İstanbul Üniversitesi İşletme Fakültesi Dergisi
Istanbul University Journal of the School of Business Administration
Cilt/Vol:39, Sayı/No:1, 2010, 1-20
ISSN: 1303-1732 – www.ifdergisi.org © 2010

Symmetric duality for multiobjective variational problems containing support functions

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Abstract

Wolfe and Mond-Weir type symmetric dual models for multiobjective variational problems with support functions are formulated. For these pairs of problems, weak, strong and converse duality theorems are validated under convexity-concavity and pseudo-convexity, pseudo-concavity assumptions on certain combination of functionals. Self duality theorems for both pairs are established. The problems with natural boundary values are formulated. It is also pointed out that our duality results can be regarded as dynamic generalizations of nonlinear programming problems having nondifferentiable terms as support functions.

Keywords: *Multiobjective, variational problems, support functions, symmetric duality, self duality, convexity-concavity, pseudoconvexity-pseudoconcavity.*

Destek fonksiyonları içeren varyasyonel problemler için simetrik dualite

Özet

Destek fonksiyonlu çok amaçlı varyasyonel problemler için Wolfe ve Mond-Weir tipi simetrik dual modeller formüle edilmiştir. Bu çeşit problemler için; zayıf, güçlü ve karşıt dualite teoremleri fonksiyonellerin belirli kombinasyonları üzerine konvekslik-konkavlık ve pseudo-konvekslik, pseudo-konkavlık varsayımları altında geçerli kılınmıştır. İki çift için de öz dualite teoremleri kurulmuştur. Doğal sınır değerli problemler formüle edilmiştir. Ayrıca dualite sonuçlarımızın, destek fonksiyoları gibi diferansiyellenemeyen ifadelere sahip olan nonlineer programlama problemlerinin dinamik genelleştirmeleri olarak kabul edilebileceğine dikkat çekilmektedir.

Anahtar Sözcükler: *Çok amaçlı, varyasyonel problemler, destek fonksiyonlar, simetrik dualite, öz dualite, konvekslik-konkavlık, pseudokonvekslik- pseudokonkavlık.*

1. Introduction

Following Dorn [1], symmetric duality results in mathematical programming have been derived by a number of authors, notably, Dantzig et al. [2], Mond [3], Bazaraa and Goode [4]. In these researches, the authors have studied symmetric duality under the hypothesis of convexity-concavity of the kernel function involved. Mond and Cottle [5] presented self duality for the problems of [2] by assuming skew symmetric of the kernel function. Later Mond-Weir [6] formulated a different pair of symmetric dual nonlinear program with a view to generalize convexity-concavity of the kernel function to pseudoconvexity-pseudoconcavity.

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Symmetric duality for variational problems was first introduced by Mond and Hanson [7] under the convexity-concavity conditions of a scalar functions like $\psi\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)$ with $x(t) \in R^n$ and $y(t) \in R^m$. Bector, Chandra and Husain [8] presented a different pair of symmetric dual variational problems in order to relax the requirement of convexity-concavity to that of pseudoconvexity-pseudoconcavity. In [9] Chandra and Husain gave a fractional analogue.

Bector and Husain [10] probably were the first to study duality for multiobjective variational problems under convexity assumptions. Subsequently, Gulati, Husain and Ahmed [11] presented two distinct pairs of symmetric dual multiobjective variational problems and established various duality results under appropriate invexity requirements. In this reference, self duality theorem is also given under skew symmetric of the integrand of the objective functional.

The purpose of this research is to present Wolfe and Mond-Weir type symmetric dual pairs of multiobjective variational problems containing support functions in order to extend the results of Gulati, Husain and Ahmed [11] to nondifferentiable cases and hence study symmetric and self duality for these pairs of nondifferentiable multiobjective variational problems. The problems, treated in this research are quite hard to solve. So to expect any immediate application of these problems would be far from reality. Unfortunately, there has not always been sufficient flow between the researchers in the multiple criteria decision making and the researchers applying it to their problems. Of course, one can find optimal control applications in galore which reflect the utility of our models. Further, the problems with natural boundary values are formulated and it is also pointed out that our results can be considered as dynamic generalizations of corresponding (static) symmetric duality results of multiobjective nonlinear programming problems involving support functions. The reduced nondiferentiable multiobjective nonlinear problems are not explicitly mentioned in the literature. The duality results for them are immediate.

2. Notations And Preliminaries

The following notation will be used for vectors in R^n.

$$x < y, \quad \Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y$$

$x \not\leq y$, is the negation of $x \leq y$

Let $I = [a, b]$ be the real interval, and $\phi^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)$, $i = 1, 2, \ldots, p$ be a scalar function and twice differentiable function where $x : I \to R^n$ and $y : I \to R^n$ with derivatives $\dot{x}$ and $\dot{y}$. In order to consider each $\phi^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)$ denote the first partial derivatives of ϕ^i with respect to $t, x(t), \dot{x}(t), y(t), \dot{y}(t)$ respectively, by $\phi^i_t, \phi^i_x, \phi^i_{\dot{x}}, \phi^i_y, \phi^i_{\dot{y}}$, that is,

$$\phi^i_t = \frac{\partial \phi^i}{\partial t}$$

$$\phi_x^i = \left[\frac{\partial \phi^i}{\partial x_1}, \frac{\partial \phi^i}{\partial x_2}, \ldots, \frac{\partial \phi^i}{\partial x_n}\right], \qquad \phi_{\dot{x}}^i = \left[\frac{\partial \phi^i}{\partial \dot{x}_1}, \frac{\partial \phi^i}{\partial \dot{x}_2}, \ldots, \frac{\partial \phi^i}{\partial \dot{x}_n}\right]$$

$$\phi_y^i = \left[\frac{\partial \phi^i}{\partial y_1}, \frac{\partial \phi^i}{\partial y_2}, \ldots, \frac{\partial \phi^i}{\partial y_n}\right], \qquad \phi_{\dot{y}}^i = \left[\frac{\partial \phi^i{}_i}{\partial \dot{y}_1}, \frac{\partial \phi^i}{\partial \dot{y}_2}, \ldots, \frac{\partial \phi^i}{\partial \dot{y}_n}\right].$$

The twice partial derivatives of ϕ^i, $i = 1, 2, \ldots, p$ with respect to $t, x(t)$, $\dot{x}(t), y(t)$ and $\dot{y}(t)$, respectively are the matrices

$$\phi_{xx}^i = \left(\frac{\partial^2 \phi^i}{\partial x_k x_s}\right)_{n\times n}, \qquad \phi_{x\dot{x}}^i = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{x}_s}\right)_{n\times n}, \qquad \phi_{xy}^i = \left(\frac{\partial^2 \phi^i}{\partial x_k y_s}\right)_{n\times n}$$

$$\phi_{x\dot{y}}^i = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s}\right)_{n\times n}, \qquad \phi_{\dot{x}y}^i = \left(\frac{\partial^2 \phi^i}{\partial \dot{x}_k y_s}\right)_{n\times n}, \qquad \phi_{x\dot{y}}^i = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s}\right)_{n\times n}$$

$$\phi_{yy}^i = \left(\frac{\partial^2 \phi^i}{\partial y_k y_s}\right)_{n\times n}, \qquad \phi_{y\dot{y}}^i = \left(\frac{\partial^2 \phi^i}{\partial y_k \dot{y}_s}\right)_{n\times n}, \qquad \phi_{\dot{y}\dot{y}}^i = \left(\frac{\partial^2 \phi^i}{\partial \dot{y}_k \dot{y}_s}\right)_{n\times n}$$

Noting that

$$\frac{d}{dt}\phi_{\dot{y}}^i = \phi_{\dot{y}t}^i + \phi_{\dot{y}y}^i \dot{y} + \phi_{\dot{y}\dot{y}}^i \ddot{y} + \phi_{\dot{y}x}^i \dot{x} + \phi_{\dot{y}\dot{x}}^i \ddot{x}$$

and hence

$$\frac{\partial}{\partial y}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{y\dot{y}}, \qquad \frac{\partial}{\partial \dot{y}}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}\dot{y}} + \phi_{\dot{y}y}, \qquad \frac{d}{d\ddot{y}}\frac{d}{dt}\phi_{\dot{y}}^i = \phi_{\dot{y}y}^i$$

$$\frac{\partial}{\partial x}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}x}, \qquad \frac{\partial}{\partial \dot{x}}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}\dot{x}} + \phi_{\dot{y}x}, \qquad \frac{\partial}{\partial \ddot{x}}\frac{d}{dt}\phi_{\dot{y}} = \phi_{\dot{y}\dot{x}}$$

In order to establish our main results, the following concepts are needed.

Definition 1. (*Support function*)**:** Let K be a compact set in R^n, then the support function of K is defined by

$$s\left(x(t)\middle|K\right) = \max\left\{x(t)^T v(t) : v(t) \in K, t \in I\right\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis i.e., there exist $z(t) \in R^n$, $t \in I$, such that

$$s\left(y(t)\middle|C\right) - s\left(x(t)\middle|C\right) \geq \left(y(t) - x(t)\right)^T z(t)$$

From [12], subdifferential of $s\left(x(t)\middle|K\right)$ is given by

$$\partial s\left(x(t)\middle|K\right) = \left\{z(t) \in K,\ t \in I \text{ such that} \middle| x(t)^T z(t) = s\left(x(t)\middle|K\right)\right\}.$$

For any set $\Gamma \subset R^n$, the normal cone to Γ at a point $x(t) \in \Gamma$ is defined by

$$N_\Gamma\left(x(t)\right)=\left\{y(t)\in R^n \middle| y(t)\left(z\left(t\right)-x\left(t\right)\right)\leq 0, \forall z\left(t\right)\in\Gamma\right\}$$

It can be verified that for a compact convex set K, $y(t)\in N_K\left(x(t)\right)$ if and only if

$$s\left(y(t)\middle|K\right)=x\left(t\right)^T y(t)\ ,\ t\in I$$

Definition 2. (*Skew Symmetric function***)**: The function $h:I\times R^n\times R^n\times R^n\times R^n\to R$ is said to be skew symmetric if for all x and y in the domain of h if

$$h\left(t,x\left(t\right),\dot{x}\left(t\right),y\left(t\right),\dot{y}\left(t\right)\right)=-h\left(t,y\left(t\right),\dot{y}\left(t\right),x\left(t\right),\dot{x}\left(t\right)\right),\ t\in I$$

where x and y (piecewise smooth are on I) are of the same dimension.

Now consider the following multiobjective variational problem (VPo):

(VP$_0$) Minimize $\int_I F\left(t,x,\dot{x}\right)dt$

Subject to

$$x\left(a\right)=\alpha,\qquad x\left(b\right)=\beta$$

$$g\left(t,x,\dot{x}\right)\leqq 0\ ,\ t\in I,$$

where $F:I\times R^n\times R^n\to R^p$ and $g:I\times R^n\times R^n\to R^m$.

Definition 3. (*Efficient Solution***)**: A feasible solution $\overline{x}$ is efficient for (VPo) if there exists no other feasible x for (VP) such that for some $i\in P=\left\{1,2,...,p\right\}$,

$$\int_I F^i\left(t,x,\dot{x}\right)dt<\int_I F^i\left(t,\overline{x},\dot{\overline{x}}\right)dt \quad \text{for all } i\in P.$$

and

$$\int_I F^j\left(t,x,\dot{x}\right)dt\leqq\int_I F^j\left(t,\overline{x},\dot{\overline{x}}\right)dt \quad \text{for all } j\in P,\ j\neq i.$$

3. Wolfe Type Symmetric Duality

We present the following pair of Wolfe type symmetric dual multiobjective variational problems containing support functions:

(SWP): Minimize: $\int_I\left(H^1,H^2,...,H^p\right)dt$

Subject to:

$$x\left(a\right)=0=x\left(b\right) \tag{1}$$

$$y\left(a\right)=0=y\left(b\right) \tag{2}$$

$$\sum_{i=1}^p\lambda^i\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\leqq 0\ ,\ t\in I \tag{3}$$

$$z^i(t) \in C^i \ , \ i = 1,\ldots,p \ , \quad t \in I \tag{4}$$

$$x(t) \geqq 0 \ , \ t \in I \tag{5}$$

$$\lambda \in \Lambda^+ = \left\{ \lambda \in R^p \,\middle|\, \lambda > 0,\ \lambda^T e = 1, e = (1,1,\ldots,1)^T \in R^p \right\} \tag{6}$$

where,

1. $H^i = f^i(t,x,\dot{x},y,\dot{y}) + s\left(x(t)\middle|C^i\right)$

$$-y(t)^T \sum_{i=1}^{p} \lambda^i \left(f_y^i(t,x,\dot{x},y,\dot{y}) + z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right) - y(t)^T z(t)$$

2. $f^i : I \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$, is continuously differentiable function.

(SWD): Maximize: $\int_I \left(G^1, G^2, \ldots, G^p\right) dt$

Subject to:

$$u(a) = 0 = u(b) \tag{7}$$

$$v(a) = 0 = v(b) \tag{8}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right) \geqq 0 \ , \ t \in I \tag{9}$$

$$\omega^i(t) \in K^i \ , \ i = 1,\ldots,p \ , \ t \in I \tag{10}$$

$$v(t) \geqq 0 \ , \ t \in I \tag{11}$$

$$\lambda \in \Lambda^+ \tag{12}$$

where,

$$G^i = f^i(t,u,\dot{u},v,\dot{v}) + s\left(v(t)\middle|K^i\right)$$

$$-u(t)^T \sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right) - x(t)^T \omega(t)$$

We present various duality results under convexity-concavity assumption.

Theorem 1 (*Weak Duality*): Let $\left(x(t), y(t), z_1(t), \ldots, z_p(t), \lambda\right)$ be feasible for the (SWP) and $\left(u(t), v(t), \omega_1(t), \ldots, \omega_p(t), \lambda\right)$ be feasible for the dual (SWD). Assume that for each i, $\int_I f^i(t,.,.,y,\dot{y})\,dt$ is convex in $(x,\dot{x})$ for fixed $(y,\dot{y})$ and $\int_I f^i(t,x,\dot{x},.,.)\,dt$ is concave in $(y,\dot{y})$ for fixed $(x,\dot{x})$. Then,

$$\int_I H dt \not\leq \int_I G dt$$

where,

$$H=\left(H^{1},H^{2},\ldots,H^{p}\right)$$

and

$$G=\left(G^{1},G^{2},\ldots,G^{p}\right).$$

Proof: Using the convexity of $\int_I f^i\left(t,.,.,y,\dot{y}\right)dt$ in $\left(x,\dot{x}\right)$ for fixed $\left(y,\dot{y}\right)$, we have

$$\int_I f^i\left(t,x,\dot{x},v,\dot{v}\right)dt-\int_I f^i\left(t,u,\dot{u},v,\dot{v}\right)dt$$

$$\geq\int_I\left[\left(x(t)-u(t)\right)^T f_u^i\left(t,u,\dot{u},v,\dot{v}\right)-\left(\dot{x}(t)-\dot{u}(t)\right)^T f_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right]dt$$

$$=\int_I\left[\left(x(t)-u(t)\right)^T\left\{f_u^i\left(t,u,\dot{u},v,\dot{v}\right)-f_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right\}\right]dt$$

$$+\left(x(t)-u(t)\right)^T f_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\Big|_{t=b}^{t=a}.$$

This on using (1) and (7) gives,

$$\int_I f^i\left(t,x,\dot{x},v,\dot{v}\right)dt-\int_I f^i\left(t,u,\dot{u},v,\dot{v}\right)dt$$

$$\geqq\int_I\left[\left(x(t)-u(t)\right)^T\left\{f_u^i\left(t,u,\dot{u},v,\dot{v}\right)-f_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right\}\right]dt. \tag{13}$$

Also by concavity of $\int_I f^i\left(t,x,\dot{x},.,.\right)dt$, we have

$$-\int_I f^i\left(t,x,\dot{x},v,\dot{v}\right)dt-\int_I f^i\left(t,x,\dot{x},y,\dot{y}\right)dt$$

$$\geqq-\int_I\left[\left(v(t)-y(t)\right)^T f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-\left(\dot{v}(t)-\dot{y}(t)\right)^T f_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right]dt$$

$$=-\int_I\left[\left(v(t)-y(t)\right)^T\left\{f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-f_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right\}\right]dt$$

$$+\left(v(t)-y(t)\right)^T f_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\Big|_{t=b}^{t=a}$$

which by using (2) and (8) we have,

$$-\int_I f^i\left(t,x,\dot{x},v,\dot{v}\right)dt-\int_I f^i\left(t,x,\dot{x},y,\dot{y}\right)dt$$

$$\geqq-\int_I\left[\left(v(t)-y(t)\right)^T\left\{f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-f_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right\}\right]dt \tag{14}$$

Adding (13) and (14) we have,

$$\int_I f^i(t,x,\dot{x},y,\dot{y})\,dt - \int_I f^i(t,u,\dot{u},v,\dot{v})\,dt \geq \int_I \Big[(x(t)-u(t))^T \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\}$$
$$-(v(t)-y(t))^T \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} \Big] dt$$
$$= \int_I \Big[(x(t))^T \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\} - (u(t))^T \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\}$$
$$-(v(t))^T \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} + (y(t))^T \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} \Big] dt$$

Multiplying this by λ^i and summing over i , $i = 1,2,..,p$, we get,

$$\sum_{i=1}^p \lambda^i \int_I f^i(t,x,\dot{x},y,\dot{y})\,dt - \sum_{i=1}^p \lambda^i \int_I f^i(t,u,\dot{u},v,\dot{v})\,dt$$
$$\geqq \int_I \Big[(x(t))^T \sum_{i=1}^p \lambda^i \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\} - (u(t))^T \sum_{i=1}^p \lambda^i \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\}$$
$$-(v(t))^T \sum_{i=1}^p \lambda^i \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} + (y(t))^T \sum_{i=1}^p \lambda^i \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} \Big] dt$$
$$= \int_I \Big[(x(t))^T \sum_{i=1}^p \lambda^i \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) + \omega_i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\}$$
$$-(u(t))^T \sum_{i=1}^p \lambda^i \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) + \omega_i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\} - \sum_{i=1}^p \lambda^i x(t)\omega_i(t) + \sum_{i=1}^p \lambda^i u(t)\omega_i(t)$$
$$-(v(t))^T \sum_{i=1}^p \lambda^i \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - z_i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\}$$
$$+(y(t))^T \sum_{i=1}^p \lambda^i \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - z_i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\} - \sum_{i=1}^p \lambda^i v(t) z_i(t) + \sum_{i=1}^p \lambda^i y(t) z_i(t) \Big] dt$$

Using (3), (5), (9) and (11), we have,

$$\sum_{i=1}^p \lambda^i \int_I \Big[f^i(t,x,\dot{x},y,\dot{y}) - (y(t))^T \sum_{i=1}^p \lambda_i \left\{ f_y^i(t,x,\dot{x},y,\dot{y}) - z_i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right\}$$
$$+\sum_{i=1}^p \lambda_i (x(t)\omega_i(t)) - \sum_{i=1}^p \lambda_i (y(t) z_i(t))$$
$$\geqq \sum_{i=1}^p \lambda_i \int_I \Big[f^i(t,u,\dot{u},v,\dot{v}) - (u(t))^T \sum_{i=1}^p \lambda_i \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) + \omega_i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right\}$$
$$+u(t)\omega_i(t) - y(t) z_i(t) \Big] dt.$$

In view of $s\left(x(t) \middle| C^i\right) \geqq x(t)^T \omega^i(t),\ i = 1,\ldots,p$ and $s\left(v(t) \middle| K^i\right) \geqq (v(t))^T z^i(t),\ i = 1,\ldots,p$, this yields,

$$\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i(t,x,\dot{x},y,\dot{y})+s\big(x(t)\big|C_i\big)$$
$$-\big(y(t)\big)^T\sum_{i=1}^{p}\lambda_i\left\{f_y^i(t,x,\dot{x},y,\dot{y})-z_i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}-\big(y(t)\big)^T z_i(t)\Big]d\ t$$
$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i(t,u,\dot{u},v,\dot{v})-s\big(v(t)\big|K_i(t)\big)$$
$$-\big(u(t)\big)^T\sum_{i=1}^{p}\lambda_i\left\{f_u^i(t,u,\dot{u},v,\dot{v})+\omega_i(t)-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}+u(t)\omega_i(t)\Big]d\ t.$$

That is,

$$\sum_{i=1}^{p}\lambda_i\int_I H^i dt\geqq\sum_{i=1}^{p}\lambda_i\int_I G^i dt\,.$$

This yields,

$$\int_I Hdt\not\leq\int_I Gdt\,.$$

Theorem 2: (*Strong Duality***):** Let $\left(x(t),y(t),z_1(t),...,z_p(t),\lambda\right)$ be an efficient solution of (SWP) and $\lambda=\overline{\lambda}$ be fixed in (SWD). Furthermore, assume that

(C_1):

$$\Big\{\big(\phi(t)\big)^T\left(\lambda^T f_{yy}(t,x,\dot{x},y,\dot{y})-D\lambda^T f_{\dot{y}y}(t,x,\dot{x},y,\dot{y})\right)-D\Big[\big(\phi(t)\big)^T\left(-D\lambda^T f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\Big]$$
$$+D^2\Big[\big(\phi(t)\big)^T\left(-f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\Big]\Big\}\big(\phi(t)\big)=0,t\in I\Rightarrow\phi(t)=0,t\in I$$

(C_2): $f_y^i(t,x,\dot{x},y,\dot{y})-z^i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})$, $i=1,..p.,\ t\in I$ are linearly independent.

Then $\left(x(t),y(t),\omega_1(t),...,\omega_p(t),\lambda\right)$ is feasible for (SWD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 1 hold, then there exists $\omega_1(t),\omega_2(t),...,\omega_p(t)$ such that $\left(u(t),v(t),\lambda,\omega_1(t),..\omega_p(t)\right)=\left(x(t),y(t),\lambda,\omega_1(t),..\omega_p(t)\right)$ is an efficient solution of the dual (SWD).

Proof: Since $\left(x(t),y(t),z_1(t),...,z_p(t),\lambda\right)$ is efficient for the problem (SWP), it is weak minimum [13]. Hence there exists $\tau\in R^p$, $\eta\in R^m$, $\gamma\in R$, $\theta(t):I\to R^n$ and $\alpha(t)\in R^n$ such that the following Fritz-John optimality conditions [14], are satisfied

$$\sum_{i=1}^{p}\tau^i\left(f_x^i(t,x,\dot{x},y,\dot{y})+\omega^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right)$$
$$+\Big(\theta(t)-\left(\tau^T e\right)y(t)\Big)^T\sum_{i=1}^{p}\left(\lambda^T f_{yx}^i(t,x,\dot{x},y,\dot{y})-D\lambda^T f_{\dot{y}x}^i(t,x,\dot{x},y,\dot{y})\right)$$
$$+D\Big[\Big(\theta(t)-\left(\tau^T e\right)y(t)\Big)^T\sum_{i=1}^{p}\lambda_i\left(f_{y\dot{x}}^i(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}\dot{x}}^i(t,x,\dot{x},y,\dot{y})-f_{\dot{y}x}^i(t,x,\dot{x},y,\dot{y})\right)\Big]$$

$$+D^2\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda_i\left(f^i_{\dot{y}\dot{x}}(t,x,\dot{x},y,\dot{y})\right)\right]=0,\ t\in I, \tag{15}$$

$$\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)\left(f^i_y(t,x,\dot{x},y,\dot{y})-z^i(t)-Df^i_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right)$$

$$+\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(f^i_{yy}(t,x,\dot{x},y,\dot{y})-Df^i_{\dot{y}y}(t,x,\dot{x},y,\dot{y})\right)$$

$$-D\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(-Df^i_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\right]$$

$$+D^2\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(-f^i_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\right]=0,\ t\in I, \tag{16}$$

$$\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f^i_y(t,x,\dot{x},y,\dot{y})-z^i(t)-Df^i_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right)-\eta=0,\ t\in I, \tag{17}$$

$$x^T(t)z^i(t)=s\left(x(t)\middle|C^i\right),\ t\in I \tag{18}$$

$$\theta(t)\sum_{i=1}^{p}\left(f^i_x(t,x,\dot{x},y,\dot{y})-z^i(t)-Df^i_{\dot{x}}(t,x,\dot{x},y,\dot{y})\right)=0,\ t\in I, \tag{19}$$

$$\tau^i y(t)-\left(\theta(t)-\left(\tau^T e\right)\lambda^i\right)\in N_{C^i}\left(z^i(t)\right),\ t\in I, \tag{20}$$

$$x^T(t)\,\alpha(t)=0\ ,t\in I \tag{21}$$

$$\eta^T\lambda=0 \tag{22}$$

$$\left(\tau,\theta(t),\alpha(t),\eta\right)\geqq 0\ ,t\in I, \tag{23}$$

$$\left(\tau,\theta(t),\alpha(t),\eta\right)\neq 0\ ,\ t\in I. \tag{24}$$

Since $\lambda>0$, (22) implies $\eta=0$. Consequently, (17) reduces to

$$\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f^i_y(t,x,\dot{x},y,\dot{y})-z_i(t)-Df^i_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right)=0\ ,\ t\in I, \tag{25}$$

Post multiplying (16) by $\left(\theta(t)-\left(\tau^T e\right)y(t)\right)$, we get,

$$\Bigg\{\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)\left(f^i_y(t,x,\dot{x},y,\dot{y})-z^i(t)-Df^i_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right)$$

$$+\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(f^i_{yy}(t,x,\dot{x},y,\dot{y})-Df^i_{\dot{y}y}(t,x,\dot{x},y,\dot{y})\right)$$

$$-D\left(\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(-Df^i_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\right)=0$$

$$+D^2\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\lambda^i\left(-f^i_{\dot y\dot y}\left(t,x,\dot x,y,\dot y\right)\right)\Bigg\}\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0\,,\ \ t\in I\,, \qquad (26)$$

Premultiplying (25) by $\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)$ and summing over i, we have

$$\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f^i_y\left(t,x,\dot x,y,\dot y\right)-z_i(t)-Df^i_{\dot y}\left(t,x,\dot x,y,\dot y\right)\right)=0 \qquad (27)$$

$t\in I$

Using (27) in (26) we have,

$$\Bigg\{\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(\lambda^T f_{yy}\left(t,x,\dot x,y,\dot y\right)-D\lambda^T f_{\dot y y}\left(t,x,\dot x,y,\dot y\right)\right)$$
$$+D\left(\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(-D\lambda^T f_{\dot y\dot y}\left(t,x,\dot x,y,\dot y\right)\right)\right)$$
$$+D^2\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(-\lambda^T f_{\dot y\dot y}\left(t,x,\dot x,y,\dot y\right)\right)\Bigg\}\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0,\ t\in I$$

This in view of hypothesis (C_1) we have,

$$\phi(t)=\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0,\ t\in I. \qquad (28)$$

Hence from (15) we have,

$$\sum_{i=1}^{p}\tau^i\left(f^i_x\left(t,x,\dot x,y,\dot y\right)+\omega^i(t)-Df^i_{\dot x}\left(t,x,\dot x,y,\dot y\right)\right)-\alpha(t)=0\quad,\ \ t\in I\,. \qquad (29)$$

Let $\tau=0$. From (29) we have $\alpha(t)=0$, $\theta(t)=0$, $t\in I$.

Therefore, $\left(\tau,\theta(t),\alpha(t),\eta\right)=0$, $t\in I$, but this contradicts (24). Hence $\tau>0$.

From (16) we have,

$$\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda_i\right)\left(f^i_y\left(t,x,\dot x,y,\dot y\right)-z^i(t)-Df^i_{\dot y}\left(t,x,\dot x,y,\dot y\right)\right)=0\ \ ,\ \ t\in I\,.$$

From hypothesis (C_2), $\left(f^i_y\left(t,x,\dot x,y,\dot y\right)-z^i(t)-Df^i_{\dot y}\left(t,x,\dot x,y,\dot y\right)\right)$ is linearly independent, hence,

$$\tau^i=\left(\tau^T e\right)\lambda^i \qquad i=1,2\ldots,p\ . \qquad (30)$$

From (29), we have

$$\sum_{i=1}^{p}\left(\tau^T e\right)\lambda^i\left(f^i_x\left(t,x,\dot x,y,\dot y\right)+\omega^i(t)-Df^i_{\dot x}\left(t,x,\dot x,y,\dot y\right)\right)-\alpha(t)=0\ \ ,\ \ t\in I$$

yielding,

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right) + \omega^i\left(t\right) - Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right) - \alpha\left(t\right) = 0 \quad , \quad t \in I \tag{31}$$

This, in view of (23) implies

$$x^T\left(t\right)\sum_{i=1}^{p} \lambda^i \left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right) + \omega^i\left(t\right) - Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right) \geqq 0 \quad , \quad t \in I \tag{32}$$

Again (31) together with (21) gives

$$x^T\left(t\right)\sum_{i=1}^{p} \lambda^i \left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right) + \omega^i\left(t\right) - Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right) = 0 \quad , \quad t \in I \tag{33}$$

(28) along with (19) yields,

$$y^T\left(t\right)\sum_{i=1}^{p} \lambda^i \left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right) - z^i\left(t\right) - Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right) = 0 \quad , t \in I \tag{34}$$

Now from (20), with $\tau^i > 0$, we have,

$$y\left(t\right) \in N_{C^i}\left(z^i\left(t\right)\right) \quad , \quad i = 1,\ldots p \quad , \quad t \in I \tag{35}$$

This implies,

$$y^T\left(t\right) z^i\left(t\right) = s\left(y\left(t\right) \middle| K^i\right) \quad , \quad t \in I \tag{36}$$

Also from (28) we have,

$$y\left(t\right) = \frac{\theta\left(t\right)}{\left(\tau^T e\right)} \geq 0 \ , t \in I \tag{37}$$

The relations (33), (37) and $\omega^i \in K^i$ yield that $\left(\bar{x}(t), \bar{y}(t), \bar{\omega}_1(t), \ldots, \bar{\omega}_p(t), \bar{\lambda}\right)$ is feasible for (SWD).

Consider,

$$\begin{aligned} H^i &= f^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right) + s\left(\bar{x}\left(t\right) \middle| C^i\right) \\ &\quad - \bar{y}\left(t\right)^T \sum_{i=1}^{p} \lambda^i \left(f_y^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right) - \bar{z}^i\left(t\right) - Df_{\dot{y}}^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right)\right) - \bar{y}\left(t\right)^T \bar{z}\left(t\right) \\ &= f^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right) + \bar{x}^T\left(t\right)\bar{\omega}^i\left(t\right) - s\left(\bar{y}\left(t\right) \middle| K^i\right) \\ &\quad - \bar{x}\left(t\right)^T \sum_{i=1}^{p} \lambda^i \left(f_x^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right) + \bar{\omega}^i\left(t\right) - Df_{\dot{x}}^i\left(t,\bar{x},\dot{\bar{x}},\dot{y},\dot{\bar{y}}\right)\right) \end{aligned}$$

or

$$H^i = G^i \quad , \quad i = 1,2,\ldots p, \quad t \in I,$$

implying $\int_I H^i dt = \int_I G^i dt \quad , \quad i = 1,2,\ldots p$

or

$$\int_I Hdt = \int_I Gdt$$

This by Theorem 1 establishes the efficiency of $\left(\bar{x}(t), \bar{y}(t), \bar{\omega}_1(t), ..., \bar{\omega}_p(t), \bar{\lambda}\right)$ for the dual problem (SWD).

Now we state the converse duality theorem whose proof follows by symmetry of the formulations of the pair of problems.

THEOREM 3: (*Converse duality*): Let $\left(x(t), y(t), \omega_1(t), ..., \omega_p(t), \lambda\right)$ be an efficient solution for (SWP) and $\lambda = \bar{\lambda}$ be fixed in (SWD). Furthermore, assume that

(A$_1$):

$$\Big\{(\psi(t))^T\left(\lambda^T f_{xx}(t,x,\dot{x},y,\dot{y}) - D\lambda^T f_{\dot{x}x}(t,x,\dot{x},y,\dot{y})\right) - D\left[(\psi(t))^T\left(-D\lambda^T f_{\dot{x}\dot{x}}(t,x,\dot{x},y,\dot{y})\right)\right]$$

$$+D^2\left[(\psi(t))^T\left(-\lambda^T f_{\dot{x}\dot{x}}(t,x,\dot{x},y,\dot{y})\right)\right]\Big\}(\psi(t)) = 0, t \in I \Rightarrow \psi(t) = 0, t \in I$$

(A$_2$): $f_x^i(t,x,\dot{x},y,\dot{y}) + \omega^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})$, $i = 1,..p$.are linearly independent.

Then $\left(x(t), y(t), z_1(t), ..., z_p(t), \lambda\right)$ is feasible for (SWD) and the objective functional values are equal. If, in addition, the hypothesis of Theorem 1 hold, then there exist $z_1(t), z_2(t), ..., z_p(t)$ such that $\left(u(t), v(t), \lambda, z_1(t),..z_p(t)\right) = \left(x(t), y(t), \lambda, z_1(t),..z_p(t)\right)$ is an efficient solution of dual (SWD).

4. Mond-Weir Type Dualıty

In this section, we present the following pair of Mond-Weir dual problems, (SM-WP) and (SM-WD):

(SM-WP): Maximize: $\int_I \left(\Phi^1, \Phi^2, ..., \Phi^p\right) dt$

Subject to:

$$x(a) = 0 = x(b)$$

$$y(a) = 0 = y(b)$$

$$\sum_{i=1}^{p} \lambda^i\left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right) \leqq 0 , \; t \in I$$

$$\int_I y^T(t) \sum_{i=1}^{p} \lambda^i\left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right) d \; t \geqq 0$$

$$z^i(t) \in C^i \;,\; i = 1,...,p, \quad t \in I$$

$$x(t) \geqq 0 \;,\; t \in I$$

$$\lambda > 0$$

(SM-WD): Minimize: $\int_I \left(\psi^1, \psi^2, ..., \psi^p\right) dt$

Subject to:

$$u(a) = 0 = u(b)$$
$$v(a) = 0 = v(b)$$
$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right) \geqq 0 \ , \ t \in I$$
$$\int_I y^T(t) \sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right) d \leqq 0$$
$$\omega^i(t) \in K^i \ , i = 1,..., p$$
$$v(t) \geqq 0 \ , \ t \in I$$
$$\lambda > 0$$

where,

1. $\Phi^i = f^i(t,x,\dot{x},y,\dot{y}) + s\left(x(t) \middle| C^i\right) - y(t)^T z(t) \quad , i = 1,..., p$
2. $\psi^i = f^i(t,u,\dot{u},v,\dot{v}) - s\left(v(t) \middle| K^i\right) + u(t)^T \omega(t) \quad , i = 1,..., p$

The duality theorems for these problems will be merely stated below for completeness as their proofs follow on the lines of [15]:

Theorem 4 (*Weak Duality***)**: Let $\left(x(t), y(t), z_1(t),..., z_p(t), \lambda\right)$ be feasible for the (SM-WP) and $\left(u(t), v(t), \omega_1(t),..., \omega_p(t), \lambda\right)$ be feasible for the d ual (SM-WD). Assume that, for each i, $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,y,\dot{y}) dt + (.) z^i \right) dt$ is pseudoconvex in $(x,\dot{x})$ for fixed $(y,\dot{y})$ and $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},.,.) - (.) z^i \right) dt$ is pseudoconcave in $(y,\dot{y})$ for fixed $(x,\dot{x})$. Then,

$$\int_I \phi(t,x,\dot{x},y,\dot{y},\lambda) dt \not\leq \int_I \psi(t,x,\dot{x},y,\dot{y},\lambda) dt$$

Theorem 5 (*Strong Duality***)**: Let $\left(x(t), y(t), z_1(t),..., z_p(t), \lambda\right)$ be an efficient solution for (SM-WP) and $\lambda = \bar{\lambda}$ be fixed in (SM-WD). Furthermore, assume that

(H_1):

$$\int_I \Big\{ \left(\phi(t)\right)^T \left(\lambda^T f_{yy}(t,x,\dot{x},y,\dot{y}) - D\lambda^T f_{\dot{y}y}(t,x,\dot{x},y,\dot{y}) \right)$$
$$-D\left[\left(\phi(t)\right)^T \left(-D\lambda^T f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y}) \right) \right]$$

$$+D^2\left[\left(\phi(t)\right)^T\left(-f_{\dot{y}\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\right)\right]\Big\}\left(\phi(t)\right)dt=0,\ t\in I\Rightarrow\phi(t)=0,\ t\in I$$

(H₂): $f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i(t)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)$, $i=1,\dots,p$, are linearly independent.

Then $\left(x(t),y(t),\omega_1(t),\dots,\omega_p(t),\lambda\right)$ is feasible for (SM-WD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 5 hold, then there exist $\omega_1(t),\omega_2(t),\dots,\omega_p(t)$ such that $\left(u(t),v(t),\lambda,\omega_1(t),\dots,\omega_p(t)\right)=\left(x(t),y(t),\lambda,\omega_1(t),\dots,\omega_p(t)\right)$ is an efficient solution of dual (SM-WD).

Theorem 6 (Converse duality**)**: Let $\left(x(t),y(t),\omega_1(t),\dots,\omega_p(t),\lambda\right)$ be an efficient solution for (SM-WP) and $\lambda=\bar{\lambda}$ be fixed in (SM-WD). Furthermore, assume that

(A₁):

$$\Big\{\int_I\left(\psi(t)\right)^T\left(\lambda^T f_{xx}\left(t,x,\dot{x},y,\dot{y}\right)-D\lambda^T f_{\dot{x}x}\left(t,x,\dot{x},y,\dot{y}\right)\right)$$

$$-D\left[\left(\psi(t)\right)^T\left(-D\lambda^T f_{\dot{x}\dot{x}}\left(t,x,\dot{x},y,\dot{y}\right)\right)\right]$$

$$+D^2\left[\left(\psi(t)\right)^T\left(-f_{\dot{x}\dot{x}}\right)\right]\Big\}\left(\psi(t)\right)dt=0,\ t\in I\Rightarrow\psi(t)=0,\ t\in I$$

(A₂): $f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+z^i(t)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)$, $i=1,\dots,p$, $t\in I$ are linearly independent.

Then $\left(x(t),y(t),z_1(t),\dots,z_p(t),\lambda\right)$ is feasible for (SM-WD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 5 hold, then there exist $z_1(t),z_2(t),\dots,z_p(t)$ such that $\left(u(t),v(t),\lambda,z_1(t),\dots,z_p(t)\right)=\left(x(t),y(t),\lambda,z_1(t),\dots,z_p(t)\right)$ is an efficient solution of dual (SM-WD).

5. Self Duality

A problem is said to be self-dual if it is formally identical with its dual, in general, the problems (SP) and (SWD) are not formally identical if the kernel function does not possess any special characteristics. Hence, skew symmetry of each f^i is assumed in order to validate the following self-duality theorems for the two pairs of problems treated in the preceding sections.

Theorem 7. (*Self Duality***):** Let $f^i, i=1,2,\dots,p$, be skew symmetric and $C^i=K^i$ and $\omega^i(t)=z^i(t)$. Then the problem (SP) is self dual. If the problems (SWP) and (SWD) are dual problems and $\left(\bar{x}(t),\bar{y}(t),z^1(t),\dots,z^p(t),\bar{\lambda}\right)$ is a joint optimal solution of (SWP) and (SWD), then so is $\left(\bar{y}(t),\bar{x}(t),z^1(t),\dots,z^p(t),\bar{\lambda}\right)$, i.e.

Minimum (SWP) = $\int_I\left(f^1\left(t,x,\dot{x},y,\dot{y}\right),f^2\left(t,x,\dot{x},y,\dot{y}\right),\dots,f^p\left(t,x,\dot{x},y,\dot{y}\right)\right)dt=0$.

Proof: By skew symmetric of f^i, we have

$$f_x^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)=-f_y^i\left(t,y(t),\dot{y}(t),x(t),\dot{x}(t)\right)$$

$$f_y^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)=-f_x^i\left(t,y(t),\dot{y}(t),x(t),\dot{x}(t)\right)$$

$$f_{\dot{x}}^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)=-f_{\dot{y}}^i\left(t,y(t),\dot{y}(t),x(t),\dot{x}(t)\right)$$

$$f_{\dot{y}}^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)=-f_{\dot{x}}^i\left(t,y(t),\dot{y}(t),x(t),\dot{x}(t)\right)$$

Recasting th e d ual p roblem (S WD) a s a m inimization p roblem a nd u sing th e a bove relations, we have,

(SWD$_1$): Minimize $-\int_I \left(G^1,G^2,...,G^p\right)dt$

Subject to:

$$x(a)=0=x(b)\ ,\quad y(a)=0=y(b)$$

$$-\sum_{i=1}^{p}\lambda^i\left[-f_x^i\left(t,y,\dot{y},x,\dot{x}\right)+\omega^i(t)-Df_{\dot{x}}^i\left(t,y,\dot{y},x,\dot{x}\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$=\quad \sum_{i=1}^{p}\lambda^i\left[f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-\omega^i(t)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right]\leqq 0\ ,\ t\in I$$

$$v(t)\geqq 0\ ,\ t\in I$$

$$\omega^i(t)\in K^i\ \ ,i=1,...,p,\ \ t\in I$$

$$\lambda\in\Lambda^+$$

$$-G^i=-f^i\left(t,x,\dot{x},y,\dot{y}\right)-s\left(y(t)\middle|K_i\right)-x(t)\omega_i(t)$$

$$-x(t)\sum_{i=1}^{p}\lambda^i\left[f_x^i\left(t,x,\dot{x},y,\dot{y}\right)-\omega_i(t)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right]$$

$$=f^i\left(t,y,\dot{y},x,\dot{x}\right)-s\left(y(t)\middle|K_i\right)-x(t)\omega_i(t)$$

$$-x(t)\sum_{i=1}^{p}\lambda^i\left[f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-\omega_i(t)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right]$$

$$=H^i\left(t,y,\dot{y},x,\dot{x},\omega_1,...,\omega_p,\lambda\right)$$

Hence we have,

(SWP-1): $\int_I \left(H^1,H^2,...,H^p\right)dt$

Subject to:

$$x(a)=0=x(b)\ ,\quad y(a)=0=y(b)$$

$$\sum_{i=1}^{p}\lambda^i\left[f_y^i\left(t,x,\dot{x},x,\dot{y}\right)-z_i(t)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$z^i(t) \in C^i \ , \quad t \in I$$

$$\lambda > 0, \ \ \lambda^T e = 1 \quad \text{where} \quad e^T = (1, \ldots\ldots, 1),$$

which is just the primal problem (SWP). Therefore $\left(\overline{x}(t), \overline{y}(t), \overline{z}^1(t), \ldots, \overline{z}^p(t), \overline{\lambda}\right)$ is an efficient solution of dual problem implies that $\left(\overline{y}(t), \overline{x}(t), \overline{z}^1(t), \ldots, \overline{z}^p(t), \overline{\lambda}\right)$ is an efficient solution of the primal. Similarly $\left(\overline{x}(t), \overline{y}(t), \overline{z}^1(t), \ldots, \overline{z}^p(t), \overline{\lambda}\right)$ is an efficient solution of (SP) implies $\left(\overline{y}(t), \overline{x}(t), \overline{z}^1(t), \ldots, \overline{z}^p(t), \overline{\lambda}\right)$ is an efficient solution of the dual problem (SWD). In view of (18), (33), (34) and (36), we get,

Minimum (SWP) = $\int_I \left\{ f^1\left(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{y}(t), \dot{\overline{y}}(t)\right), . \quad f^p .\left(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{y}(t), \dot{\overline{y}}(t)\right) \right\} d \ \ t$

Corresponding, to the solution $\left(\overline{y}(t), \overline{x}(t), \overline{z}^1(t), \ldots, \overline{z}^p(t), \overline{\lambda}\right)$, we have,

Minimum (SWP) = $\int_I \left\{ f^1\left(t, \overline{y}(t), \dot{\overline{y}}(t), \overline{x}(t), \dot{\overline{x}}(t)\right), . \quad f^p \left(t, \overline{y}(t), \dot{\overline{y}}(t), \overline{x}(t), \dot{\overline{x}}(t)\right) \right\} d \ \ t$

By the skew-symmetry of each f^i

$$\int_I \left\{ f^1\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}\right), \ldots, f^p\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}\right) \right\} d \ t = \int_I \left\{ f^1\left(t, \overline{y}, \dot{\overline{y}}, \overline{x}, \dot{\overline{x}}\right), \ldots, f^p\left(t, \overline{y}, \dot{\overline{y}}, \overline{x}, \dot{\overline{x}}\right) \right\} d \ \ t$$

$$= -\int_I f^1 \left\{ \left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}\right), . \quad . f^p \left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}\right) \right\} d \ t = 0$$

The following self duality theorem for the pair of Mond-Weir type self duality theorem will be merely stated for completeness and its proof runs parallel to that of Theorem 4.

Theorem 8. **(***Self Duality***)** Let $f^i, i = 1, 2, \ldots, p$, be skew symmetric and $C^i = K^i$ and $\omega^i(t) = z^i(t)$. Then the problem (SM-WP) is self dual. If the problems (SM-WP) and (SM-WD) are dual problems and $\left(\overline{x}(t), \overline{y}(t), z^1(t), \ldots, z^p(t), \overline{\lambda}\right)$ is a joint optimal solution of (SM-WP) and (SM-WD), then so is $\left(\overline{y}(t), \overline{x}(t), z^1(t), \ldots, z^p(t), \overline{\lambda}\right)$**,** i.e.

Minimum (SM-WP) = $\int_I f\left(t, x, \dot{x}, y, \dot{y}\right) dt = 0$.

6. Natural Boundary Values

The pairs of Wolfe type and Mond-Weir type symmetric multiobjective variational problem can be formulated with natural boundary values rather than fixed end points. The problems with natural boundary conditions are needed to establish well defined relationship between the pairs of continuous programming problems and nonlinear programming problems.

Following is the pair of Wolfe type symmetric dual problems with natural boundary values.

Primal (SWPo): Maximize $\int_I \left(H^1, H^2, ..., H^p\right) dt$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z^i\left(t\right) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right] \leqq 0 ,$$

$$\left(x\left(t\right)\right) \geqq 0 , \; t \in I$$

$$z^i\left(t\right) \in C^i , \qquad t \in I$$

$$\lambda > 0, \quad \lambda^T e = 1 \; , \; e^T = \left(1, ...,1\right)$$

$$f_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right)\Big|_{t=a} = 0 \; , \; f_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right)\Big|_{t=b} = 0$$

Dual ($\mathbf{SWD_0}$): Maximize $\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) + \omega_i\left(t\right) - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right] \geqq 0 , \; t \in I ,$$

$$v\left(t\right) \geqq 0 , \; t \in I$$

$$\omega^i\left(t\right) \in K^i \quad , \; t \in I$$

$$\lambda > 0, \quad \lambda^T e = 1 \; , \; e^T = \left(1, ...,1\right)$$

$$f_{\dot{x}}^i\left(t, x, \dot{x}, y, \dot{y}\right)\Big|_{t=a} = 0 \; , \; f_{\dot{x}}^i\left(t, x, \dot{x}, y, \dot{y}\right)\Big|_{t=b} = 0$$

The duality results for each of the above pairs of dual problems can be proved easily on the lines of the proofs of the Theorems 1-8, with slight modifications in the arguments, as in Mond and Hanson [7].

Following is the pair of Mond-Weir type symmetric dual problems with natural boundary values.

Primal ($\mathbf{SM\text{-}WP_0}$): Maximize $\int_I \left(\Phi^1, \Phi^2, ..., \Phi^p\right) dt$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left(f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z^i\left(t\right) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right) \leqq 0 , \; t \in I$$

$$\int_I y^T\left(t\right) \sum_{i=1}^{p} \lambda^i \left(f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z^i\left(t\right) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right) d \; \geqq 0$$

$$z^i\left(t\right) \in C^i \; , \; i = 1, ..., p , \quad t \in I$$

$$x\left(t\right) \geqq 0 \; , \; t \in I$$

$$\lambda > 0$$

$$\lambda^T f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\Big|_{t=a} = 0 \ , \ \lambda^T f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\Big|_{t=b} = 0$$

Dual (SM-WD$_0$): Minimize: $\int_I \left(\psi^1,\psi^2,...,\psi^p\right)dt$

Subject to:

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},v,\dot{v}\right)+\omega^i\left(t\right)-Df_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right)\geqq 0 \ , \ t\in I$$

$$y^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},v,\dot{v}\right)+\omega^i\left(t\right)-Df_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right)\leqq 0 \ , \ t\in I$$

$$\omega^i\left(t\right)\in K^i \ , i=1,...,p$$

$$v\left(t\right)\geqq 0 \ , \ t\in I$$

$$\lambda\in\Lambda^+$$

$$\lambda^T f_{\dot{u}}\left(t,u,\dot{u},v,\dot{v}\right)\Big|_{t=a} = 0 \ , \ \lambda^T f_{\dot{u}}\left(t,u,\dot{u},v,\dot{v}\right)\Big|_{t=b} = 0$$

7. Nondifferentiable Multiobjective Nonlinear Programming

If the time dependency of Wolfe type symmetric pair of dual problems, (SWPo) and (SWDo) is removed and $b-a=1$, we obtain following pair of Wolfe type static nondifferentiable multiobjective dual problems with support functions which are not explicitly reported in literature.

(Primal SWP-2): Minimize $\hat{H}^i = \hat{H}^1,\hat{H}^2,...,\hat{H}^p$

Subject to:

$$\sum_{i=1}^{p}\lambda^i\left[f_y^i\left(x,y\right)-z^i\right]\leqq 0 \ ,$$

$$z^i\in K^i \ , \ i=1,...,p$$

$$\lambda\in\Lambda^+$$

where,

$$\hat{H}^i = f^i\left(x,y\right)+s\left(x\middle|C^i\right)-y^T\sum_{i=1}^{p}\lambda^i\left(f_y^i\left(x,y\right)-z^i\right)-y^Tz \ , \quad i=1,...,p$$

(Dual SWD-2): Maximize: $\hat{G}^i = \left(\hat{G}^1,\hat{G}^2,...,\hat{G}^p\right)$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_u^i(u,v) + \omega^i \right] \geqq 0,$$

$$\omega^i \in C^i, \quad i = 1, \dots, p$$

$$\lambda \in \Lambda^+$$

Where

$$\hat{G}^i = f^i(u,v) + s\left(v \middle| K^i\right) - u^T \sum_{i=1}^{p} \lambda^i \left(f_u^i(u,v) + \omega^i \right) - x^T \omega$$

As in the case of pair of Wolfe type dual problems, the pair of Mond-Weir type dual problems (SM-WPo) and (SM-WDo) reduce to the following static counterparts in nonlinear programming.

Primal (SM-WP-2): Maximize $\hat{\Phi}^i = \left(\hat{\Phi}^1, \hat{\Phi}^2, \dots, \hat{\Phi}^p\right)$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_y^i(x,y) - z^i \right] \leqq 0,$$

$$y^T \sum_{i=1}^{p} \lambda^i \left(f_y^i(x,y) - z^i \right) \geqq 0,$$

$$z^i \in C^i, \ i = 1, \dots, p,$$

$$x \geqq 0,$$

$$\lambda > 0$$

where

$$\hat{\Phi}^i = f^i(x,y) + s\left(x \middle| C^i\right) - y^T z$$

Dual (SM-WD-2): Maximize: $\hat{\psi}^i = \hat{\psi}^1, \hat{\psi}^2, \dots, \hat{\psi}^p$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_u^i(u,v) + \omega^i \right] \geqq 0,$$

$$y^T \sum_{i=1}^{p} \lambda^i \left(f_u^i(u,v) + \omega^i \right) \leqq 0$$

$$\omega^i \in K^i \quad i = 1, \dots, p$$

$$v \geqq 0$$

$$\lambda \in \Lambda^+$$

where,

$$\hat{\psi}^i = f^i(x, y) + s\left(y \middle| K^i\right) - x^T w$$

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