Novel Lyapunov-type inequality for fractional boundary value problem

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Abstract — *In this paper, we establish a new Lyapunov-type inequality for a differential equation involving Caputo fractional derivatives subject to non-local boundary conditions. As an application to the corresponding eigenvalue problem is also discussed.*

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1 Introduction

In this work, we obtain Lyapunov type inequality for the following fractional differential equation

$$^{C}D_{0^+}^{\alpha}u(t)+q(t)u(t)=0, a\leq t\leq b, \tag{1}$$

with the boundary conditions

$$u(a)=0, u'(a)=\xi u(b), \tag{2}$$

where $1<\alpha\leq 2, 0<\xi<\frac{1}{b-a}$, $^{C}D_{a^+}^{\alpha}$ denotes the Caputo derivative of order α and q is a continuous function on $[a,b]$.

The Lyapunov inequality and its generalizations have many applications in different fields such in oscillation theory, asymptotic theory, disconjugacy, eigenvalue problems. The classical Lyapunov inequality [7] states that, if u is a nontrivial solution of the Hill's equation

$$u''(t)+q(t)u(t)=0,\ a<t<b,$$

subject to Dirichlet-type boundary conditions:

$$u(a)=u(b)=0,$$

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then

$$\int_a^b |q(t)|\, dt > \frac{4}{b-a},$$

where $q : [a, b] \to \mathbb{R}$ is a real and continuous function.

Recently, more generalized Lyapunov-type inequalities have been obtained for fractional differential equations [2,3,4] and conformable derivative differential equations in [1,5]. For more results on Lyapunov-type inequalities for fractional differential equations, we refer to the recent survey of Ntouyas et al. [8].

2 Preliminaries

We recall the concept of fractional integral and derivative of order $p > 0$. For details on the subject we refer the reader to [6,9,10]

Definition 1 *The Riemann-Liouville fractional integral of a function f is defined by*

$$I_{a^+}^p f(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{f(s)}{(t-s)^{1-p}} ds.$$

Definition 2 *The Caputo derivatives of order $p > 0$, of a function f is defined by:*

$${}^C D_{a^+}^p f(t) = I_{a^+}^{n-p} f^{(n)}(t),$$

where n is the smallest integer greater or equal than p.

We also recall the following composition rule of fractional operators.

Proposition 3 *Let $n - 1 < p < n$ and $f \in C^n [a, b]$. Then*

$$I_{a^+}^{p C} D_{a^+}^p f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

3 Lyapunov inequality

Next we transform the problem (1)-(2) to an equivalent integral equation.

Lemma 4 *Assume that $1 < \alpha \leq 2$ and $0 < \xi < \frac{1}{b-a}$. The function u is a solution to the boundary value problem (1)-(2) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t, r) q(r) u(r) dr, \tag{3}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} + \frac{\xi(t-a)}{(1-\xi(b-a))} (b-s)^{\alpha-1}, a \leq s \leq t \leq b \\ \frac{\xi(t-a)}{(1-\xi(b-a))} (b-s)^{\alpha-1}, a \leq t \leq s \leq b. \end{cases} \tag{4}$$

is the Green's function of problem (1)-(2).

Proof. Using Proposition3 we obtain that u is a solution of (1)-(2) if and only if it satisfies the following equation:

$$u(t) = I_{a^+}^{\alpha} q(t) u(t) + a_0 + a_1 (t-a), \tag{5}$$

from the conditions $u(a) = 0$, we obtain $a_0 = 0$ and the boundary condition $u'(a) = \xi u(b)$ yields

$$a_1 = \frac{\xi}{1-\xi(b-a)} \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s) u(s) ds$$

Substituting a_0 and a_1 in (5), we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) u(s) ds \\ &+ \frac{\xi(t-a)}{(1-\xi(b-a))\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s) u(s) ds, \end{aligned}$$

from which the intended result follows. ∎

In the next Lemma we give the property of the Green function G that will be needed in the sequel.

Lemma 5 *Assume that* $1 < \alpha \leq 2$ *and* $0 < \xi < \frac{1}{b-a}$,. *Then, for all* $s, t \in [0,1]$ *the Green function* $G(t,r)$ *given in (4) is non negative, continuous and satisfies the following property:*

$$\max_{(s,t)\in[a,b]^2} G(t,s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))} \max\left\{1, \frac{\xi(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)\right\}.$$

Moreover,

$$G(t,s) = \frac{\xi(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))\alpha^{\alpha}}(b-a)^{\alpha},$$

if and only if, $s = t = \frac{b+(\alpha-1)a}{\alpha}$ *and*

$$G(t,s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))},$$

if and only if $(t,s) = (b,0)$.

Proof. Obviously $G(t,s)$ is continuous and $G(t,s) \geq 0$ for $s,t \in [0,1]$.

Firstly, For $a \leq s \leq t \leq b$ and fix $s \in [a,b]$,we have

$$\max_{s\leq t\leq b}\left[(t-s)^{\alpha-1} + \frac{\xi(t-a)}{(1-\xi(b-a))}(b-s)^{\alpha-1}\right] = \frac{(b-s)^{\alpha-1}}{(1-\xi(b-a))},$$

so,

$$\max_{a\leq s\leq t\leq b} G(t,s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))}. \tag{6}$$

Now, for $a \leq t \leq s \leq b$, and fix $t \in [a, b]$,we have

$$\max_{t \leq s \leq b} \frac{\xi (t-a)}{(1-\xi(b-a))} (b-s)^{\alpha-1} = \frac{\xi (t-a)}{(1-\xi(b-a))} (b-t)^{\alpha-1}.$$

Define the function

$$h(t) = (t-a)(b-t)^{\alpha-1},$$

by differentiating the function h, it yields

$$h'(t) = (b-t)^{\alpha-2} [b + (\alpha-1) a - \alpha t].$$

We can see that $h'(t) = 0$ for $t_0 = \frac{b+(\alpha-1)a}{\alpha} \in (a, b)$, $h'(t) < 0$ for $t > t_0$ and $h'(t) > 0$ for $t < t_0$. Hence, the function $h(t)$ has a unique maximum given by

$$\begin{aligned} \max_{t \in [a,b]} h(t) &= h(t_0) \\ &= \frac{(b-a)}{\alpha} \left[\frac{(\alpha-1)}{\alpha} (b-a) \right]^{\alpha-1}. \end{aligned}$$

So,

$$\begin{aligned} \max_{a \leq t \leq s \leq b} G(t,s) &= \max_{a \leq t \leq b} G(t,t) = G(t_0, t_0) \\ &= \frac{\xi}{\Gamma(\alpha)(1-\xi(b-a))} \frac{(b-a)}{\alpha} \left[\frac{(\alpha-1)}{\alpha} (b-a) \right]^{\alpha-1} \\ &= \frac{\xi (\alpha-1)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a)) \alpha^{\alpha}} (b-a)^{\alpha}. \end{aligned} \tag{7}$$

Finally, from (6) and (7) we get

$$\max_{(s,t) \in [a,b]^2} G(t,s) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))} \max \left\{ 1, \frac{\xi (\alpha-1)^{\alpha-1}}{\alpha^{\alpha}} (b-a) \right\}$$

Next, we state and prove the Lyapunov inequality for problem (1)-(2). ■

Theorem 6 *(Lyapunov Inequality). Assume that* $1 < \alpha \leq 2, 0 < \xi < \frac{1}{b-a}$*. If the fractional boundary value problem (1)-(2) has a nontrivial continuous solution, then*

$$\int_a^b |q(r)| \, dr > \frac{\Gamma(\alpha)(1-\xi(b-a))}{(b-a)^{\alpha-1} \max \left\{ 1, \frac{\xi(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}} (b-a) \right\}}. \tag{8}$$

Proof. Let $X = C[a, b]$ be the Banach space endowed with norm $||u|| = \max_{t \in [a,b]} |u(t)|$. It follows from Lemma 4 that a solution $u \in X$ to the boundary value problem (1)-(2) satisfies

$$\begin{aligned} |u(t)| &\leq \int_a^b |G(t,r)| \, |q(r)| \, |u(r)| \, dr \\ &\leq \|u\| \int_a^b |G(t,r)| \, q(r) \, dr, \end{aligned}$$

Now, applying Lemma 5 to equation (3), it yields

$$|u(t)| < \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))} \max\left\{1, \frac{\xi(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)\right\} \|u\| \int_a^b |q(r)|\, dr$$

Hence,

$$\|u\| < \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)(1-\xi(b-a))} \max\left\{1, \frac{\xi(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)\right\} \|u\| \int_a^b |q(r)|\, dr,$$

from which the inequality (8) follows. This completes the proof of Theorem. ■

3.0.1 Application to a fractional eigenvalue problem

We give an application of the Lyapunov-type inequality (8) for the following eigenvalue problem (EVP)

$$\begin{gathered} {}^{C}D_{b^-}^{\alpha} D_{a^+}^{\beta} u(t) = \lambda u(t), \; a < t < b, \lambda \in \mathbb{R}, \\ u(a) = 0, u'(a) = \xi u(b), \end{gathered}$$

From Theorems, the next result immediately follows.

Corollary 7 *Assume that* $1 < \alpha \leq 2, 0 < \xi < \frac{1}{b-a}$. *If* λ *is an eigenvalue to the fractional boundary value problem (EVP), then*

$$|\lambda| > \frac{\Gamma(\alpha)(1-\xi(b-a))}{(b-a)^{\alpha} \max\left\{1, \frac{\xi(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)\right\}}.$$

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