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RESEARCH ARTICLE

Oscillatory behavior of third-order nonlinear differential equations with mixed neutral terms

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Abstract

This paper deals with the oscillation of third-order nonlinear differential equations with neutral terms involving positive and negative nonlinear parts. An example is provided to illustrate the results.

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1. Introduction

We are concerned with oscillatory properties of all solutions of the third-order nonlinear differential equation with mixed neutral terms

$$\frac{d}{dt}\left(a(t)\left(\frac{d^2}{dt^2}\left[x(t)+p_1(t)x^{\beta}(\sigma(t))-p_2(t)x^{\delta}(\sigma(t))\right]\right)^{\alpha}\right)=q(t)x^{\gamma}(\tau(t))+c(t)x^{\lambda}(\omega(t)) \tag{1.1}$$

for $t \geq t_0 > 0$. For convenience in what follows we set $y(t) := x(t) + p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))$. We assume throughout that the following conditions are satisfied:

- (C_1) α, β, γ, δ, and λ are the ratios of odd positive integers;
- (C_2) a, p_1, p_2, q, $c : [t_0, \infty) \to (0, \infty)$ are continuous functions;
- (C_3) τ, σ, $\omega : [t_0, \infty) \to \mathbb{R}$ are continuous and nondecreasing functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$, $\omega(t) \geq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \omega(t) = \infty$;
- (C_4) $h(t) := \sigma^{-1}(\tau(t)) \leq t$ and $\lim_{t\to\infty} h(t) = \infty$, where σ^{-1} is the inverse of σ.

We also assume

$$A(t, t_0) := \int_{t_0}^{t} a^{-1/\alpha}(s)ds \to \infty \quad \text{as } t \to \infty. \tag{1.2}$$

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A *solution* of equation (1.1) is a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ with $y \in C^2([t_x, \infty), \mathbb{R})$, $a(y'')^\alpha \in C^1([t_x, \infty), \mathbb{R})$, and (1.1) is satisfied on $[t_x, \infty)$. Only those solutions of (1.1) existing on a half-line $[t_x, \infty)$ and satisfying

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any} \quad T_1 \geq t_x$$

are under consideration here. Moreover, it is assumed that (1.1) in fact has such solutions. A solution $x(t)$ of (1.1) is called *oscillatory* if it has arbitrarily large zeros, and *nonoscillatory* otherwise.

The study of the oscillatory behavior of solutions of functional differential equations has been a very active area of research due in part to its applications in science and engineering. We refer to the monographs [1,2,13,22] and the papers [3,4,11,12,14–17,19–21,26,27,29] for recent results of this type.

Applications of neutral delay differential equations can be found in the study of high speed electrical networks involving lossless transmission lines as those that can be found in computers (see also [28]). They also arise, for example, as the Euler equation for variational problems involving delay equations.

Beginning with the classic work of Sturm on second-order linear equations, the oscillation of solutions of differential equations has been the object of study by many authors using many different techniques. In the last three decades, oscillation theory for neutral delay differential equations of the second order and retarded delay equations of the third order has been well developed; for example, see the monographs [5,13,22], the papers [3,4,10–12,15,17,21,25–27,29], and the included references. By comparison to second-order neutral delay differential equations, considerably less work has appeared on the oscillation and asymptotic behavior of solutions of third-order neutral differential equations [17,18]. As best we can tell, there appears to be no results for the type of third order differential equations with mixed nonlinear neutral terms considered here. Our aim here is to initiate the study of oscillation of (1.1) with $\beta < 1$ and $\delta > 1$ as well as for the case $\beta < \delta \leq 1$ by making comparisons to first order differential inequalities whose oscillatory behaviors are known. Our results here are new even in the case of equation (1.1) with $p_1(t) = 0$, or $p_2(t) = 0$, or $p_1(t) = p_2(t) = 0$.

2. Oscillation of (1.1) for $\beta < 1$ and $\delta > 1$

In this section we present some oscillation criteria for equation (1.1) in the case where

$$\beta < 1 \quad \text{and} \quad \delta > 1. \tag{2.1}$$

To obtain our results, we need the following lemma.

Lemma 2.1 ([23]). *If X and Y are nonnegative, then*

$$X^\varphi + (\varphi - 1)Y^\varphi - \varphi X Y^{\varphi-1} \geq 0 \quad \textit{for } \varphi > 1 \tag{2.2}$$

and

$$X^\varphi - (1 - \varphi)Y^\varphi - \varphi X Y^{\varphi-1} \leq 0 \quad \textit{for } 0 < \varphi < 1, \tag{2.3}$$

where equality holds if and only if $X = Y$.

For notational purposes, let

$$A(v, u) := \int_u^v a^{-1/\alpha}(s) ds,$$

and for any function $p \in C([t_0, \infty), (0, \infty))$, we set

$$g_1(t) := (\delta - 1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t),$$

$$g_2(t) := (1 - \beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(t) p_1^{1/(1-\beta)}(t),$$

and

$$Q(t) := \frac{q(t)}{(p_2(h(t)))^{\gamma/\delta}}.$$

The first of our oscillation results is as follows.

Theorem 2.2. *Let conditions (C_1)–(C_4),* (1.2)*, and* (2.1) *hold. Assume that there exist a function* $p \in C([t_0, \infty), (0, \infty))$ *and nondecreasing functions* μ, ξ, $\eta \in C([t_0, \infty), \mathbb{R})$ *such that*

$$\lim_{t\to\infty} (g_1(t) + g_2(t)) = 0, \tag{2.4}$$

and

$$\mu(t) < t, \ \ \rho(t) := \omega(\mu(\mu(t))) > t \ \ \text{and} \ \ h(t) \le \xi(t) \le \eta(t) \le t. \tag{2.5}$$

If for all constants κ_0, $\kappa_1 \in (0, 1)$ *the first-order delay differential inequality*

$$Y'(t) + \kappa_0 q(t) \left[\tau(t) A(\xi(t), \tau(t))\right]^{\gamma} Y^{\gamma/\alpha}(\xi(t)) \le 0, \tag{2.6}$$

the first-order advanced differential inequality

$$y'(t) - \kappa_1 \left(\int_{\mu(t)}^{t} a^{-1/\alpha}(u) \left(\int_{\mu(u)}^{u} c(s) ds \right)^{1/\alpha} du \right) y^{\lambda/\alpha}(\rho(t)) \ge 0, \tag{2.7}$$

and the first-order delay differential inequalities

$$W'(t) + Q(t) \left(\int_{t_0}^{h(t)} A(s, t_0) ds \right)^{\gamma/\delta} W^{\gamma/\alpha\delta}(h(t)) \le 0, \tag{2.8}$$

and

$$X'(t) + Q(t) \left[(\xi(t) - h(t)) A(\eta(t), \xi(t))\right]^{\gamma/\delta} X^{\gamma/\alpha\delta}(\eta(t)) \le 0, \tag{2.9}$$

have no positive solutions, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, and $x(\omega(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. If the solution $x(t)$ is eventually negative the proof is similar, so we omit the details here as well as in other proofs in the paper. Then, for $t \ge t_1$, it follows from (1.1) that

$$\left(a(t) \left(y''(t)\right)^{\alpha}\right)' = q(t) x^{\gamma}(\tau(t)) + c(t) x^{\lambda}(\omega(t)) > 0, \tag{2.10}$$

hence $a(t)\left(y''(t)\right)^{\alpha}$ is increasing and eventually does not change its sign, say on $[t_2, \infty)$ for some $t_2 \ge t_1$. Therefore, $y''(t)$ eventually has a fixed sign on $[t_2, \infty)$, and so we shall distinguish the following four cases:

$$\text{(I)} \ \ y(t) > 0 \ \text{ and } \ y''(t) < 0, \quad \text{(II)} \ \ y(t) > 0 \ \text{ and } \ y''(t) > 0,$$

$$\text{(III)} \ \ y(t) < 0 \ \text{ and } \ y''(t) > 0, \quad \text{(IV)} \ \ y(t) < 0 \ \text{ and } \ y''(t) < 0.$$

First, we consider the cases where $y(t) > 0$ for $t \ge t_2$, i.e., Cases (I) and (II). Clearly we see that $y'(t) > 0$ for $t \ge t_2$. Next, from the definition of $y(t)$, we get

$$x(t) = y(t) - \left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right] - \left[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\right]. \tag{2.11}$$

Applying (2.2) to $\left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right]$ with

$$\varphi = \delta > 1, \ X = p_2^{1/\delta}(t) x(\sigma(t)), \ \ \text{and} \ \ Y = \left(\frac{1}{\delta} p(t) p_2^{-1/\delta}(t)\right)^{1/(\delta-1)},$$

we see that

$$\left[p(t)x(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right] \le (\delta - 1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t) := g_1(t). \tag{2.12}$$

Applying (2.3) to $\Big[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\Big]$ with

$$\varphi = \beta < 1,\ X = p_1^{1/\beta}(t)x(\sigma(t)), \ \text{ and } \ Y = \left(\frac{1}{\beta}p(t)p_1^{-1/\beta}(t)\right)^{1/(\beta-1)},$$

we obtain

$$\Big[p_1(t)x^{\beta}(\sigma(t)) - p(t)x(\sigma(t))\Big] \le (1-\beta)\beta^{\beta/(1-\beta)}p^{\beta/(\beta-1)}(t)p_1^{1/(1-\beta)}(t) := g_2(t). \tag{2.13}$$

Using (2.12) and (2.13) in (2.11) gives

$$x(t) \ge \left[1 - \frac{g_1(t)+g_2(t)}{y(t)}\right] y(t) \quad \text{for } t \ge t_2. \tag{2.14}$$

Since $y(t)$ is positive and increasing on $[t_2,\infty)$, there exist a $t_3 \ge t_2$ and a constant $c_1 > 0$ such that $y(t) \ge c_1$ for $t \ge t_3$, and so, inequality (2.14) can be written as

$$x(t) \ge \left[1 - \frac{g_1(t)+g_2(t)}{c_1}\right] y(t) \quad \text{for } t \ge t_3.$$

Now, in view of (2.4), for any $\kappa \in (0,1)$ there exists $t_\kappa \ge t_3$ such that

$$x(t) \ge \kappa y(t) \quad \text{for } t \ge t_\kappa. \tag{2.15}$$

Choose $\kappa \in (0,1)$ and select t_κ so (2.15) holds. Since $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\omega(t) = \infty$, we can find $t_5 \ge t_\kappa$ such that $\tau(t) \ge t_\kappa$ and $\omega(t) \ge t_\kappa$ for $t \ge t_5$. Now (2.15) implies

$$x(\tau(t)) \ge \kappa y(\tau(t)) \ \text{ and } \ x(\omega(t)) \ge \kappa y(\omega(t)) \ \text{ for } t \ge t_5. \tag{2.16}$$

Using (2.16) in (2.10) yields

$$\left(a(t)\left(y''(t)\right)^{\alpha}\right)' \ge \kappa^{\gamma}q(t)y^{\gamma}(\tau(t)) + \kappa^{\lambda}c(t)y^{\lambda}(\omega(t)) \quad \text{for } t \ge t_5. \tag{2.17}$$

We now consider Case (I). From (2.17) we obtain

$$\left(a(t)\left(y''(t)\right)^{\alpha}\right)' \ge \kappa^{\gamma}q(t)y^{\gamma}(\tau(t)) \quad \text{for } t \ge t_5. \tag{2.18}$$

Since $y'(t) > 0$ and $y''(t) < 0$ for $t \ge t_5$, for $v \ge u \ge t_5$, we may write

$$y'(u) - y'(v) = -\int_u^v a^{-1/\alpha}(s)\left(a(s)\left(y''(s)\right)^{\alpha}\right)^{1/\alpha} ds \ge A(v,u)\left(a(v)\left(-y''(v)\right)^{\alpha}\right)^{1/\alpha}.$$

Letting $u = \tau(t)$ and $v = \xi(t)$ in the last inequality, we see that

$$y'(\tau(t)) \ge A(\xi(t),\tau(t))\left(a(\xi(t))\left(-y''(\xi(t))\right)^{\alpha}\right)^{1/\alpha}. \tag{2.19}$$

In view of the fact that $y(t) > 0$, $y'(t) > 0$ and $y''(t) < 0$ on $[t_5,\infty)$, there exist a constant $\theta \in (0,1)$ such that

$$y(t) = y(t_5) + \int_{t_5}^{t} y'(s)ds \ge (t - t_5)y'(t) \ge \theta t y'(t),$$

and so, we obtain

$$y(\tau(t)) \ge \theta\tau(t)y'(\tau(t)) \ \text{ for } t \ge t_6 \tag{2.20}$$

for some $t_6 \ge t_5$. Using (2.19) in (2.20) yields

$$y(\tau(t)) \ge \theta\tau(t)A(\xi(t),\tau(t))\left(a(\xi(t))\left(-y''(\xi(t))\right)^{\alpha}\right)^{1/\alpha} \quad \text{for } t \ge t_6. \tag{2.21}$$

Letting $Y(t) = a(t)\left(-y''(t)\right)^{\alpha} > 0$, we see from (2.18) and (2.21) that $Y(t)$ is a positive solution of the first-order delay differential inequality

$$Y'(t) + (\kappa\theta)^{\gamma}q(t)\left[\tau(t)A(\xi(t),\tau(t))\right]^{\gamma} Y^{\gamma/\alpha}(\xi(t)) \le 0, \tag{2.22}$$

which contradicts assumption (2.6).

If Case (II) holds, then from (2.17) we have

$$\left(a(t)\left(y''(t)\right)^{\alpha}\right)' \ge \kappa^{\lambda}c(t)y^{\lambda}(\omega(t)) \quad \text{for } t \ge t_5. \tag{2.23}$$

Integrating (2.23) from $\mu(t)$ to t, we see that

$$a(t)\left(y''(t)\right)^{\alpha} \geq \kappa^{\lambda}\int_{\mu(t)}^{t} c(s)y^{\lambda}(\omega(s))ds \geq \kappa^{\lambda}y^{\lambda}(\omega(\mu(t)))\int_{\mu(t)}^{t} c(s)ds,$$

from which we get

$$y''(t) \geq \kappa^{\lambda/\alpha}y^{\lambda/\alpha}(\omega(\mu(t)))a^{-1/\alpha}(t)\left(\int_{\mu(t)}^{t} c(s)ds\right)^{1/\alpha}. \tag{2.24}$$

Integrating (2.24) from $\mu(t)$ to t yields

$$y'(t) \geq \kappa^{\lambda/\alpha}y^{\lambda/\alpha}(\rho(t))\int_{\mu(t)}^{t} a^{-1/\alpha}(u)\left(\int_{\mu(u)}^{u} c(s)ds\right)^{1/\alpha} du.$$

Thus, $y(t)$ is a positive solution of the advanced differential inequality of the first order

$$y'(t) - \kappa^{\lambda/\alpha}\left(\int_{\mu(t)}^{t} a^{-1/\alpha}(u)\left(\int_{\mu(u)}^{u} c(s)ds\right)^{1/\alpha} du\right)y^{\lambda/\alpha}(\rho(t)) \geq 0, \tag{2.25}$$

which contradicts assumption (2.7).

Next, we consider the cases where $y(t) < 0$ for $t \geq t_2$, i.e., Cases (III) and (IV). Letting $z(t) = -y(t) > 0$, from the definition of $y(t)$ we see that

$$z(t) = -y(t) = -x(t) - p_1(t)x^{\beta}(\sigma(t)) + p_2(t)x^{\delta}(\sigma(t)) \leq p_2(t)x^{\delta}(\sigma(t)),$$

from which we obtain

$$x(\sigma(t)) \geq \left(\frac{z(t)}{p_2(t)}\right)^{1/\delta},$$

or

$$x(t) \geq \left(\frac{z(\sigma^{-1}(t))}{p_2(\sigma^{-1}(t))}\right)^{1/\delta} \quad \text{for } t \geq t_2. \tag{2.26}$$

Using (2.26) in (2.10), we see that

$$\left(a(t)\left(-z''(t)\right)^{\alpha}\right)' \geq Q(t)z^{\gamma/\delta}(h(t)) \quad \text{for } t \geq t_3 \tag{2.27}$$

for some $t_3 \geq t_2$. Now, we consider Case (III). Letting $z(t) = -y(t) > 0$ for $t \geq t_3$, we see that $z''(t) = -y''(t) < 0$ for $t \geq t_3$. This is impossible since if $y''(t) \geq 0$, then (2.10) and condition (1.2) would imply that y is eventually positive.

Finally, we consider case (IV). Now $z''(t) = -y''(t) > 0$ for $t \geq t_3 \geq t_2$, so we distinguish the two cases:

(i) $z(t) > 0$, $z'(t) > 0$, and $z''(t) > 0$,
(ii) $z(t) > 0$, $z'(t) < 0$, and $z''(t) > 0$.

For case (i), from (2.27), we obtain

$$z'(t) = z'(t_3) + \int_{t_3}^{t} a^{-1/\alpha}(s)\left(a(s)\left(z''(s)\right)^{\alpha}\right)^{1/\alpha} ds \geq A(t,t_3)\left(a(t)\left(z''(t)\right)^{\alpha}\right)^{1/\alpha}. \tag{2.28}$$

Integrating (2.28) from t_3 to t, we get

$$z(t) \geq \left(\int_{t_3}^{t} A(s,t_3)ds\right)\left(a(t)\left(z''(t)\right)^{\alpha}\right)^{1/\alpha}. \tag{2.29}$$

Using (2.29) in (2.27) and taking $W(t) = a(t)\left(z''(t)\right)^{\alpha}$, we see that $W(t)$ is a positive solution of the first-order delay differential inequality

$$W'(t) + Q(t)\left(\int_{t_3}^{h(t)} A(s,t_3)ds\right)^{\gamma/\delta} W^{\gamma/\alpha\delta}(h(t)) \leq 0, \tag{2.30}$$

which contradicts assumption (2.8).

We are now left with case (ii). For $v \geq u \geq t_3$, we see that

$$z(u) - z(v) \geq (v - u)(-z'(v)).$$

Letting $u = h(t)$ and $v = \xi(t)$ in the last inequality, we obtain

$$z(h(t)) \geq (\xi(t) - h(t))(-z'(\xi(t))). \tag{2.31}$$

In view of (ii) and (2.27), we see that

$$\begin{aligned} -z'(u) \geq z'(v) - z'(u) &= \int_u^v a^{-1/\alpha}(s) \left(a(s) \left(z''(s)\right)^{\alpha}\right)^{1/\alpha} ds \\ &\geq A(v, u) \left(a(v) \left(z''(v)\right)^{\alpha}\right)^{1/\alpha}. \end{aligned} \tag{2.32}$$

Setting $u = \xi(t)$ and $v = \eta(t)$ in (2.32) gives

$$-z'(\xi(t)) \geq A(\eta(t), \xi(t)) \left(a(\eta(t)) \left(z''(\eta(t))\right)^{\alpha}\right)^{1/\alpha}. \tag{2.33}$$

Using (2.33) in (2.31) yields

$$z(h(t)) \geq (\xi(t) - h(t)) A(\eta(t), \xi(t)) \left(a(\eta(t)) \left(z''(\eta(t))\right)^{\alpha}\right)^{1/\alpha}. \tag{2.34}$$

Using (2.34) in (2.27) and taking $X(t) = a(t) \left(z''(t)\right)^{\alpha}$, we see that $X(t)$ is a positive solution of the delay differential inequality

$$X'(t) + Q(t) \left[(\xi(t) - h(t)) A(\eta(t), \xi(t))\right]^{\gamma/\delta} X^{\gamma/\alpha\delta}(\eta(t)) \leq 0, \tag{2.35}$$

which contradicts assumption (2.9) and completes the proof of the theorem. ☐

Next, we let

$$Q^*(t) = \min\left\{ Q(t) \left(\int_{t_0}^{h(t)} A(s, t_0) ds \right)^{\gamma/\delta}, Q(t) \left[(\xi(t) - h(t)) A(\eta(t), \xi(t))\right]^{\gamma/\delta} \right\}.$$

Then it is easy to see that Theorem 2.2 takes the following form.

Theorem 2.3. *Let conditions (C$_1$)–(C$_4$),* (1.2), *and* (2.1) *hold and assume that there exist a function* $p \in C\left([t_0, \infty), (0, \infty)\right)$ *and nondecreasing functions* μ, ξ, $\eta \in C\left([t_0, \infty), \mathbb{R}\right)$ *such that* (2.4) *and* (2.5) *hold. If for all constants* κ_0, $\kappa_1 \in (0, 1)$ *the first-order differential inequalities* (2.6)–(2.7) *and the first-order delay differential inequality*

$$Z'(t) + Q^*(t) Z^{\gamma/\alpha\delta}(\eta(t)) \leq 0$$

have no positive solutions, then equation (1.1) *is oscillatory.*

Proof. The proof is straightforward and so is omitted. ☐

The following oscillation result is a consequence of Theorem 2.2.

Corollary 2.4. *Let conditions (C$_1$)–(C$_4$),* (1.2), *and* (2.1) *hold. Assume that there exist a function* $p \in C\left([t_0, \infty), (0, \infty)\right)$ *and nondecreasing functions* μ, ξ, $\eta \in C\left([t_0, \infty), \mathbb{R}\right)$ *such that* (2.4) *and* (2.5) *hold. If*

$$\int_{t_0}^{\infty} q(s) \left[\tau(s) A(\xi(s), \tau(s))\right]^{\gamma} ds = \infty, \quad \text{for } \gamma < \alpha, \tag{2.36}$$

$$\int_{t_0}^{\infty} \left(\int_{\mu(v)}^{v} a^{-1/\alpha}(u) \left(\int_{\mu(u)}^{u} c(s) ds \right)^{1/\alpha} du \right) dv = \infty, \quad \text{for } \lambda > \alpha, \tag{2.37}$$

$$\int_{t_0}^{\infty} Q(u) \left(\int_{t_0}^{h(u)} A(s, t_0) ds \right)^{\gamma/\delta} du = \infty, \quad \text{for } \gamma < \alpha\delta, \tag{2.38}$$

and

$$\int_{t_0}^{\infty} Q(s) \left[(\xi(s) - h(s)) A(\eta(s), \xi(s))\right]^{\gamma/\delta} ds = \infty, \quad \text{for } \gamma < \alpha\delta, \tag{2.39}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, and $x(\omega(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.2, we again arrive at (2.22) for $t \geq t_6$, (2.25) for $t \geq t_5$, (2.30) for $t \geq t_3$, and (2.35) for $t \geq t_3$, respectively. Using the fact that $Y(t) = a(t)\,(-y''(t))^{\alpha}$ is positive and decreasing, and noting that $\xi(t) \leq t$, we have

$$Y(\xi(t)) \geq Y(t),$$

and so, inequality (2.22) can be written as

$$Y'(t) + (\kappa\theta)^{\gamma} q(t)\,[\tau(t) A(\xi(t), \tau(t))]^{\gamma}\, Y^{\gamma/\alpha}(t) \leq 0,$$

or

$$\frac{Y'(t)}{Y^{\gamma/\alpha}(t)} + (\kappa\theta)^{\gamma} q(t)\,[\tau(t) A(\xi(t), \tau(t))]^{\gamma} \leq 0 \quad \text{for } t \geq t_6. \tag{2.40}$$

An integration of (2.40) from t_6 to ∞ gives

$$\int_{t_6}^{\infty} q(s)\,[\tau(s) A(\xi(s), \tau(s))]^{\gamma}\, ds \leq \frac{1}{(\kappa\theta)^{\gamma}} \frac{Y^{1-\frac{\gamma}{\alpha}}(t_6)}{1 - \frac{\gamma}{\alpha}} < \infty,$$

which contradicts (2.36). Using the similar arguments as in the above, the remainder of proof follows from the fact that $h(t) \leq t$, $\eta(t) \leq t$, $\rho(t) > t$, and inequalities (2.25), (2.30), and (2.35); we omit the details. □

3. Oscillation of (1.1) for $\beta < \delta \leq 1$

This section is devoted to the oscillatory behavior of solutions of equation (1.1) in the case where the exponents in the neutral term satisfy

$$\beta < \delta \leq 1. \tag{3.1}$$

In order to obtain our results in this section, we do not need the existence of the functions p, g_1, or g_2 utilized in the previous section. We should also note that the results obtained in this section can be applied to the cases where $\delta = 1$ and $\delta < 1$. We begin with the following lemma.

Lemma 3.1 (Young's inequality)**.** *Let X and Y be nonnegative, $n > 1$, and $\frac{1}{n} + \frac{1}{m} = 1$. Then*

$$XY \leq \frac{1}{n} X^{n} + \frac{1}{m} Y^{m}, \tag{3.2}$$

where equality holds if and only if $Y = X^{n-1}$.

For notational purposes; we let

$$P(t) = \left(\frac{\delta - \beta}{\beta}\right) \left[\frac{\beta}{\delta} p_1(t)\right]^{\delta/(\delta-\beta)} p_2^{\beta/(\beta-\delta)}.$$

Theorem 3.2. *Let conditions (C_1)–(C_4), (1.2), and (3.1) hold. Assume that there exist nondecreasing functions μ, ξ, $\eta \in C\left([t_0, \infty), \mathbb{R}\right)$ such that (2.5) holds and*

$$\lim_{t\to\infty} P(t) = 0. \tag{3.3}$$

If for all constants κ_0, $\kappa_1 \in (0, 1)$ the first-order differential inequalities (2.6)–(2.9) have no positive solutions, then equation (1.1) is oscillatory.

Proof. Again let $x(t)$ be a nonoscillatory solution of equation (1.1) with $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, and $x(\omega(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, as in the proof of Theorem 2.2, (2.10) holds, and so again we have the following four cases to consider for $t \geq t_2$ for some $t_2 \geq t_1$:

$$\text{(I)}\ \ y(t) > 0 \ \text{ and } \ y''(t) < 0, \quad \text{(II)}\ \ y(t) > 0 \ \text{ and } \ y''(t) > 0,$$
$$\text{(III)}\ \ y(t) < 0 \ \text{ and } \ y''(t) > 0, \quad \text{(IV)}\ \ y(t) < 0 \ \text{ and } \ y''(t) < 0.$$

First, consider the cases where $y(t) > 0$ for $t \geq t_2$, i.e., Cases (I) and (II). Clearly we see that $y'(t) > 0$ for $t \geq t_2$. From the definition of $y(t)$, we have

$$x(t) = y(t) - \left[p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right]. \tag{3.4}$$

Applying (3.2) to $\left[p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right]$ with

$$n = \frac{\delta}{\beta} > 1, \ X = x^{\beta}(\sigma(t)), \ Y = \frac{\beta}{\delta}\frac{p_1(t)}{p_2(t)}, \ \text{ and } \ m = \frac{\delta}{\delta - \beta},$$

we see that

$$\begin{aligned}\left[p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\delta}(\sigma(t))\right] &= \frac{\delta}{\beta}p_2(t)\left[x^{\beta}(\sigma(t))\frac{\beta}{\delta}\frac{p_1(t)}{p_2(t)} - \frac{\beta}{\delta}\left(x^{\beta}(\sigma(t))\right)^{\delta/\beta}\right] \\ &= \frac{\delta}{\beta}p_2(t)\left[XY - \frac{1}{n}X^n\right] \leq \frac{\delta}{\beta}p_2(t)\left(\frac{1}{m}Y^m\right) \\ &= \left(\frac{\delta-\beta}{\beta}\right)\left[\frac{\beta}{\delta}p_1(t)\right]^{\delta/(\delta-\beta)} p_2^{\beta/(\beta-\delta)} = P(t).\end{aligned} \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$x(t) \geq \left(1 - \frac{P(t)}{y(t)}\right) y(t). \tag{3.6}$$

Since $y(t)$ is positive and increasing on $[t_2, \infty)$, there exist a $t_3 \geq t_2$ and a constant $c_2 > 0$ such that $y(t) \geq c_2$ for $t \geq t_3$, and so, inequality (3.6) can be written as

$$x(t) \geq \left[1 - \frac{P(t)}{c_2}\right] y(t) \quad \text{for } t \geq t_3. \tag{3.7}$$

Now, in view of (3.3), for any $\kappa \in (0,1)$ there exists $t_\kappa \geq t_3$ such that

$$x(t) \geq \kappa y(t) \quad \text{for } t \geq t_\kappa. \tag{3.8}$$

Fix $\kappa \in (0,1)$ and choose t_κ by (3.8). Since $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\omega(t) = \infty$, we can choose $t_5 \geq t_\kappa$ such that $\tau(t) \geq t_\kappa$ and $\omega(t) \geq t_\kappa$ for all $t \geq t_5$. Thus, from (3.8) we have

$$x(\tau(t)) \geq \kappa y(\tau(t)) \ \text{ and } \ x(\omega(t)) \geq \kappa y(\omega(t)) \ \text{ for } t \geq t_5. \tag{3.9}$$

Using (3.9) in (2.10), we again arrive at (2.17). The rest of the proof is the same as that of Theorem 2.2 and hence is omitted. □

Remark 3.3. Results analogous to those in Theorem 2.3 and Corollary 2.4 can also be obtained in the case where $\beta < \delta \leq 1$; the details are left to the reader.

It is well known from [24] (see also [2, Lemma 2.2.9] that if

$$\liminf_{t\to\infty} \int_{\zeta(t)}^{t} R(s)ds > \frac{1}{e}, \tag{3.10}$$

then the first-order delay differential inequality

$$x'(t) + R(t)x(\zeta(t)) \leq 0 \tag{3.11}$$

has no eventually positive solution, where $R, \zeta \in C([t_0,\infty),\mathbb{R})$ with $R(t) \geq 0$, $\zeta(t) \leq t$, and $\lim_{t\to\infty}\zeta(t) = \infty$.

For $\zeta(t) \geq t$, and $\zeta'(t) \geq 0$, we have the the following result (see [2, Lemma 2.2.10]). If

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} R(s)ds > \frac{1}{e}, \tag{3.12}$$

then the first-order advanced differential inequality

$$x'(t) - R(t)x(\zeta(t)) \geq 0 \tag{3.13}$$

has no eventually positive solution.

Thus, from Theorem 3.2, we have the following result for equation (1.1) in the case where $\delta = 1$.

Corollary 3.4. *Let conditions* (C_1)*–*(C_4)*,* (1.2)*, and* (3.1) *hold. Assume that there exist nondecreasing functions* μ*,* ξ*,* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* (2.5) *and* (3.3) *hold. If*

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} q(s)\left[\tau(s)A(\xi(s),\tau(s))\right]^{\gamma} ds > \frac{1}{e}, \quad \text{if } \gamma = \alpha, \tag{3.14}$$

$$\liminf_{t\to\infty} \int_t^{\rho(t)} \left(\int_{\mu(v)}^{v} a^{-1/\alpha}(u) \left(\int_{\mu(u)}^{u} c(s)ds \right)^{1/\alpha} du \right) dv > \frac{1}{e}, \quad \text{if } \lambda = \alpha, \tag{3.15}$$

$$\liminf_{t\to\infty} \int_{h(t)}^{t} Q(u) \left(\int_{t_0}^{h(u)} A(s,t_0)ds \right)^{\gamma/\delta} du > \frac{1}{e}, \quad \text{if } \gamma = \alpha\delta, \tag{3.16}$$

and

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q(s)\left[(\xi(s)-h(s))A(\eta(s),\xi(s))\right]^{\gamma/\delta} ds > \frac{1}{e}, \quad \text{if } \gamma = \alpha\delta, \tag{3.17}$$

then equation (1.1) *is oscillatory.*

Proof. From (3.14), we can choose a positive constant κ_0 with $0 < \kappa_0 < 1$ such that

$$\liminf_{t\to\infty} \kappa_0 \int_{\xi(t)}^{t} q(s)\left[\tau(s)A(\xi(s),\tau(s))\right]^{\gamma} ds > \frac{1}{e}. \tag{3.18}$$

Now, in view of (3.10)–(3.11), inequality (3.18) ensures that inequality (2.6) has no positive solutions in the case where $\gamma = \alpha$. Again, in view of (3.10)–(3.11), inequalities (3.16) and (3.17) ensure that inequalities (2.8) and (2.9) have no positive solutions in case $\gamma = \alpha\delta$, respectively. In view of (3.12)–(3.13), inequality (3.15) ensures that inequality (2.7) has no positive solutions if $\lambda = \alpha$. So, by Theorem 3.2, the conclusion of Corollary 3.4 holds. ☐

To illustrate our results, we have the following example.

Example 3.5. Consider the equation

$$(ty''(t))' = (1+t^3)x^{1/3}(t/8) + (2t)x^{\lambda}(12t), \quad t \geq 1, \tag{3.19}$$

with

$$y(t) = x(t) + \frac{1}{t}x^{1/3}(t/2) - tx^3(t/2).$$

Here we have $\alpha = 1$, $\gamma = 1/3$, $\beta = 1/3$, $\delta = 3$, $\lambda > 1$ is the ratio of positive odd integers, $\tau(t) = t/8$, $\sigma(t) = t/2$, $\omega(t) = 12t$, $a(t) = t$, $q(t) = 1 + t^3$, $c(t) = 2t$, $p_1(t) = 1/t$ and $p_2(t) = t$. Then, it is easy to see that conditions (C_1)–(C_3) and (1.2) hold. Letting $p(t) = 1$, we see that condition (2.4) holds. Letting $\xi(t) = t/3$, $\eta(t) = t/2$ and $\mu(t) = t/2$, we see that $\rho(t) = 3t$, and (2.5) holds with $h(t) = \sigma^{-1}(\tau(t)) = t/4$. Since

$$A(t,t_0) = A(t,1) = \int_1^t \frac{ds}{s} = \ln t,$$

$$A(\xi(t),\tau(t)) = \ln\frac{8}{3}, \quad \text{and} \quad A(\eta(t),\xi(t)) = \ln\frac{3}{2},$$

we see that

$$\int_{t_0}^{\infty} q(s)\left[\tau(s)A(\xi(s),\tau(s))\right]^{\gamma} ds = \frac{(\ln 8/3)^{1/3}}{2}\int_1^{\infty}(1+s^3)s^{1/3}ds = \infty,$$

$$\int_{t_0}^{\infty} Q(u)\left(\int_{t_0}^{h(u)} A(s,t_0)ds\right)^{\gamma/\delta} du = \int_1^{\infty}\frac{4^{1/9}(1+u^3)}{u^{1/9}}\left(\frac{u}{4}\ln\frac{u}{4}-\frac{u}{4}+1\right)^{1/9} du = \infty,$$

and

$$\int_{t_0}^{\infty} Q(s)\left[(\xi(s)-h(s))A(\eta(s),\xi(s))\right]^{\gamma/\delta} ds = \frac{(\ln 3/2)^{1/9}}{3^{1/9}}\int_1^{\infty}(1+s^3)ds = \infty,$$

i.e., conditions (2.36), (2.38) and (2.39) hold. Since

$$\int_{t_0}^{\infty}\left(\int_{\mu(v)}^{v} a^{-1/\alpha}(u)\left(\int_{\mu(u)}^{u} c(s)ds\right)^{1/\alpha} du\right) dv = \frac{9}{32}\int_1^{\infty} v^2 dv = \infty,$$

condition (2.37) holds. Thus, by Corollary 2.4, equation (3.19) is oscillatory.

Remark 3.6. It would be of interest to extend the results here to the higher-order nonlinear differential equations with mixed neutral terms of the form

$$\left(a(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' = q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)),$$

or

$$\left(a(t)\left(y''(t)\right)^{\alpha}\right)^{(n-2)} = q(t)x^{\gamma}(\tau(t)) + c(t)x^{\mu}(\omega(t)),$$

where $n \geq 3$ is an odd positive integer, and the functions a, c, q, and y are as in this paper.

4. Conclusions

In this paper the authors have obtained some new results on the oscillation of all solutions of a third order neutral differential equation in which the neutral term involves both positive and negative parts with a delay. The right hand side of the equation contains both advanced and delayed arguments, so the equation studied is quite general.

The results are obtained by comparing the equation under discussion to some first order differential inequalities whose asymptotic behavior is known. It would be of interest in future work to try to extend the results here to equations of fourth and higher orders such as those studied in [6–9].

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