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Elastica in Galilean 3-Space

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Abstract

In this work, we aim to develop classical Euler-Bernoulli elastic curves in a non-Euclidean space. So, we study the curvature energy action under some boundary conditions in the Galilean $3-$space G_3. Then, we derive the Euler-Lagrange equation for bending energy functional acting on suitable curves in G_3. We solve this differential equation by using some solving methods in applied mathematics. Finally, we give an example for elastic curves in Galilean $3-$space G_3.

Keywords: *Elastic curves, Euler-Lagrange equations, Galilean 3-space*
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1. Introduction

The differential geometry of curves plays important role in many area. Especially, the curvature theory of curves and surfaces are useful tool in many different branch of science such as engineering, chemistry, biology, etc. Curves are also encountered in the solution of some important physics problems. Also, the curvature theory of curves is often used to describe complicated systems that arising in kinematic and mechanic problems.

In 1870, Cayley-Klein stated that there are nine different geometries in the plane. These geometries are determined by parabolic, elliptic, and hyperbolic measures of angles and lengths. The Galilean geometry is the simplest of all Cayley-Klein geometries. For example, some problems can not be solved in Euclidean geometry, but they can be solved easily in Galilean geometry. Also, another importance of Galilean geometry is that it is associated with the Galilean principle of relativity, [1].

One of the most familiar topics in the calculus of variation is the problem of elastic curves (or known elastica) which are defined as curves satisfying a variational condition appropriate for interpolation problems. In a more geometric manner, elastic curve is a variational problem minimizing the bending energy functional given by $\int_{\gamma}\kappa(s)^2ds$ with fixed length and boundary conditions, [2]. This type of curve, motivated by the physical statement of a thin, inextensible elastic rod, is characterized by basic techniques of differential geometry and variational calculations and has a great area in geometry, physics, analysis, engineering and chemistry, today. With this work, we study the elastic curves which are critical points of the bending energy functional under some boundary conditions in a Galilean $3-$space G_3. Then, we get the Euler-Lagrange equation for bending energy functional acting on suitable curves. Then we get solution of the differential equation and give an application for elastic curves in Galilean $3-$space.

2. Preliminaries

In this section, we recall some general facts about the structure of the Galilean $3-$space G_3 and notations needed throughout the paper, and repeat some of the definitions in G_3.

In Galilean $3-$space, the Galilean scalar product of two vectors $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ is defined by

$$<u,v>_{G_3}=\begin{cases} u_1v_1, & \text{if } u_1\neq 0 \text{ or } v_1\neq 0\\ u_2v_2+u_3v_3, & \text{if } u_1 \text{ and } v_1 \text{ are zero.}\end{cases}$$

If $<u,v>_{G_3}=0$, then u and v are perpendicular.

The Galilean cross product of the vectors u and v is given by

$$(u\times v)_{G_3}=\begin{vmatrix} 0 & e_2 & e_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix},$$

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where e_1, e_2 and e_3 are Euclidean standard basis. The Galilean norm of the vector $u=(u_1,u_2,u_3)$ is defined by

$$\|u\|_{G_3}=\begin{cases}|u_1|,\ u_1\neq 0,\\ \sqrt{u_2^2+u_3^2},\ u_1=0.\end{cases}$$

If the first component of a vector is zero, then the vector is called as isotropic, otherwise it is called non- isotropic [3, 4]. All unit non-isotropic vectors are of the form $(1,u_2,u_3)$. Let $\gamma: I\subset\mathbb{R}\to G_3$ be a unit speed curve and parametrized by the arc-length parameter s. So, the curve γ is given by in the coordinate form

$$\gamma(s)=(s,\gamma_2(s),\gamma_3(s)).$$

The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\begin{aligned}\kappa(s)&=\|\gamma''(s)\|_{G_3}=\sqrt{\gamma_2''(s)^2+\gamma_3''(s)^2},\\ \tau(s)&=\frac{\det(\gamma'(s),\ \gamma''(s),\ \gamma''(s))}{\kappa^2(s)}\end{aligned}\tag{2.1}$$

and the moving trihedron is given by

$$\begin{aligned}T(s)&=\gamma'(s),\\ N(s)&=\frac{\gamma''(s)}{\kappa(s)},\\ B(s)&=\tfrac{1}{\kappa(s)}(0,-\gamma_3''(s),\gamma_2''(s)).\end{aligned}$$

The vector fields T, N and B are known the tangent vector field, the principal normal and the binormal vector field, respectively. Therefore, the Frenet-Serret formulas can be given in matrix form as

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\0&0&\tau\\0&-\tau&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix},$$

[4, 5].

3. Elastic Curves in Galilean $3-$space

In this section, we get the Euler-Lagrange equation for bending energy functional acting on suitable curves in Galilean $3-$space G_3.
We consider the set of regular space curves in Galilean $3-$space G_3 given by

$$\Omega=\{\gamma\,|\,\gamma(a_i)=\gamma_i,\ \gamma'(a_i)=\gamma_i',\ \gamma_i,\ \gamma_i'\in G_3,\ i=1,2,3\}$$

and the subspace of Ω as

$$\Omega_u=\{\gamma\in\Omega\,\Big|\,\|\gamma'\|_{G_3}=1\}.$$

For the bending energy functional

$$\begin{aligned}\mathscr{F}^\lambda:\ &\Omega\ \to\ G_3\\ &\gamma\ \to\ \mathscr{F}^\lambda=\frac{1}{2}\int_\gamma\Big(\|\gamma''\|_{G_3}^2+\Lambda(s)(\|\gamma'\|_{G_3}-1)\Big)ds,\end{aligned}$$

we will obtain critical values of the functional $\mathscr{F}^\lambda$ by using the techniques of differential geometry and calculus of variations. Here, Λ is a Lagrange multiplier and $\mathscr{F}^\lambda$ on Ω_u has a minimum value with respect to Lagrange multiplier principle.
Now, we suppose that γ is a critical point of the functional $\mathscr{F}^\lambda$. If γ is a critical point of the functional $\mathscr{F}^\lambda$, we have

$$\partial\mathscr{F}^\lambda(W)=\frac{\partial}{\partial\varepsilon}\mathscr{F}^\lambda(\gamma+\varepsilon W)\Big|_{\varepsilon=0}=0.$$

If we use standard calculation involving some integrations by parts, we get the first variation formula of the functional $\mathscr{F}^\lambda$ as follows

$$\int_0^\ell <E(\gamma),W>_{G_3}ds+\Big(<\gamma'',W'>_{G_3}-<\gamma'''-\Lambda\gamma',W>_{G_3}\Big)\Big|_0^\ell=0,$$

where

$$E(\gamma)=\gamma''''-\frac{d}{ds}(\Lambda\gamma').\tag{3.1}$$

So, for some function $\Lambda(t)$, the elastic curve in G_3 must satisfy the equation

$$E(\gamma)=\gamma''''-\frac{d}{ds}(\Lambda\gamma')\equiv 0.\tag{3.2}$$

By using Frenet equations in G_3, we obtain

$$\gamma''=\kappa' N+\kappa\tau B$$

and

$$\gamma'''=(\kappa''-\kappa\tau^2)N+(2\kappa'\tau+\kappa\tau')B. \tag{3.3}$$

Substituting Eq. (3.3) into (3.2), we get

$$-\Lambda' T+(\kappa''-\kappa\tau^2-\Lambda\kappa)N+(2\kappa'\tau+\kappa\tau')B=0. \tag{3.4}$$

From (3.4), we have $\Lambda'=0$, and so, we can choose

$$\Lambda=\frac{\lambda}{2}=const.. \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.4), the Euler-Lagrange equations are derived as follows

$$\kappa''-\kappa\tau^2-\frac{\lambda}{2}\kappa=0 \tag{3.6}$$

and

$$2\kappa'\tau+\kappa\tau'=0. \tag{3.7}$$

We obtain from Eq. (3.7)

$$\kappa^2\tau=c,\ c=const.. \tag{3.8}$$

If we combine the Eqs. (3.6) and (3.8), we have the Euler-Lagrange equation as follow

$$\kappa''-\frac{c^2}{\kappa^3}-\frac{\lambda}{2}\kappa=0. \tag{3.9}$$

Theorem 3.1. *The characterization for an elastic curve γ in Galilean $3-$space G_3 is given by the Euler-Lagrange equation (3.9).*

Multiplying $2\kappa'$ Eq. (3.9) and then integrating once, we find

$$(\kappa')^2+\frac{c^2}{\kappa^2}-\frac{\lambda}{2}\kappa^2=A \tag{3.10}$$

where A is a integration constant. One can see that $\kappa=const.$ is a solution of (3.10). Now, we assume that κ has a non-constant value. For solving Eq. (3.10), we suppose that $\kappa^2=u$. Then, Eq. (3.10) reduces to

$$(u')^2-2\lambda u^2-4Au+4c^2=0. \tag{3.11}$$

For solving Eq. (3.11), we suppose that $u'=p$. Then, Eq. (3.11) reduces to

$$p^2-2\lambda u^2-4Au+4c^2=0. \tag{3.12}$$

The first derivative of Eq. (3.12) is found as

$$p\left(p'-2\lambda u-2A\right)=0. \tag{3.13}$$

If $p=0$ in the Eq. (3.13) we get $\kappa=const.$. From Eq. (3.9), we have the solution for $\lambda<0$

$$\kappa^4=-2\frac{c^2}{\lambda}.$$

If $p\neq 0$, then we find the solution of the differential equation

$$p'-2\lambda u-2A=0.$$

The last equation can be written as follows

$$\frac{d^2u}{2\lambda u+2A}=ds^2.$$

Then we have the solution in the following

$$\left(\frac{u}{2\lambda}+\frac{A}{2\lambda^2}\right)ln(2\lambda u+2A)-\frac{u}{2\lambda^2}=\frac{s^2}{2}+c_1s+c_2,$$

or

$$\left(\frac{\kappa^2}{2\lambda}+\frac{A}{2\lambda^2}\right)ln(2\lambda\kappa^2+2A)-\frac{\kappa^2}{2\lambda^2}=\frac{s^2}{2}+c_1s+c_2.$$

Example 3.2. *Let $\gamma(s)=\left(s,\frac{s-\sin s\cos s}{4},\frac{\sin s^2-s^2}{4}\right)$ be a curve in G_3. From (2.1) the curvature and the torsion of γ is found as $\kappa(s)=\sin s$ and $\tau=1$, [7]. Then γ is found an elastic curve with $\lambda=-4$ and $c=\sin^2 s$.*

References

[1] T. Şahin, 2013. Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space, Acta Mathematica Scientia, Vol: 33, No.3 (2013), 701-711.
[2] D.A. Singer, Lectures on elastic curves and rods, In AIP Conference Proceedings Vol: 1002, No.1 (2008), 3-32.
[3] I.M. Yaglom, A Simple Non-Euclidean Geometry and Physical Basis, (1979), Springer-Verlag, New York, 307p.
[4] Y. Keleş, Galilean and Pseudo-Galilean Space Curves, Master Thesis, Karadeniz Technical University The Graduate School of Natural and Applied Sciences, (2004), Trabzon, TURKEY.
[5] B.J. Pavkovic and I. Kamenarovic, The equiform differential geometry of curves in the Galilean space. Glasnik Matematicki Vol:22, No.2 (1987), 449-457.
[6] P. A. Griffiths, P. A., Exterior differential systems and the calculus of variations, Vol: 25, (2013), Springer Science & Business Media.
[7] T. Şahin, and B. C. Dirişen, Position vectors of curves with recpect to Darboux frame in the Galilean space G_3, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, Vol: 68, No. 2 (2019), 2079-2093.