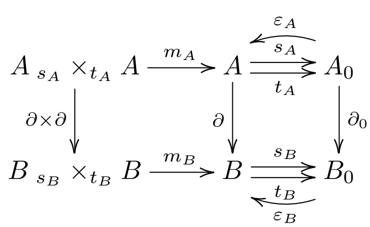
The following results can be obtained as restrictions of this equivalence.

Corollary 2.1. *[15] The category* SIG *of Schreier internal groupoids in* MON *is equivalent to the category of crossed semimodules (A, B, ∂) where A is a group.*

Theorem 2.2. *[5] The category of group-groupoids is equivalent to the category of crossed modules over groups.*

3. Internal categories in crossed semimodules

Some properties of internal categories in the category CM of crossed modules are examined in [17]. In this section, we shall characterize internal categories in the category CSM of crossed semimodules and generalise some results given in [17]. Since the category MON of monoids has pullbacks then we can talk about internal categories in MON. Let $\mathcal{C} = (C_0, C, s, t, \varepsilon, m)$ be an internal category in CSM. This means that $\mathcal{C}$ consists of a pair of crossed semimodules $C_0 = (A_0, B_0, \partial_0)$, $C = (A, B, \partial)$ and four morphisms of crossed semimodules $s = \langle s_A, s_B \rangle$, $t = \langle t_A, t_B \rangle$, $\varepsilon = \langle \varepsilon_A, \varepsilon_B \rangle$, $m = \langle m_A, m_B \rangle$ which are called the source and the target maps, the identity object map and the composition map, respectively, as shown in the following diagram:



Since $s\varepsilon = \langle s_A, s_B \rangle \langle \varepsilon_A, \varepsilon_B \rangle = \langle s_A \varepsilon_A, \varepsilon_B s_B \rangle = \langle 1_{A_0}, 1_{B_0} \rangle = 1_{C_0}$, the condition [IC 1] is satisfied. The other conditions [IC 1]-[IC 4] can be proved using a similar way.

Example 3.1. Let $C_0 = (A, B, \partial)$ be a crossed semimodule. Since $C = (A \times A, B \times B, \partial \times \partial)$ is also a crossed semimodule where the action of $B \times B$ on $A \times A$ is defined by $(b, b') \bullet (a, a') = (b \bullet a, b' \bullet a')$, then $\mathcal{C} = (C_0, C, s, t, \varepsilon, m)$ becomes an internal category in CSM where the structure maps are defined by $s = \langle \pi_1, \pi_1 \rangle$, $t = \langle \pi_2, \pi_2 \rangle$, $\varepsilon_A(a) = (a, a)$, $\varepsilon_B = (b, b)$ and $(a', a'') \circ (a, a') = (a, a'')$, $(b', b'') \circ (b, b') = (b, b'')$, for $a, a', a'' \in A$, $b, b', b'' \in B$.

Example 3.2. Let $C = (A, B, \partial)$ be a crossed semimodule. Then $\mathcal{C} = (C, C, s, t, \varepsilon, m)$ is an internal category in CSM where the structure maps are identities.

Lemma 3.1. *Let $\mathcal{C} = (C_0, C, s, t, \varepsilon, m)$ be an internal category in* CSM*. Then*

(i) $s_A(aa') = s_A(a)s_A(a')$, $t_A(aa') = t_A(a)t_A(a')$, $s_B(bb') = s_B(b)s_B(b')$, $t_B(bb') = t_B(b)t_B(b')$,

(ii) $\varepsilon_A(xx') = \varepsilon_A(x)\varepsilon(x')$, $\varepsilon_B(yy') = \varepsilon_B(y)\varepsilon_B(y')$,

(iii) $\partial_0 s_A = s_B \partial$, $\partial_0 t_A = t_B \partial$, $\partial \varepsilon_A = \varepsilon_B \partial_0$,

(iv) $s_A(b \bullet a) = s_B(b) \bullet s_A(a), t_A(b \bullet a) = t_B(b) \bullet t_A(a), \varepsilon_A(y \bullet x) = \varepsilon_B(y) \bullet \varepsilon_A(x)$,

for $a, a' \in A, x, x' \in A_0, b, b' \in B, y, y' \in B_0$.

Proof: The proof is clear, since $s = \langle s_A, s_B \rangle$, $t = \langle t_A, t_B \rangle$, $\varepsilon = \langle \varepsilon_A, \varepsilon_B \rangle$ are morphisms of crossed semimodules. □

Lemma 3.2. *Let* $\mathcal{C} = (C_0, C, s, t, \varepsilon, m)$ *be an internal category in* CSM. *Then*

(i) $(a_1 a_1')\circ(aa') = (a_1 \circ a)(a_1' \circ a'), (b_1 b_1') \circ (bb') = (b_1 \circ b)(b_1' \circ b')$

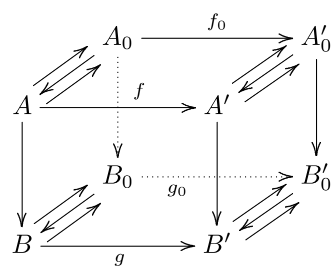
(ii) $(b_1 \circ b) \bullet (a_1 \circ a) = (b_1 \bullet a_1) \circ (b \bullet a)$,

(iii) $\partial m_A = m_B(\partial \times \partial)$,

for $a, a', a_1, a_1' \in A, b, b', b_1, b_1' \in B$ *such that* $s_A(a_1) = t_A(a), s_A(a_1') = t_A(a'), s_B(b_1) = t_B(b)$ *and* $s_B(b_1') = t_B(b')$.

Proof: The proof is clear, since $m = \langle m_A, m_B \rangle$ is a morphism of crossed semimodules. □
The conditions **(i)-(ii)** are called interchange rules.

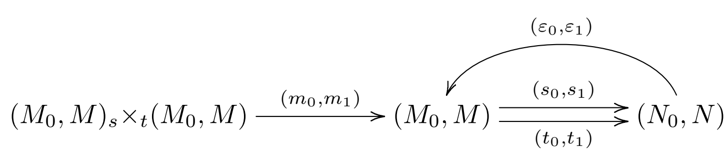
Definition 3.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two internal categories in CSM. An internal functor $F = (f, g)\colon \mathcal{C} \to \mathcal{C}'$ is called morphism of internal categories in CSM, if $f_0\colon A_0 \to A_0'$, $f\colon A \to A'$, $g_0\colon B_0 \to B_0'$, $f\colon B \to B'$ such that $f = (f_0, f), g = (g_0, g)$ are functors and $\langle f_0, g_0 \rangle$, $\langle f, g \rangle$ are crossed semimodule morphisms.



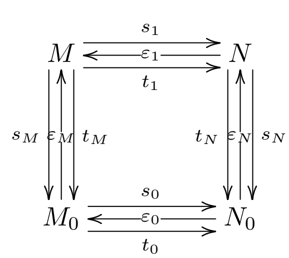
Hence we form the category of internal categories in CSM, which we denote by CAT(CSM).

4. Internal categories in Schreier internal categories

We know that double categories are internal categories in the category CAT of small categories. In this section, we characterize internal categories in the category SIC of Schreier internal categories in MON and generalize some results given in [18]. Let $\mathcal{K} = (\mathcal{M}, \mathcal{N}, s, t, \varepsilon, m)$ be an internal category in SIC. This means that $\mathcal{K}$ contains a pair of Schreier internal categories $\mathcal{M} = (M_0, M), \mathcal{N} = (N_0, N)$ in MON and four morphisms $s = (s_0, s_1), t = (t_0, t_1), \varepsilon = (\varepsilon_0, \varepsilon_1)$ and $m = (m_0, m_1)$ of SIC as internal functors which are called the source and the target maps, the identity object map, and the composition map, respectively, as partially shown in the following diagrams:



Here the set of squares is M, the sets of horizontal and vertical morphisms are N and M_0, respectively, and the set of points is N_0.

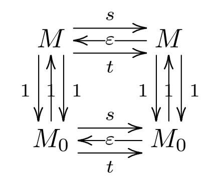


The composition m_M is the horizontal composition of squares, and it will be denoted by m_h or $\circ_h$ when no confusion arises. The composition m_1 of the category $\mathcal{M}$ is the vertical composition of squares, and it will be denoted by m_v or $\circ_v$ when no confusion arises. Hence we replace $s_1, t_1, \varepsilon_1, s_M, t_M, \varepsilon_M$ by $s_v, t_v, \varepsilon_v, s_h, t_h, \varepsilon_h$, respectively. Due to Schreier condition, we write

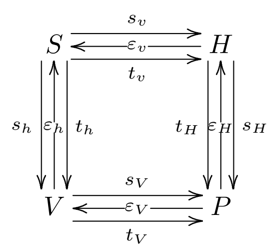
$$\alpha = \widehat{\alpha} \cdot \varepsilon_h s_h(\alpha), \quad n = \widehat{n} \cdot \varepsilon_N s_N(n)$$

for $\alpha \in M, n \in N$. This structure denoted by $\mathfrak{C} = (M, N, M_0, N_0)$ is a generalization of the double group-groupoid concept, which is defined in [18].

Example 4.1. Given a Schreier internal category $\mathfrak{C}$ in MON, then $\mathfrak{C} = (M, M_0, M, M_0)$ is an internal category in SIC with the trivial structural maps.



Lemma 4.1. *Let $\mathfrak{C} = (S, H, V, P)$ be an internal category in* SIC.



Then the following interchange rules are satisfied:

(i) $(\alpha_4 \circ_v \alpha_2) \circ_h (\alpha_3 \circ_v \alpha_1) = (\alpha_4 \circ_h \alpha_3) \circ_v (\alpha_2 \circ_h \alpha_1)$

(ii) $(\alpha_2 \circ_h \alpha_1) \cdot (\alpha_2' \circ_h \alpha_1') = (\alpha_2 \cdot \alpha_2') \circ_h (\alpha_1 \cdot \alpha_1')$

(iii) $(\alpha_3 \circ_v \alpha_1) \cdot (\alpha_3' \circ_v \alpha_1') = (\alpha_3 \cdot \alpha_3') \circ_v (\alpha_1 \cdot \alpha_1')$

whenever one side of the equations makes sense, for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in S$.

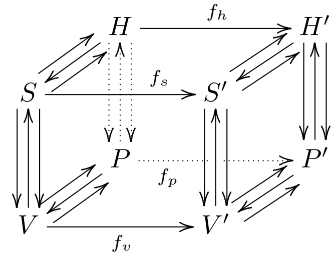
Lemma 4.2. *Let $\mathfrak{C} = (S, H, V, P)$ be an internal category in* SIC*. Then the compositions of morphisms can be written in terms of the monoid operation as follows:*

(i) $\beta \circ_h \alpha = \widehat{\beta} \cdot \widehat{\alpha} \cdot \varepsilon_h s_h(\alpha)$,

(ii) $k \circ h = \widehat{k} \cdot \widehat{h} \cdot \varepsilon_H s_H(h)$.

for $\alpha, \beta, \delta \in S$ and $k, h \in H$, whenever all compositions above are defined.

Definition 4.1. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be two internal categories in SIC. A morphism $F = (f_s, f_h, f_v, f_p) \colon \mathfrak{C} \to \mathfrak{C}'$ of double categories such that $f_s \colon S \to S'$, $f_h \colon H \to H'$, $f_v \colon V \to V'$ and $f_p \colon P \to P'$ are monoid homomorphisms is called morphism of internal categories in SIC.



Hence we form the category of internal categories in SIC which we denote by CAT(SIC).

Theorem 4.1. *The categories* CAT(SIC) *and* CAT(CSM) *are equivalent.*

Proof: A functor $\delta\colon \text{CAT(SIC)} \to \text{CAT(CSM)}$ is defined as in the following way. Let $\mathfrak{C} = (S, H, V, P)$ be an internal category in SIC. Then we obtain crossed semimodules (A, B, ∂), (A_0, B_0, ∂_0) from Schreier internal categories (V, S), (P, H), respectively, where $A = \ker s_v, B = V, \partial = t_v|_{\ker s_v}$ and $A_0 = \ker s_H, B = P, \partial_0 = t_H|_{\ker s_H}$.

$$\begin{array}{ccc} \ker s_h & \overset{s_v}{\underset{t_v}{\overset{\xleftarrow{\varepsilon_v}}{\rightrightarrows}}} & \ker s_H \\ \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial_0} \\ V & \overset{s_V}{\underset{t_V}{\overset{\xleftarrow{\varepsilon_V}}{\rightrightarrows}}} & P \end{array}$$

Here the action of V on $\ker s_v$ is defined by such that

$$(v \bullet \alpha) \cdot \varepsilon_h(v) = \varepsilon_h(v) \cdot \alpha$$

and the action of P on $\ker s_H$ is defined by such that

$$(p \bullet h) \cdot \varepsilon_H(p) = \varepsilon_H(p) \cdot h$$

where $p \in P, h \in \ker s_H, v \in V$ and $\alpha \in \ker s_h$. We shall prove that $\langle s_v, s_V \rangle$ is a morphism of crossed semimodules. Clearly $s_V\partial = \partial_0 s_v$, $t_V\partial = \partial_0 t_v$ and $\partial\varepsilon_v = \varepsilon_H\partial_0$. Since $\varepsilon_H s_V = s_v\varepsilon_h$, we write

$$\begin{aligned} \Big(s_V(v) \bullet s_v(\alpha)\Big) \cdot \varepsilon_H s_V(v) &= \varepsilon_H s_V(v) \cdot s_v(\alpha) \\ &= s_v\varepsilon_h(v) \cdot s_v(\alpha) \\ &= s_v(\varepsilon_h(v) \cdot \alpha) \\ &= s_v((v \bullet \alpha) \cdot \varepsilon_h(v)) \\ &= s_v(v \bullet \alpha) \cdot s_v\varepsilon_h(v) \\ &= s_v(v \bullet \alpha) \cdot \varepsilon_H s_V(v). \end{aligned}$$

Let $h = \Big(s_V(v) \bullet s_v(\alpha)\Big) \cdot \varepsilon_H s_V(v)$. Since $h \in H$, $s_H\varepsilon_H = 1_H$ and $s_V(v) \bullet s_v(\alpha) \in \ker s_H$, we have

$$\varepsilon_H s_H(h) = \varepsilon_H s_H\Big(s_V(v) \bullet s_v(\alpha)\Big) \cdot \varepsilon_H s_H \varepsilon_H s_V(v) = \varepsilon_H s_V(v).$$

Let $k = s_v(v \bullet \alpha) \cdot \varepsilon_H s_V(v)$. Similarly we get

$$\varepsilon_H s_H(k) = \varepsilon_H s_V(v).$$

Therefore, under the Schreier condition, we have

$$s_v(v \bullet \alpha) = s_V(v) \bullet s_v(\alpha).$$

Using a similar way, we get $\langle t_v, t_V \rangle$ and $\langle \varepsilon_v, \varepsilon_V \rangle$ are morphisms of crossed semimodules.

Let $F = (f_s, f_h, f_v, f_p)\colon (S, H, V, P) \to (S', H', V', P')$ be a morphism of CAT(SIC). Then $\delta(F) = (f, g)$ is a morphism of CAT(CSM) where $f = (f_h, f_s|_{\ker s_h})$, $g = (f_p, f_v|_{\ker s_H})$. We will only prove that $\langle f_s|_{\ker s_h}, f_v|_{\ker s_H} \rangle$ is a morphism of crossed semimodules. Since $f_s\varepsilon_h = \varepsilon_H f_v$, we get

$$\begin{aligned} f_s(v \bullet \alpha) \cdot f_s\varepsilon_h(v) &= f_s((v \bullet \alpha) \cdot \varepsilon_h(v)) \\ &= f_s(\varepsilon_h(v) \cdot \alpha) \\ &= f_s\varepsilon_h(v) \cdot f_s(\alpha) \\ &= \varepsilon_H f_v(v) \cdot f_s(\alpha) \\ &= (f_v(v) \bullet f_s(\alpha)) \cdot \varepsilon_H f_v(v) \\ &= (f_v(v) \bullet f_s(\alpha)) \cdot f_s\varepsilon_h(v). \end{aligned}$$

Let $\alpha' = f_s(v \bullet \alpha) \cdot f_s\varepsilon_h(v)$. Since $\alpha' \in S'$, $s'_h f_s = f_v s_h$, $\varepsilon'_h f_v = f_s \varepsilon_h$, $s_h \varepsilon_h = 1_S$ and $f_s(v \bullet \alpha) \in \ker s'_H$, we have

$$\varepsilon'_h s'_h(\alpha') = \varepsilon'_h s'_h f_s(v \bullet \alpha) \cdot \varepsilon'_h s'_h f_s \varepsilon_h(v) = \varepsilon'_h f_v s_h \varepsilon_h(v) = \varepsilon'_h f_v(v) = f_s\varepsilon_h(v).$$

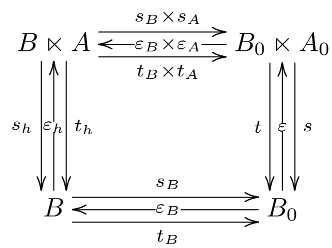
Let $\beta' = (f_v(v) \bullet f_s(\alpha)) \cdot f_s\varepsilon_h(v)$. Using a similar way, we get

$$\varepsilon'_h s'_h(\beta') = f_s\varepsilon_h(v).$$

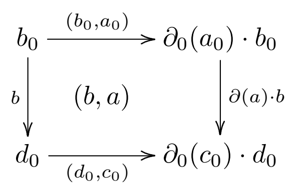
Hence, under the Schreier condition, we have

$$f_s(v \bullet \alpha) = f_v(v) \bullet f_s(\alpha).$$

Now we define a functor $\gamma\colon \mathrm{CAT}(\mathrm{CSM}) \to \mathrm{CAT}(\mathrm{SIC})$ as a weak inverse of the functor δ. Given an internal category $\mathcal{C} = (C_0, C, s, t, \varepsilon, m)$ in CSM where $C_0 = (A_0, B_0, \partial_0)$, $C = (A, B, \partial)$ are crossed semimodules and $s = \langle s_A, s_B\rangle$, $t = \langle t_A, t_B\rangle$, $\varepsilon = \langle \varepsilon_A, \varepsilon_B\rangle$, $m = \langle m_A, m_B\rangle$ are morphisms of crossed semimodules, we obtain Schreier internal categories $(B, B \ltimes A)$ and $(B_0, B_0 \ltimes A_0)$ by the Theorem 2.1 as partially shown in the following diagram:



Here pairs (b, a) are squares and $(b_0, a_0), (d_0, c_0)$ are horizontal morphisms as



for $b_0, d_0 \in B_0, b \in B, a_0, c_0 \in A_0, a \in A$ where $a_0 \xrightarrow{a} c_0$. The horizontal composition of squares is defined by

$$(\partial(a) \cdot b, c) \circ_h (b, a) = (b, c \cdot a)$$

where

$$(\partial_0(a_0) \cdot b_0, c_0) \circ (b_0, a_0) = (b_0, c_0 \cdot a_0).$$

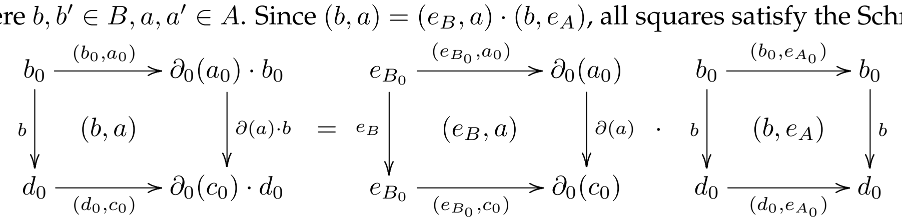
The vertical composition of squares is defined by

$$(d, c) \circ_v (b, a) = (d \circ b, c \circ a)$$

whenever $d \circ b, c \circ a$ are defined. The product of squares are defined by

$$(b, a) \cdot (b', a') = (b \cdot b', a \cdot (b \bullet a'))$$

where $b, b' \in B, a, a' \in A$. Since $(b, a) = (e_B, a) \cdot (b, e_A)$, all squares satisfy the Schreier condition.



Let $F = (f, g)$ be a morphism of CAT(CSM) where $f = (f_0, f), g = (g_0, g)$. Then

$$\gamma(F) = (g \times f, g_0 \times f_0, g, g_0)$$

is a morphism of CAT(CSM).

In order to prove $1_{\text{CAT(SIC)}} \cong \gamma\delta$, we define a natural transformation

$$\eta\colon \mathfrak{C} \to \gamma\delta(\mathfrak{C})$$

through a map $\eta_{\mathfrak{C}}$ such that to be identity on points and on vertical morphisms,

$$\eta_{\mathfrak{C}}(h) = (s_H(h), \widehat{h}), \quad \eta_{\mathfrak{C}}(\alpha) = (s_h(\alpha), \widehat{\alpha}),$$

for $h \in H, \alpha \in S$. We will verify that $\eta_{\mathfrak{C}}$ preserves monoid multiplication and vertical and horizontal compositions.

$$\begin{aligned}
\eta_{\mathfrak{C}}(\alpha \cdot \alpha') &= \eta_{\mathfrak{C}}\Big(\widehat{\alpha} \cdot \varepsilon_h s_h(\alpha) \cdot \widehat{\alpha'} \cdot \varepsilon_h s_h(\alpha')\Big) \\
&= \eta_{\mathfrak{C}}\Big(\widehat{\alpha} \cdot (s_h(\alpha) \bullet \widehat{\alpha'}) \cdot \varepsilon_h s_h(\alpha) \cdot \varepsilon_h s_h(\alpha')\Big) \\
&= \eta_{\mathfrak{C}}\Big(\widehat{\alpha} \cdot (s_h(\alpha) \bullet \widehat{\alpha'}) \cdot \varepsilon_h s_h(\alpha \cdot \alpha')\Big) \\
&= (s_h(\alpha \cdot \alpha'), \widehat{\alpha} \cdot (s_h(\alpha) \bullet \widehat{\alpha'})) \\
&= (s_h(\alpha) \cdot s_h(\alpha'), \widehat{\alpha} \cdot (s_h(\alpha) \bullet \widehat{\alpha'})) \\
&= (s_h(\alpha), \widehat{\alpha}) \cdot (s_h(\alpha'), \widehat{\alpha'}) \\
&= \eta_{\mathfrak{C}}(\alpha) \cdot \eta_{\mathfrak{C}}(\alpha')
\end{aligned}$$

for $\alpha, \alpha' \in S$,

$$\eta_{\mathfrak{C}}(\beta \circ_h \alpha) = \eta_{\mathfrak{C}}(\widehat{\beta} \cdot \widehat{\alpha} \cdot \varepsilon_h s_h(\alpha)) = (s_h(\alpha), \widehat{\beta} \cdot \widehat{\alpha}) = (s_h(\beta), \widehat{\beta}) \circ_h (s_h(\alpha), \widehat{\alpha}) = \eta_{\mathfrak{C}}(\beta) \circ_h \eta_{\mathfrak{C}}(\alpha)$$

whenever $\beta \circ_h \alpha$ is defined for $\beta, \alpha \in S$. Since

$$\delta \circ_v \alpha = (\widehat{\delta} \cdot \varepsilon_h s_h(\delta)) \circ_v (\widehat{\alpha} \cdot \varepsilon_h s_h(\alpha)) = (\widehat{\delta} \circ_v \widehat{\alpha}) \cdot (\varepsilon_h s_h(\delta) \circ_v \varepsilon_h s_h(\alpha)) = (\widehat{\delta} \circ_v \widehat{\alpha}) \cdot \varepsilon_h s_h(\delta \circ_v \alpha),$$

under the Schreier condition, we can write

$$\eta_{\mathfrak{C}}(\delta \circ_v \alpha) = (s_h(\delta) \circ s_h(\alpha), \widehat{\delta} \circ_v \widehat{\alpha}) = (s_h(\delta), \widehat{\delta}) \circ_v (s_h(\alpha), \widehat{\alpha}) = \eta_{\mathfrak{C}}(\delta) \circ_v \eta_{\mathfrak{C}}(\alpha).$$

In order to prove $1_{\text{CAT(CSM)}} \cong \delta\gamma$, we define a natural transformation

$$\mu\colon 1_{\text{CAT(CSM)}} \to \delta\gamma,$$

through a map $\mu_{\mathcal{C}}$ which is defined to be identity on A_0 and B_0, $\mu_{\mathcal{C}}(a) = (e_B, a)$ and $\mu_{\mathcal{C}}(a_0) = (e_{B_0}, a_0)$ for $a \in A, a_0 \in A_0$.

Other details are straightforward, and so is omitted. □

Then we can write the following corollary as a restriction of this theorem.

Corollary 4.1. *The category of double group-groupoids is equivalent to the category of internal categories in the category* CM *of crossed modules.*

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