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Simple approach of obtaining the optimal pricing and lot-sizing policies for an EPQ model on deteriorating items with shortages under inflation and time-discounting

M. Valliathal [1]
Department of Mathematics,
Chikkaiah Naicker College, Erode, Tamil Nadu, India-624302

R. Uthayakumar [2]
Department of Mathematics,
Gandhigram University, Gandhigram, Tamil Nadu, India-624302.

Abstract

In this paper, the optimal pricing and lot-sizing policies of a production lot-sizing model for deteriorating item under inflation and time discounting by considering two different decision making policies such as coordinated and decentralized decision making policies under which we derive the solution of the multivariate maximization problem are discussed. In addition, shortages are allowed and the unsatisfied demand is partially backlogged. The backlogging rate is a constant fraction of the on-hand inventory. The model is studied under inventory followed by shortages. We take demand as a function of selling price and time. The objective of this model is to maximize the total profit (TP) which includes the sales revenue, purchase cost, the set up cost, holding cost, shortage cost and opportunity cost due to lost sales. Theoretical results are given to justify the model. Finally, numerical examples are presented to determine the developed model and the solution procedure. Sensitivity analysis of the optimal solution with respect to major parameters is carried out. We propose a solution procedure to find the solution and obtain some managerial results by using sensitivity analysis.

Keywords: *Inventory Control, Deterioration, Inflation, Time – Discounting, Partial Backlogging.*

Enflasyon etkisi altında ve kıtlık gösteren bozulabilir ürünlere yönelik ekonomik sipariş büyüklüğü modeli için optimum fiyatlandırma ve sipariş büyüklüğü belirleme politikaları için basit yaklaşım

Özet

Bu çalışmada, enflasyon etkisi altındaki kısa ömürlü ürünler için üretim sipariş büyüklüğü modelinin optimum fiyatlandırma ve sipariş büyüklüğü politikaları ele alınmıştır. Çok değişkenli maksimizasyon problemleri olarak sonuçları elde edilen iki farklı karar verme politikası kullanılmıştır: Eşgüdümlü ve eşgüdümsüz karar verme politikaları. Ayrıca talep karşılayamamaya izin verilmiş ve karşılanamayan talepler kısmi olarak biriktirilmiştir. Biriktirme oranı eldeki mevcut stokun sabit bir oranı olarak belirlenmiştir. Model, karşılanamayan taleplerin stokta takibi yoluyla çalışılmıştır. Talep, satış fiyatı ve zamanın bir fonksiyonu olarak ele alınmıştır. Modelde amaç, satış geliri, satın alma maliyeti, hazırlık (set up) maliyeti, elde bulundurma maliyeti, 'yok' satıştan kaynaklanan elde bulundurmama ve fırsat maliyetlerini içeren toplam kârın maksimizasyonudur. Model verilen teorik sonuçlarla sınanmıştır. Son olarak, geliştirilen model ve çözüm prosedürüne ait sayısal örnekler verilmiştir. Temel parametrelere ilişkin en uygun çözüme ait duyarlılık

[1] *balbal_ba@yahoo.com (M.Valliathal)*
[2] *udhayagri@gmail.com (R. Uthayakumar)*

analizi gerçekleştirilmiştir. Çözüm bulmak amacıyla bir prosedür önerilmiş ve duyarlılık analizi kullanarak yönetsel bazı sonuçlar elde edilmeye çalışılmıştır.

Anahtar Sözcükler: *Stok Kontrol, Yıpranma, Enflasyon, Kısa Ömür, Kısmi Bekletme.*

1. Introduction

Generally, goods in inventory do not always safeguard their physical characteristics because there are some items which are subject to risks like breakage, evaporation, obsolescence etc. The first attempt for assessing deteriorating items was made by Ghare and Schrader [1], who derived the revised form of the EOQ model assuming exponential decay. Dave and Patel [2] considered the inventory models for deteriorating items with linear increasing demand when shortages are not allowed over a finite horizon. Goyal and Giri [3] provided a detailed review of deteriorating inventory literatures. Tsao and Sheen [4] considered a replenishment model for deteriorating items with lot-size and time-dependent purchasing cost under credit period. Recent research in this area includes:

In contrast to the above EOQ models, Balkhi [5] first generalized the EPQ model for deteriorating items in which the demand, production and deterioration rates are continuous functions of time. After that Goyal and Giri [6] extend Balkhi's [5] model to allow for partial backlogging.

An increase in the general price level results in a reduction in the consumption power of money. Most of the classical inventory models did not consider the effects of inflation. But many of countries suffer from large scale inflation. It is therefore necessary to investigate how inflation influences various inventory policies. The inflationary effect on an inventory policy has been examined by several authors. The pioneer in this field was Buzacott [7]. Brahmbhatt [8] studied an economic order quantity under variable rate of inflation and mark-up prices.

Recently, Mirzazadeh et al. [9] studied an EPQ model for deteriorating items with inflation-dependent demand under uncertain inflationary conditions and shortages.

In classical economic order quantity model, it is often assumed that shortages are either completely backlogged or completely lost. As a physical phenomenon, some customers may like to prefer backlogging during the shortage period, while the others would not. In 1995, Padmanabhan and Vrat [10] evolved an EOQ model for perishable items with stock-dependent demand. Chang and Dye [11] studied an EOQ model for deteriorating items with partial backordering. Goyal and Giri [6] established an EPQ model for deteriorating items with partial backlogging. Chen and Chen [12] developed an EPQ model for deteriorating items with complete backlogging.

In this paper, we discuss the inventory policies for the production of a lot-sizing model for deteriorating items with selling price and time dependent demand. Our review of relevant literature failed to identify any model which considers the selling price under two different policies and time dependent demand function, over an infinite time horizon under partial backlogging with inflation and time discounting, and extends it to a finite horizon model. We discuss the model under the two types of policies by Chen and Chen [12] (the sufficient conditions cannot be verified analytically in their model) and finally compare them. Here we study the model under infinite horizon and extend it to the finite horizon proposed by Goyal and Giri [6]. Instead of using dynamic programming approach, for easy understanding , application and the verification of second order sufficient conditions, we solve the EPQ problems here by Lagrangian multipliers method. The rest of this paper is organized as follows. In section 2, the notations and assumptions used are listed. In section 3, we present the mathematical model. In section 4, numerical examples are given to illustrate the model. Finally, we conclude the paper.

2. Notations and Assumptions

To develop the Mathematical model, the following notations and assumptions are being made:

2.1 Notations

The following notations are defined:

K	the fixed production set up cost per run
H	the holding cost per unit per unit time
P	the purchasing cost per unit
P'	the selling price per unit, where $P' > P$ (a decision variable)
$\theta(t)$	the deterioration rate at time t, $t \in [0,T]$
S	the shortage cost per unit per unit time
Π	the opportunity cost due to lost sales per unit
$I(t)$	the inventory level at time t, $t \in [0,T]$
$R(P',t)$	the demand rate at time t, $t \in [0,T]$
$P_p(t)$	the production rate at time t, $P_p(t) > R(P', t)$where $t \in [0,T]$
i''	the discount rate of net inflation
S_o	the time at which the cycle begins with zero stock and the shortage begins to accumulate (a decision variable)
S_1	the time at which the shortage reaches its maximum and the production process starts to meet the demand (a decision variable)
T	the time where the shortage level reaches zero and the inventory starts to accumulate (a decision variable)
T_1	the time at which the inventory level reaches its maximum and the production process stops (a decision variable)
S	the time where the cycle ends with zero number of stocks (a decision variable)
Q	the production lot-size (a decision variable)
TP	the total relevant profit of the inventory

2.2 Assumptions

In addition, we use the following basic assumptions.

- A single item inventory is considered over an infinite planning horizon.

- The demand rate R (P', t) is a nonnegative and continuous function of time and a decreasing, convex function of selling price in the planning horizon.
- The production rate P_p (t) is a known functions of time with $P_p(t) > R(P', t)$.
- The items deteriorate continuously over time which is a known function of time.
- There is no repair or replacement of the deteriorated items during the production cycle.
- The lead time is assumed to be zero.
- Shortages are allowed, only a fraction B ($0 \leq B \leq 1$) of the demand is backlogged and the remaining fraction ($1-B$) is lost.

3. Model Formulation

There are two types of inventory models: Type i. Inventory followed by shortages, and Type ii. Shortages followed by inventories. Throughout our study we use inventory followed by shortages. In the proposed model a cycle can be divided into four periods. Here [S_o, S] is taken as one cycle duration. During [S_o, T] the inventory is on the negative side and [T, S] the inventory is on the positive side.

A typical behaviour of the inventory in a cycle is depicted in the following fig.1.The inventory starts and ends with zero stock. So the shortage begins to accumulate at the early stage in inventory. The production starts only at time S_1 to meet the current and backlog demands. T is the time when shortage level reaches zero; afterwards the positive level of the inventory begins to build up.T_1 is the time when the production process stops; the inventory level then starts declining. Finally, the cycle ends with zero stock at time S.

During the time duration [S_o ,S_1], the inventory level starts with zero at S_o and from onwards the shortage begins to accumulate and reaches its maximum at S_1 and only a fraction B is backlogged partially and the remaining fraction ($1-B$) is lost. Hence the inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -B\,R(P',t); \quad S_o \leq t \leq S_1; \tag{1}$$

with the boundary condition $I(S_0)=0$. Solving the differential equation (1), we get the inventory level as

$$I(t) = \int_t^{S_0} B\,R(P',u)du \;; \qquad S_0 \leq t \leq S_1; \tag{2}$$

During the time duration [S_1, T], the production starts at S_1 and meets the current and backlog demands so that the shortages become zero at T. Hence the inventory level is governed by the following differential equation:

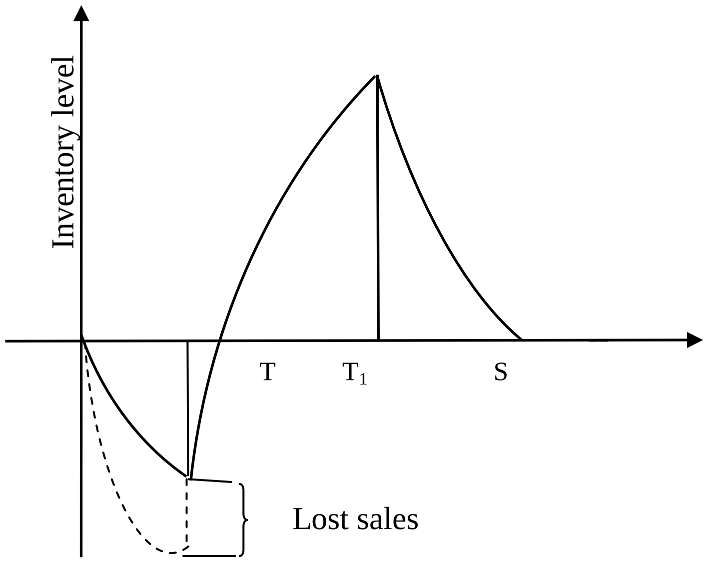

Figure 1 A typical behaviour of the inventory in a cycle.

$$\frac{dI(t)}{dt} = P_p(t) - R(P',t); \qquad S_1 \le t \le T; \tag{3}$$

with the boundary condition $I(T) = 0$. Solving the differential equation (3), we get the inventory level as

$$I(t) = \int_T^t \left[P_p(t) - R(P',t)\right]dt; \qquad S_1 \le t \le T \tag{4}$$

During the time duration [T, T_1], the production of the item is continued and since the demand rate is lesser than the production rate, the inventory level increases and meets the current demand, and the excess of the inventory is stored, so that deterioration takes place as soon as the product reaches the stock. The on hand inventory reaches maximum level at T_1, where the production process comes to an end. Hence the inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -\theta(t)I(t) + P_p(t) - R(P',t); \qquad T \le t \le T_1; \tag{5}$$

with the boundary condition $I(T) = 0$. Solving the differential equation (5), we get the inventory level as

$$I(t) = e^{-\int \theta(t)dt} \int_T^t \left[P_p(u) - R(P',u)\right] e^{\int \theta(u)du} du; \quad T \le t \le T_1; \tag{6}$$

During the time duration [T_1, S], the production of the item is stopped at T_1, so that the on-hand inventory decreases due to the demand and deterioration, and the inventory reaches zero at S. Hence the inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -\theta(t)I(t) - R(P',t); \qquad T_1 \le t \le S; \tag{7}$$

with the boundary condition $I(S) = 0$. Solving the differential equation (7), we get the inventory level as

$$I(t) = e^{-\int \theta(t)dt} \int_t^S R(P', u) e^{\int \theta(u)du} du; T_1 \le t \le S; \qquad (8)$$

When $t = S_1$, we have

$$\int_{S_1}^{S_0} B\, f(u)du = \int_T^{S_1} \left(P_P\ (u) - R(P', u)\right) du \qquad (9)$$

and $t = T_1$, we have

$$\int_T^{S_1} \left(P_P(u) - R(P', u)\right) e^{\int \theta(u)du} du = \int_{T_1}^S R(P', u) e^{\int \theta(u)du} du \qquad (10)$$

The present worth of the total inventory cost during the period [S_0, S] is the sum of

- Present worth of holding cost (HC_1, HC_2)
- Present worth of shortage cost (SC_1, SC_2)
- Present worth of opportunity cost and (OC)
- Present worth of setup cost $K\ e^{-i''S_1}$
- Present worth of production cost (PC_1, PC_2)

The present worth of the production cost during the production periods [S_1, T] and [T, T_1] are given respectively by

$$PC_1 = \int_{S_1}^T P(t) e^{-i''t} P_P(t) dt \qquad (11)$$

$$PC_2 = \int_T^{T_1} P(t) e^{-i''t} P_P(t)\ dt \qquad (12)$$

The present worth of the holding cost for carrying inventory over the periods [T, T_1] and [T_1, S] are given respectively by

$$HC_1 = \int_T^{T_1} he^{-i''t} I(t) dt \qquad (13)$$

$$HC_2 = \int_{T_1}^S h\, e^{-i''t} I(t) dt \qquad (14)$$

The present worth of the shortage cost over the period [S_0, S_1] and [S_1, T] are respectively given by

$$SC_1 = S \int_{S_0}^{S_1} e^{-i''t} [-I(t)\ ] dt \qquad (15)$$

$$SC_2 = S \int_{S_1}^T e^{-i''t} \left(\int_t^T \left(P_P\ (u) - R(P', u) du\right) \right) dt \qquad (16)$$

The present worth of the opportunity cost due to lost sales during the period [S_0, S_1] is given by

$$OC = \pi \int_{S_0}^{S_1} e^{-i''t} R(P',t)[1-B]dt \tag{17}$$

Therefore, the present worth of the total profit in [S_0, S] is given by

TP = the present worth of

$$\left[\begin{array}{l} \text{sales revenue - production cost - set up cost - holding cost} \\ \text{- shortage cost - opportunity cost due to lost sales} \end{array}\right]$$

$$TP = \left[\begin{array}{l} \int_{S_0}^{S_1} P' BR(P',t)e^{-i''t}dt + \int_{S_1}^{S} P' R(P',t)e^{-i''t}dt - \int_{S_1}^{T_1} e^{-i''t} P(t)P_P(t)dt \\ -Ke^{-i''S_1} - \int_{T}^{T_1} he^{-i''t}\left[e^{-\int\theta(t)dt}\int_{T}^{t}\left[P_p(u) - R(P',u)\right]e^{\int\theta(u)du}du\right]dt \\ -\int_{T_1}^{S} he^{-i''t}\left[e^{-\int\theta(t)dt}\int_{t}^{S} R(P',u)e^{\int\theta(u)du}du\right]dt - s\int_{S_0}^{S_1} e^{-i''t}\left[\int_{S_0}^{t} BR(P',u)du\right]dt \\ -s\int_{S_1}^{T} e^{-i''t}\left[\int_{t}^{T}\left(P_P(u) - R(P',u)du\right)\right]dt - \pi\int_{S_0}^{S_1} e^{-i''t}(1-B)R(P',t)dt \end{array}\right] \tag{18}$$

Equation (18) is a function of the selling price P′ and the time variables S_1, T, T_1, and S, which can either be determined by the decentralized policy or by the centralized, coordinated policy.

3.1 Model A: Decentralized Policy

In the decentralized decision process the marketing department sets the price by maximizing its gross profit function disregarding the production cost, the market responds with a specific demand and the production department makes the lot sizing and scheduling decision with the objective of minimizing the total production cost while satisfying the demand [12].

The multivariate maximum problem under the decentralized policy can be formulated as follows:

Maximize TP

Subject to the constraints (9), (10), $0 < P' < P'_{\max}$ and

$$S_o < S_1 < T < T_1 < S. \tag{19}$$

The gross profit function over[S_0, S] can be expressed as

$$\varphi_{TP}(P') = \int_{S_0}^{S_1}\left(P' - p(t)\right)BR(P',t)e^{-i''t}dt + \int_{S_1}^{S}\left(P' - p(t)\right)R(P',t)e^{-i''t}dt \tag{20}$$

The optimal price over the cycle can be obtained by differentiating equation (20) with respect to P' and setting the results equal to zero:

$$\frac{d\varphi_{TP}(P')}{dP'} = \begin{bmatrix} \int_{S_0}^{S_1} e^{-i''t}\left[B\left(R(P',t)+(P'-p(t))\frac{dR(P',t)}{dP'}\right)\right]dt + \\ \int_{S_1}^{S} e^{-i''t}\left[\left(R(P',t)+(P'-p(t))\frac{dR(P',t)}{dP'}\right)\right]dt \end{bmatrix} = 0 \qquad (21)$$

Let P'^* be the solution of equation (21) which represents the optimal price during the cycle. To show the optimality of the solution, we shall demonstrate that the gross profit function in equation (20) is concave in P', i.e., its second order sufficient condition is strictly less than zero:

$$\frac{d^2\varphi_{TP}(P')}{dP'^2} = \begin{pmatrix} \int_{S_0}^{S_1} e^{-i''t}\left[B\left(2\frac{dR(P',t)}{dP'}+(P'-p(t))\frac{d^2R(P',t)}{dP'^2}\right)\right]dt + \\ \int_{S_1}^{S} e^{-i''t}\left[\left(2\frac{dR(P',t)}{dP'}+(P'-p(t))\frac{d^2R(P',t)}{dP'^2}\right)\right]dt \end{pmatrix} < 0 \qquad (22)$$

Proposition 1 The gross profit function $\varphi_{TP}(P')$ is concave in P′, provided the demand function is the form: $R(P',t) = (a - bP'^{\gamma})$f (t).

Proof Substituting the demand function with $(a - bP'^{\gamma})$f (t) into equation (22), we have

$$\frac{d^2\varphi_{TP}(P')}{dP'^2} = \begin{bmatrix} \int_{S_0}^{S_1} e^{-i''t}\left[B\left(-2b\gamma P'^{(\gamma-1)} f(t) + (P'-P(t))\left(-b\gamma(\gamma-1)P'^{(\gamma-2)}\right)f(t)\right)\right]dt + \\ \int_{S_1}^{S} e^{-i''t}\left[\left(-2b\gamma P'^{(\gamma-1)} f(t) + (P'-P(t))\left(-b\gamma(\gamma-1)P'^{(\gamma-2)}\right)f(t)\right)\right]dt \end{bmatrix} \qquad (23)$$

Since B > 0, b > 0, $\gamma \geq 1$, $(P'-P(t))$ > 0 and f(t) > 0 for $S_o < S_1 < T < T_1 < S$, equation (23) is strictly negative.

The objective of the decentralized policy is to minimize the total cost.

Our problem is to find S_1, T, T_1, P' and S to minimize the total cost.

To develop a solution to the problem, we reduce the problem into an equality constrained Lagrangian as

$$L(S_0, S_1, T, T_1, \lambda, \mu) = \begin{pmatrix} TP + \lambda\left[\int_{S_1}^{S_0} B\, f(u)du - \int_{T}^{S_1}\left(P_P(u) - R(P',u)\right)du\right] + \\ \mu\left[\int_{T}^{S_1}\left(P_P(u) - R(P',u)\right)e^{\int\theta(u)du}du - \int_{T_1}^{S} R(P',u)e^{\int\theta(u)du}du\right] \end{pmatrix} \qquad (24)$$

The first order conditions for having maximum are for any given selling price P' are :

$$\frac{\partial TP}{\partial S_1} = 0,\ \frac{\partial TP}{\partial T_1} = 0, \frac{\partial TP}{\partial S} = 0,\ \frac{\partial TP}{\partial T} = 0, \frac{\partial TP}{\partial \lambda} = 0,\ \frac{\partial TP}{\partial \mu} = 0$$

and for the sufficiency condition, their corresponding Hessian matrix is negative definite.

$$\frac{\partial TP}{\partial \lambda}=0 \Rightarrow \int_{S_1}^{S_0} B\,R(P',u)du-\int_{T}^{S_1}\left(P_P\ (u)-R(P',u)\right)du = 0 \qquad (25)$$

$$\frac{\partial TP}{\partial \mu}=0 \Rightarrow \int_{T}^{S_1}\left(P_P\ (u)-R(P',u)\right)e^{\int \theta(u)du}du-\int_{T_1}^{S} R(P',u)e^{\int \theta(u)du}du = 0 \qquad (26)$$

$$\frac{\partial TP}{\partial S_1}=0 \Rightarrow \begin{bmatrix} s\left[\mathrm{e}^{-i''S_1}\int_{S_0}^{S_1} BR(P',u)du - \mathrm{e}^{-i''S_1}\int_{S_1}^{T}(P_p(u)-R(P',u))du\right]-\pi(1-B)\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1} \\ +\mathrm{i}''\mathrm{K}\mathrm{e}^{-i''S_1}-\lambda\left(\mathrm{BR}(\mathrm{P}',\mathrm{S}_1)+P_p(\mathrm{S}_1)-\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\right)+\mathrm{P}'\mathrm{BR}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1} \\ -\mathrm{P}'\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1}+P\,(\mathrm{S}_1)P_p(\mathrm{S}_1)\mathrm{e}^{-i''S_1} \end{bmatrix} = 0$$

$$\frac{\partial TP}{\partial S_1}=0 \Rightarrow \begin{bmatrix} s\left[\mathrm{e}^{-i''S_1}\int_{S_0}^{S_1} BR(P',u)du - \mathrm{e}^{-i''S_1}\int_{S_1}^{T}(P_p(u)-R(P',u))du\right]-\pi(1-B)\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1} \\ +\mathrm{i}''\mathrm{K}\mathrm{e}^{-i''S_1}-\lambda\left(\mathrm{BR}(\mathrm{P}',\mathrm{S}_1)+P_p(\mathrm{S}_1)-\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\right)+\mathrm{P}'\mathrm{BR}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1} \\ -\mathrm{P}'\mathrm{R}(\mathrm{P}',\mathrm{S}_1)\mathrm{e}^{-i''S_1}+P\,(\mathrm{S}_1)P_p(\mathrm{S}_1)\mathrm{e}^{-i''S_1} \end{bmatrix} = 0 \qquad (27)$$

$$\frac{\partial TP}{\partial T}=0 \Rightarrow \left(h\left[\int_{T}^{T_1} e^{\int_t^T \theta(v)dv} e^{-i''t}dt\right]-s\int_{S_1}^{T_1} e^{-i''t}dt+\lambda-\mu e^{\int_{T_1}^{T}\theta(v)dv}\right) = 0 \qquad (28)$$

$$\frac{\partial TP}{\partial T_1}=0 \Rightarrow \mu=P(T_1)e^{-i''T_1} \qquad (29)$$

$$\frac{\partial TP}{\partial S}=0 \Rightarrow \left(-h\left(\int_{T_1}^{S} e^{\left(-i''t+\int_t^S \theta(v)dv\right)}dt\right)+P'e^{-i''S}+\lambda B-\mu e^{\int_{T_1}^{S}\theta(v)dv}\right) = 0 \qquad (30)$$

$$\frac{\partial^2 TP}{\partial {S_1}^2} = \begin{bmatrix} -\mathrm{s}\mathrm{e}^{-i''S_1}\left(B\left(R(P',S_1)\right)+P_p(S_1)-R(P',S_1)\right)-K(i'')^2\mathrm{e}^{-i''S_1} \\ (\mathrm{B}-1)(\mathrm{P}'+\pi)\left(R(P',S_1)\mathrm{e}^{-i''S_1}(i'')+R(P',S_1)\mathrm{e}^{-i''S_1}\right) \\ \lambda\left(BR'(P',S_1)+P_p'(S_1)-R'(P',S_1)\right)+P\,(S_1)P_p(S_1)\mathrm{e}^{-i''S_1}(-i'') \\ +P'(S_1)P_p(S_1)\mathrm{e}^{-i''S_1}+P(S_1)P_p'(S_1)\mathrm{e}^{-i''S_1} \end{bmatrix} \qquad (31)$$

$$\frac{\partial^2 TP}{\partial T^2} = \left(h\left[\int_{T}^{T_1}\theta(T)e^{\int_t^T \theta(v)dv} e^{-i''t}dt-e^{-i''T}\right]-\mu\theta(T)e^{\int_{T_1}^{T}\theta(v)dv}\right) \qquad (32)$$

$$\frac{\partial^2 TP}{\partial T_1^{\ 2}} = \begin{bmatrix} \left(h - P'(T_1)\right)P_p(T_1)e^{-i''T_1} + \left(\mu - P(T_1)e^{-i''T_1}\right)P_p'(T_1) \\ + i''P(T_1)e^{-i''T_1}P_p(T_1) \end{bmatrix} \tag{33}$$

$$\frac{\partial^2 TP}{\partial S^2} = \begin{pmatrix} -h\left(\int_{T_1}^{S}\theta(S)e^{\left(-i''t+\int_t^S \theta(v)dv\right)}dt + e^{-i''S}\right) - \mu\theta(S)e^{\left(\int_{T_1}^{S}\theta(v)dv\right)} \\ -(i'')e^{-i''S}P' \end{pmatrix} \tag{34}$$

From the above results it is clear that:

$$\frac{\partial^2 TP}{\partial T^2} < 0 \text{ if } \left(h\left[\int_T^{T_1}\theta(T)e^{\int_t^T \theta(v)dv} e^{-i''t}dt\right] < e^{-i''T} + \mu\theta(T)e^{\int_{T_1}^T \theta(v)dv} \quad and\ \mu > 0 \right)$$

$$\frac{\partial^2 TP}{\partial S^2} < 0,$$

$$\frac{\partial^2 TP}{\partial T_1^{\ 2}} < 0 \text{ if } \begin{bmatrix} \begin{pmatrix} \left(h - P'(T_1)\right)P_p(T_1)e^{-i''T_1} \\ + \left(\mu - P(T_1)e^{-i''T_1}\right)P_p'(T_1) \end{pmatrix} > i''P(T_1)e^{-i''T_1}P_p(T_1), \\ \left(h < P'(T_1) and\ \mu < P(T_1)\right) \end{bmatrix}$$

$$\frac{\partial^2 TP}{\partial S_1^{\ 2}} < 0 \text{ if } \begin{bmatrix} \begin{pmatrix} -\lambda\left(BR'(P',S_1) + P_p'(S_1) - R'(P',S_1)\right) \\ +P(S_1)P_p(S_1)e^{-i''S_1}(-i'') + P(S_1)P_p'(S_1)e^{-i''S_1} \\ +P'(S_1)P_p(S_1)e^{-i''S_1} \end{pmatrix} < 0, \\ \left(B\left(R(P',S_1)\right) + P_p(S_1) - R(P',S_1))\right) > 0\ and \\ \left(\mathrm{R}(\mathrm{P'},\mathrm{S}_1)e^{-i''S_1}(-i'') + \mathrm{R'}(\mathrm{P'},\mathrm{S}_1)e^{-i''S_1}\right) > 0 \end{bmatrix}$$

Since the second order sufficiency conditions for maximisation are also obtained the optimum solution obtained here becomes the unique optimal solution.

The production lot-size can be obtained by integrating the production rates over $\left[S_1, T_1\right]$:

$$Q = \int_{S_1}^{T_1} P_p(t)dt \tag{35}$$

Summarizing the above results, we can now establish the following solution procedure to obtain the optimal solution of our problem.

3.2 Solution procedure for model A

Step 1: Fix P'.

Step 2: Initially take $S_0 = 0$.

Step 3: Use equation (25) to find T as function of S_1(Use MATLAB).

Step 4: Use equation (26) to find T_1 as function of S and S_1.

Step 5: Use equation (28) to find λ as a function of S and S_1.

Step 6: Use equation (27) to find S as a function of S_1.

Step 7: Use equation (29) to find μ as a function of T_1.

Step 8: Use equation (30) to find S_1.

Step 9: Use S_1 value in equation (27) to find S.

Step10: Use S_1 value in equation (25) to find T.

Step 11: Use S, S_1 values in equation (26) to find T_1.

Step 12: Use S and S_1 values in equation (28) to find λ.

Step 13: Use T_1 value in equation (29) to find μ.

Step 14: If S_1, T, T_1 and S values satisfies Equations (31) to (34) go to next step; otherwise go to step 1.

Step 15: Using (21), find P' value.

Step 16: If P' satisfies (22), go to next step; otherwise go to step 1.

Step 17: If the difference between the two P' values are sufficiently small stop the procedure; otherwise go to step 2 and repeat the steps 2 to 16 till P' converges. If yes go to step 18.

Step 18: Enter this P' as optimal one and repeat step 2 to step to find S_1, T, $T_1$and S values for the P' value given in step 14.

Step 19: Using (18), find TP value.

Step 20: Using (35), find Q value.

Step 21: For the second cycle, take $S_0 = S$ and continue the above procedure till $S = H$.

With the help of above given prescriptions one can easily solve the given model A.

3.3 Model B: Coordinated Policy

In contrast to the sequential process, the coordinated policy makes the pricing and production decisions at a time [12].

The multivariate maximum problem under the coordinated policy can be formulated as follows:

Maximize TP Subject to the constraints (9), (10), $0 < P' < P'_{\max}$ and

$$S_o < S_1 < T < T_1 < S. \tag{36}$$

Under coordinated policy, production lot-size and selling price are determined simultaneously by solving the first order differential equations of model A.

$$\frac{dTP}{dP'} = \begin{bmatrix} \int_{S_0}^{S_1}\left(R(P',t)+P'\frac{dR(P',t)}{dP'}\right)Be^{-i''t}dt+\int_{S_1}^{S}\left(R(P',t)+P'\frac{dR(P',t)}{dP'}\right)e^{-i''t}dt- \\ Ke^{-i''S_1}-\int_{T}^{T_1}he^{-i''t}\left[e^{t-\int\theta(t)dt}\int_{T}^{t}-e^{-\int\theta(u)du}\frac{dR(P',u)}{dP'}du\right]dt \\ -\int_{T_1}^{S}he^{-i''t}\left[-e^{-\int\theta(u)dt}\int_{t}^{S}\frac{dR(P',u)}{dP'}e^{+\int\theta(u)du}du\right]dt-s\int_{S_0}^{S_1}e^{-i''t}\left[\int_{S_0}^{t}B\frac{dR(P',u)}{dP'}du\right]dt \\ -s\int_{S_1}^{T}e^{-i''t}\left[\int_{t}^{T}\frac{dR(P',u)}{dP'}du\right]dt-\pi\left[\int_{S_0}^{S_1}e^{-i''t}(1-B)\frac{dR(P',t)}{dP'}dt\right] \end{bmatrix} \tag{37}$$

$$\frac{d^2TP}{dP'^2} =$$

$$\begin{pmatrix} \int_{S_0}^{S_1}\left(2\frac{dR(P',t)}{dP'}+P'\frac{d^2R(P',t)}{dP'^2}\right)Be^{-i''t}dt+\int_{S_1}^{S}\left(2\frac{dR(P',t)}{dP'}+P'\frac{d^2R(P',t)}{dP'^2}\right)e^{-i''t}dt-Ke^{-i''S_1} \\ -\int_{T}^{T_1}he^{-i''t}\left[e^{-\int\theta(t)dt}\int_{T}^{t}-e^{-\int\theta(u)du}\frac{d^2R(P',u)}{dP'^2}du\right]dt-\int_{T_1}^{S}he^{-i''t}\left[-e^{-\int\theta(u)dt}\int_{t}^{S}\frac{d^2R(P',u)}{dP'^2}e^{+\int\theta(u)du}du\right]dt \\ -s\int_{S_1}^{T}e^{-i''t}\left[\int_{t}^{T}\frac{d^2R(P',u)}{dP'^2}du\right]dt-s\int_{S_0}^{S_1}e^{-i''t}\left[\int_{S_0}^{t}B\frac{d^2R(P',u)}{dP'^2}du\right]dt-\pi\left[\int_{S_0}^{S_1}e^{-i''t}(1-B)\frac{d^2R(P',t)}{dP'^2}dt\right] \end{pmatrix}$$

Since $\left(2\frac{dR(P',t)}{dP'}+P'\frac{d^2R(P',t)}{dP'^2}\right) < 0$ and $\frac{d^2R(P',t)}{dP'^2} > 0$, we have $\frac{d^2TP}{dP'^2}<0$.

So it satisfies the second order sufficiency condition $\frac{d^2TP}{dP'^2}<0$ (38)

Summarizing the above results, we can now establish the following solution procedure to obtain the optimal solution to our problem.

3.4 Solution procedure for model B

Step 1: Fix P'.

Step 2: Initially take S_0=0.

Step 3: Use equation (25) to find T as function of S_1 (Use MATLAB).

Step 4: Use equation (26) to find T_1 as function of S and S_1.

Step 5: Use equation (28) to find λ as a function of S and S_1.

Step 6: Use equation (27) to find S as a function of S_1.

Step 7: Use equation (29) to find μ as a function of T_1.

Step 8: Use equation (30) to find S_1.

Step 9: Use S_1 value in equation (27) to find S.

Step 10: Use S_1 value in equation (25) to find T.

Step 11: Use S, S_1 values in equation (26) to find T_1.

Step 12: Use S and S_1 values in equation (28) to find λ.

Step 13: Use T_1 value in equation (29) to find μ.

Step 14: If S_1, T, T_1 and S values satisfies Equations (31) to (34) go to next step; otherwise go to step 1.

Step 15: Using (37), find P' value.

Step 16: If P' satisfies (38), go to next step; otherwise go to step 1.

Step 17: If the difference between the two P' values are sufficiently small stop the procedure; otherwise go to step 2 and repeat the steps 2 to 16 till P' converges. If yes go to step 18.

Step 18: Enter this P' as optimal one and repeat step 2 to step to find S_1, T, $T_1$and S values for the P' value given in step 14.

Step 19: Using (18), find TP value.

Step 20: Using (35), find Q value.

Step 21: For the second cycle, take $S_0 = S$ and continue the above procedure till $S = H$.

With the help of above given prescriptions one can easily solve the given model B.

3.5 The finite time horizon model

The model developed in the pervious section can also be considered over a finite planning horizon H by suitably adjusting the production cycles. The following steps are followed:

1. Continue to find the optimal replenishment policies of the successive cycles until $\sum_{i=1}^{n} CL_i \le H \le \sum_{i=1}^{n+1} CL_i$ where CL_i denotes the i^{th} cycle-length, i =1, 2…n+1.

2. For n production-cycles, increase each cycle proportionally to finish the cycle exactly at H. Then modify the values of the decision variables accordingly and evaluate the total cost TC (n) by summing up the cost over n cycles.

3. Similarly for (n +1) cycles, decrease each cycle proportionally to finish the (n +1) cycle at H and evaluate TC (n+1).

4. Finally, determine $TC_{near\ opt}$ = min {TC (n), TC (n+1)} [6].

4. Numerical Example

In this section, numerical examples are proposed to illustrate the proposed model and its solution procedure. Optimal replenishment policies for the decentralized and coordinated policies are shown in Table 1 and Table 2 respectively. Sensitivity analyses are also reported for the model mentioned above and are shown in Table 3. Here we consider the same example of [12] to see the optimal replenishment policy while considering partial backlogging rate under inflation and time discounting.

Example 1

Let us consider H = 5, K = 4, h = .3, s = .1, B = .05, i'' = .12, π = .01, H = 5,

$\theta(v)$ = .1v, $P(t)$ = 10e$^{-.02t}$, $P_p(t)$= 150e$^{-.02t}$, $R(p',t)$= $(100-5P')$e$^{-.02t}$ in appropriate units. The optimum values for the decentralized policy and coordinated policy are shown in Table 1 and Table 2.

Table 1 Optimum values for the decentralized policy

S_0	S_1	T	T_1	S	P′	TP
0.000	0.0200	0.0202	0.4240	2.1026	14.5	109.8405
2.1026	2.1226	2.1228	2.3752	3.3242	14.4	58.7820
3.3242	3.3624	3.3626	3.5764	4.3543	14.4	43.4194
4.3543	4.3751	4.3745	4.5674	5.2535	14.4	34.8126

Table 2 Optimum values for the coordinated policy

S_0	S_1	T	T_1	S	P′	TP
0.0000	0.0200	0.0201	0.3871	2.5945	16.1	117.0486
2.5945	2.6145	2.6147	2.8288	3.9118	15.8	57.6949
3.9118	3.9318	4.1155	3.5764	4.9909	15.7	41.8980

From the numerical results obtained by us, which are shown in Table 1 and Table 2, we conclude that the selling price and the profit of the product using model B are better than that of model A.

In order to find out how various parameters affect the profit of our model we are going to study the sensitivity analysis. Sensitivity analyses on various parameters for model A and model B are shown in Table 3**.**

Table 3 Sensitivity analyses on various parameters

Table 3.1 Effect of *B* on TP

B	Case1 TP	Case2TP
0.05	109.8405	117.0486
0.09	112.1558	120.9544
0.50	94.7561	113.7430

Table 3.2 Effect of π on TP

π	Case1TP	Case2TB
0.001	109.9454	117.1798
0.010	109.8405	117.0486
0.120	108.4700	113.9620

Table 3.3 Effect of i″ on TP

i″	Case1TP	Case2TP
0.11	113.7489	120.6577
0.12	109.8405	117.0486
0.22	79.4760	87.1963

Table 3.4 Effect of *s* on TP

s	Case1TP	Case2TP
0.01	111.4780	119.9926
0.10	109.8405	117.0486
0.20	107.4087	112.9808

Table 3.5 Effect of *h* on TP

h	Case1TP	Case2TP
0.90	109.8405	117.0486
2	98.0489	111.7579
4	72.0444	85.6157

Table 3.6 Effect of *K* on TP

K	Case1TP	Case2TP
2	111.7323	118.8586
4	109.8405	117.0486
6	107.9476	115.2373

Table 3.7 Effect of *a* on TP

a	Case1 TP	Case2TP
100	123.6275	126.4560
150	109.8405	117.0486
200	117.0486	114.2357

Table 3.8 Effect of *b* on TP

b	Case1 TP	Case2TP
9	156.8051	152.1266
10	109.8405	117.0486
11	69.0763	85.3767

4.1 Managerial Implications

Based on our numerical results, we obtain the following managerial phenomena:

- Manufacturers who use coordinated policy have more profit than those having decentralized policy.
- During shortages, the manufacturers who minimize their shortage cost have better results.
- In order to maximize the profit, the manufacturers could reduce the cost for holding the items.
- When the manufacturers backlog the items partially, they must reduce the opportunity cost, which gives them more benefit.
- Manufacturers who have high backlogging parameter value suffer more than that those having less backlogging parameter value.
- Increasing the unit cost of the items and the inflation rate minimizes the profit of the company.

4.2 Special cases

- By taking i'' = 0, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for deteriorating items with partial backlogging.
- If we take i'' = 0, and B =1, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for deteriorating items with complete backlogging.
- If we take B =1, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for deteriorating items with complete backlogging under inflation and time discounting.
- If we take $\theta(t)$ = constant, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for constant deteriorating items with partial backlogging under inflation and time discounting.
- If we take $R\left(p',t\right) = 100\,e^{-.02t}$ i.e. P' = 0, our model reduces to that of a lot-sizing policies of an EPQ model for deteriorating items with partial backlogging under inflation and time discounting.
- If we take $P_p\left(t\right)$ =a constant, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for deteriorating items with constant production and partial backlogging under inflation and time discounting.
- If we take $P\left(t\right)$= a constant, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for deteriorating items with constant purchasing cost and partial backlogging under inflation and time discounting.

Similarly it is possible for us to derive some more models from the prescribed model. In practical life after production, the value of certain items like wine increase with time. Such items are known as ameliorating items (See [13], [14], [15]).

- If we take $\theta(t)$ = -$\theta(t)$, our model reduces to that of an optimal pricing and lot-sizing model of an EPQ model for ameliorating items with partial backlogging.

For ameliorating model all the special cases discussed above (for deteriorating items) also holds true.

5. Conclusions

In this paper, we discussed the effects of partial backlogging on the production inventory problem of a product with time-varying demand under inflation and time discounting. Our model is suitable for any given time horizon in any product life cycle including hi-tech products. Inflation and time discounting are important factors in recent time. In keeping with this reality, these factors are incorporated in the discussed model. From the numerical values we concluded that coordinated policies are better than decentralized policy under inflation and time discounting. We provided some useful practical applications for the manufacturers. Some special cases of our model are also discussed which elaborate upon the importance of our model and its wider applications in manufacturing. Optimum production quantity can also be obtained from our model. Our model is also suitable for ameliorating items also. Finally, the sensitivity of the solution to changes in the values of different parameters has been discussed.

The proposed model can be extended in several ways. For instance, we may apply the deterministic demand function to stochastic fluctuating demand patterns. Finally, we could generalize the model to allow for quantity discounts, permissible delay in payments etc.

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