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On the coordinated search problem on the plane

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Abstract

Two unit-speed searchers at (0,0) seek a randomly located target on the plane according to a known unsymmetric continous distribution. The objective is to minimize the expected time for the searchers to return to (0,0) after one of them has found the target. We find a necessary condition which make the search strategy be optimal when the target has a bivariate Balakrishnan skew-normal distribution. The search strategy is derived using a dynamic programming algorithm. An example is given to show the applications of this technique. The problem has applications to parallel processing and to the optimal choice of drilling depths in the search for an underground mineral.

AMS Subject Classification: *60K30, 90B40.*

Keywords: *Coordinated Search, Optimal Search, Unsymmetric Distribution*

Düzlemde koordineli arama problemi

Özet

(0,0)'da iki birim hızlı arayıcı, bilinen simetrik olmayan bir sürekli dağılıma göre düzlemde rastgele yerleştirilmiş bir hedef aramaktadırlar. Amaç, arayıcılardan birinin hedefi bulmasının ardından, (0,0)'a dönmek için beklenen süreyi her ikisi için de minimize etmektir. Hedefin iki değişkenli Balakrishnan çarpık-normal (skew-normal) dağılımına uygun olması halinde arama stratejisinin optimal düzeye getiren önemli bir koşul bulunmuştur. Arama stratejisi dinamik programlama algoritması kullanılarak türetilmiştir. Bu tekniğin uygulamasını göstermek amacıyla bir örnek verilmiştir. Problemin yer altı madenlerinin aranmasında paralel hesaplama ve delme derinliğinin optimal seçiminde uygulamaları bulunmaktadır.

AMS Konu Sınıflandırması: *60K30, 90B40.*

Anahtar Sözcükler: *Koordineli Arama, Optimal Arama, Simetrik Olmayan Dağılım.*

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1. Introduction

Work on search theory began in the US Navy's Antisubmarine Warfare Operations Research Group in 1942 in response to the German submarine threat in the Atlantic. After several decades of development, search problems are still largely of the same form as in 1942: a single target is lost, the problem is to find it efficiently. So the searching for a lost target either located or moving is often a time-critical issue that is when the target is very important such as searching for a bomb in known region or a life raft on the ocean. And the prime focus is to search for and find the cast ways in the smallest possible amount of time.

In an earlier work, many variants and extensions of the above problem, in a wide variety of directions, have been presented in both statistical and operations research literature since Koopman solved this problem for the unidimensional case, see [1], under some specific hypotheses, see [2]. However, as pointed out by Koopman, there is so much complexity in real search and rescue missions that any statistical model can only reflect part of the real-life situation and ours is no exception, see [3, 4]. One of these model is the graphical methods in the classical stationary search theory. These methods are extended to a two-stage search of a submerged target, subject to an overall budgetary constraint and with possible budget transfer between the stages, see [5].

On search theory in general, Stone has been given a good account of various results presently available, with some informative examples, and also has been provided a rigorous mathematical treatment of the subject, for both discrete and continuous cases, see [6]. On the other hand, Stone has been provided an overview of different areas in the development of search theory, which could be designated as classical, mathematical, algorithmic and dynamic, see [7]. Also an exhaustive surveys of works realized on this topic has been given, see [8, 9].

Recently, this problem has been illustrated by using co-ordinated search technique on open area when the located target has symmetric distribution, see [10]. Also, some papers concerned with this problem when the target moves on the plane with random process, like missing boats, submarines and missing system by applying many search techniques such as Bayesian Search and Tracking (SAT). The Bayesian approach would formulate for a target whose prior distribution and probabilistic motion model are known and generalized the approach for coordinated multi-vehicle search, see [11, 12].

The primary concern of the paper thus lies in the coordinated search technique which allows two searchers S_1 and S_2 start together and looking for the target from the point $(0,0)$, which is the center of the known region. The region is divided by two roads and they are intersected in the center of this region as indicated in **Figure 1.** One of these roads is vertical $y-$axis and the other is horizontal $x-$axis.

Each searcher of them start looking for the target from $(0,0)$ using a continuous path. The target being sought for might be in either direction from $y-$axis, so the searcher has in general to retrace his steps many times before he attains the target. The position is given by the value of independent random variables X, Y which has known (or unknown) unsymmetric distribution W. Each searcher of them would change his direction at suitable points on $y-$axis before attaining his goal. At these points, the region will be divided into many sectors h_i and g_i, $i=1, 2, ...,$

in the right and the left part, respectively. Where the sectors h_i and g_i have tracks with width $a_i - a_{i-1}$ and $b_i - b_{i-1}$.

Figure 1 gives an illustration of such search paths. The search process is the continuous space and time. It is clear that, the two searchers go different distances on $y-$axis because the position of the target has unsymmetric distribution on the plane. Then, the two searchers must go different distance through $y-$axis and search the two parts as in the following:

The searcher S_1 would conduct his search in the right part of $y-$axis as in the following manner:

(i) Start at $(0,0)$ and go to the $-ve$ part of $y-$axis as far as a_1 to the point $(0,-a_1)$.

(ii) Search the sector h_1 and its track until he reaches to the point $(0, a_1)$ on $y-$axis. And, then he returns again to $(0,0)$ through the $+ve$ part of $y-$axis to tell the other searcher if he met the target or not.

If the target is not found there, go with a distance a_2 towards the $-ve$ part of $y-$axis to the point $(0,-a_2)$ and explore the sector h_2 and its track until he reaches to the point $(0, a_2)$ on $y-$axis. He returns again to $(0,0)$ through the $+ve$ part of $y-$axis to tell the other searcher if he met the target or not. And, if the tagret is still not found, retrace the steps again to the $-ve$ part of $y-$axis to explore the sector h_3 and its track, and so fourth until the position of the target be detected.

Also, the searcher S_2 would conduct his search in the left part of $y-$axis as in the following manner:

(a) Start at $(0,0)$ and go to the $+ve$ part of $y-$axis If the target is not found there, go with a distance b_2 towards the $+ve$ part of $y-$axis to the point $(0, b_2)$ and explore the sector g_2 and its track until he reaches to the point $(0,-b_2)$ on $y-$axis. He returns again to $(0,0)$ through the $-ve$ part of $y-$axis to tell the other searcher if he met the target or not. And, if the tagret is still not found, retrace the steps again to the $+ve$ part of $y-$axis to explore the sector g_3 and its track, and so fourth until the target be detected.

A problem is similar to the frequently encountered one of determining the location of the black box of an aircraft lost at sea, with, supposedly, no survivor.

as far as b_1 to the point $(0, b_1)$.

(b) Search the sector g_1 and its track until he reaches to the point $(0,-b_1)$ on $y-$axis. And, then he returns again to $(0,0)$ through the $-ve$ part of $y-$axis to tell the other searcher if he met the target or not.

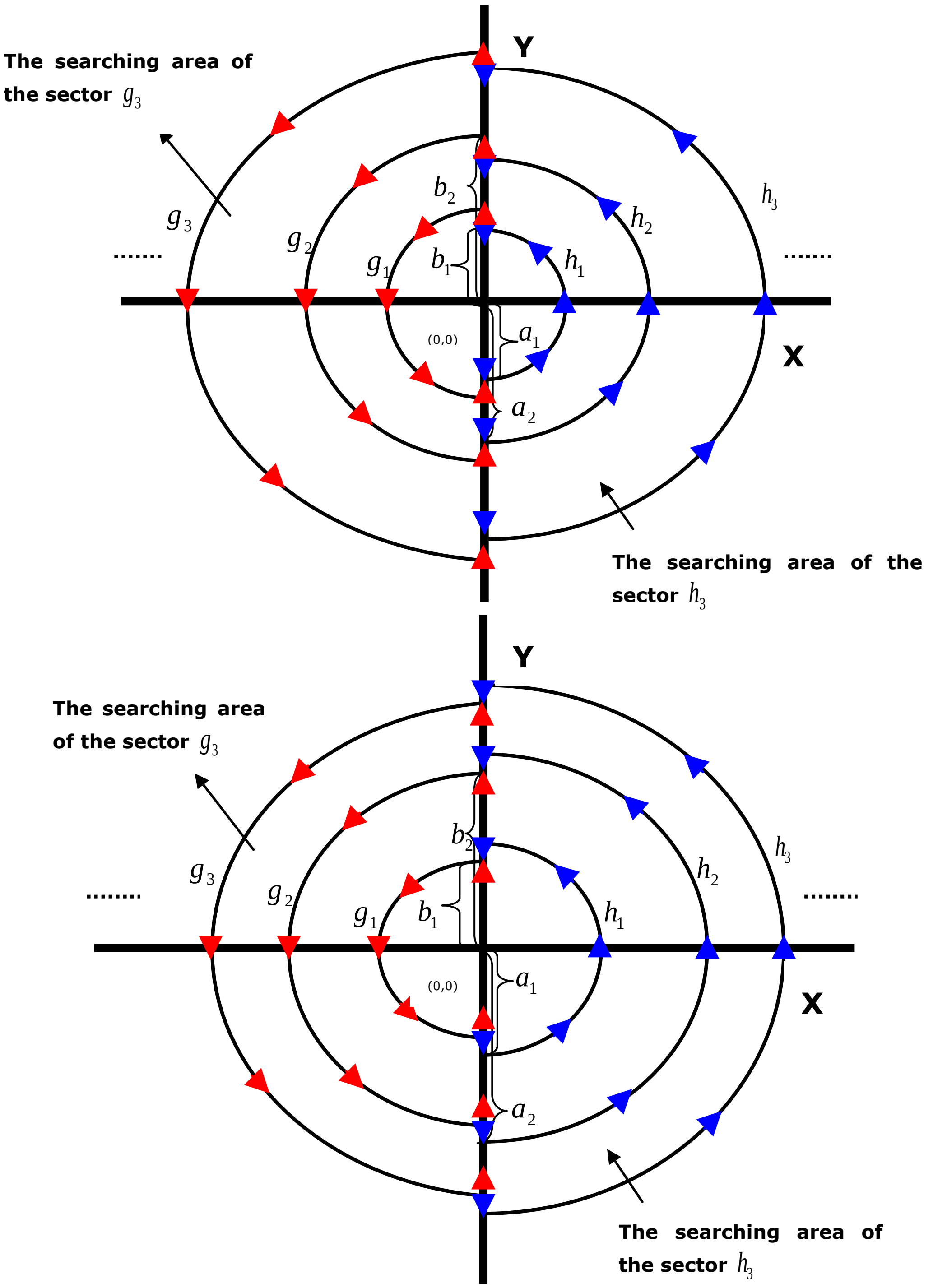

Figure 1 The Two Search Paths Which Will Give Us The Case Of Search When We Consider All Relative Positions Of The Starting Point (0,0) And The Located Target

Our aim is to calculate the expected value of the time for detecting the target; also we wish to find the necessary conditions which give the optimal search plan (O.S.P.) to detect it.

The rest of the paper is organized as follows. In Section 2, we disscus the problem and a recursive solution. In Section 3, we study the necessary conditions which make the search plan be optimal. Section 4 develops the dynamic programming algorithm for obtaining the minimum expected value of the time of detection. Section 5 gives simulation example with numerical resluts which can show the effectiveness of this technique and demonstrates the applicablity of it to real world search scenarios. Finally, section 6 concludes the paper.

2. The Problem and a Recursive Solution

Here, assumptions of the coordinate search problem with two searchers S_1 and S_2 for a located target are described and the problem is mathematically formulated as an allocation of searching effort which is the expected value of the first meeting time to detect the target. The surface of the plane be a "*Standard Euclidean* $2-$*space* E ", with points designated by ordered pairs (x, y).

The searchers S_1 and S_2 follow search paths e and f respectively to detect the target. The first search path e_1 of S_1 is defined as in the above steps (i) and (ii). Also the second search path e_2 of S_1 is defined as in the above steps (i) and (ii), but after the searcher S_1 goes a distance a_2 and searches the sector h_2 and its track and so on. Then, the search path e of S_1 is completely defined by a sequence $\{e_i,\ i \geq 0\}$.

The first search path f_1 of S_2 is defined as in the above steps (a) and (b). And the second search path f_2 of S_2 is defined as in the above steps (a) and (b), but after the searcher S_2 goes a distance b_2 and searches the sector g_2 and its track and so on. Then the search path f of S_2 is completely defined by a sequence $\{f_i,\ i \geq 0\}$.

Let (X, Y) be two independent random variables which they are represent the position of the target on the plane. Any track i has width $a_i - a_{i-1}$ and $b_i - b_{i-1}$ in the right and the left part such that the searchers S_1 and S_2 cover tracks with width $a_i - a_{i-1}$ and $b_i - b_{i-1}$. By virtue of the randomness of the position of the target, it is clear that the cost of the search is, also, a random variable.

The searchers go different distances on $y-$axis, then we have unequal two sectors h_i and g_i, $i = 1, 2, ...,$ as in **Figure 2**. These sectors made two unequal searching area (tracks of the sectors h_i, g_i with width $a_i - a_{i-1}$ and $b_i - b_{i-1}$ respectively) in the two parts. Let the target has unsymmetric distribution. We consider, the searchers go on $y-$axis with equal speeds $(v_1 = v_2 = 1)$ and they search the sectors and its tracks with "*regular speed*" β, where the searching process done only on the sectors and its tracks. The time which the searchers taked it through going on $y-$axis will added to the time of the searching process. The searchers wish to minimize the expected cost to detect the target, so that if any searcher of them can detect the target before the other, he will return to the origin in the shortest path to wait him and he will tell him that he finds the target.

Let the probability density function of the target position on the region is $w(x,y)$ and the distribution function is $W(x,y)$. Each part is divided into sectors as in **Figure 2** and each sector is also divided into an equal small sectors $l_k,\ k=1,\ 2,\ ...,\ n$. These small sectors make small search area of the track which the search done on it by which we mean for the moment that the searcher searches for every thing from his position, and nothing beyond that.

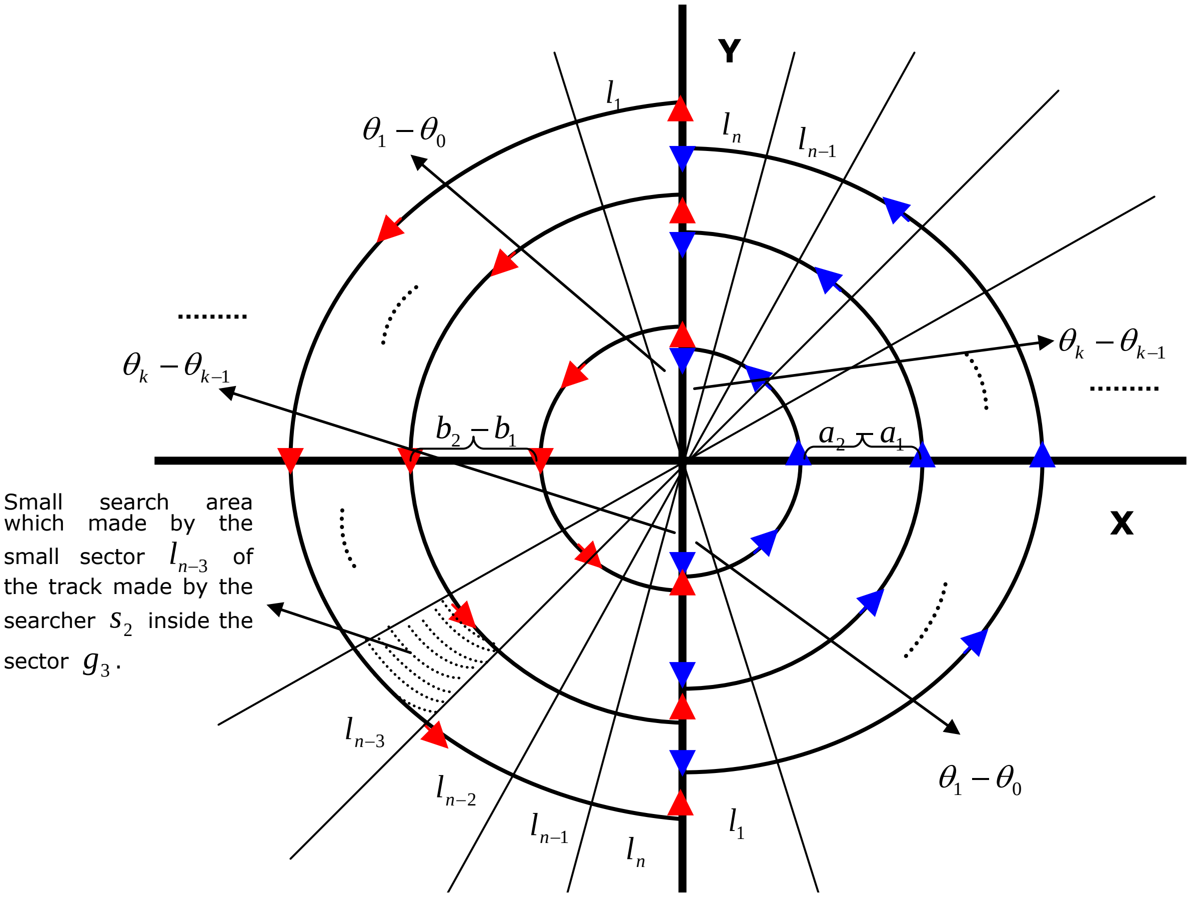

Figure 2 The Small Search Area Which Made By Small Sectors $l_k,\ k=1,\ 2,\ ...,\ n$ made by the searchers inside the sectors with radiuses a_i *and* $b_i,\ i=2,\ 3,\ 4,\ 5,\ ....$

Let $t_j,\ j=1,\ 2$ be the time which the searchers S_1 and S_2 take them in the search paths $\{e_i,\ i\geq 0\}$ and $\{f_i,\ i\geq 0\}$ in the right and the left parts respectively to $(0,0)$. They go on $y-$axis from the origin before searching the sectors.

And, they return after finishing on the sectors to the origin with equal speeds $(v_1 = v_2 = 1)$, then in this case the time of going through $y-$axis is equal to the distances which done. They searching on the sectors h_i, g_i, $i = 1, 2, ...,$ and its tracks (searching areas of the sectors) with "*regular speed*" β. Then, we consider, the searching time on the sectors and inside them is the "*time league*". Assuming that the time league on the right part is equal to $\tau_i = \frac{2\pi}{\omega_i}$ and in the left part is $\xi_i = \frac{2\pi}{\Gamma_i}$, where ω_i and Γ_i are called "*the angular velocity*". The searching time τ_i and ξ_i are depend on ω_i and Γ_i respectively, which they depend on the radiuses a_i and b_i. Let $t(\psi)$ be the time of detecting the target.

Theorem 1 *The expected value of the time for the searchers to return to the point (0,0) after one of them has detected unsymmetric distributed target is given by :*

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(2a_i + \frac{\pi}{\omega_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right) + \left(2b_i + \frac{\pi}{\Gamma_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right]. \quad (1)$$

Proof.

IF the target lies in any point of the track of g_1, then $t_1 = a_1 + \frac{1}{2}.\frac{2\pi}{\omega_1} + a_1 = 2a_1 + \frac{\pi}{\omega_1}$.

IF the target lies in any point of the track of g_2, then $t_1 = 2(a_1 + a_2) + \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)$.

IF the target lies in any point of the track of g_3, then $t_1 = 2(a_1 + a_2 + a_3) + \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3}\right)$, and so on.

IF the target lies in any point of the track of h_1, then $t_2 = b_1 + \frac{1}{2}.\frac{2\pi}{\Gamma_1} + b_1 = 2b_1 + \frac{\pi}{\Gamma_1}$.

IF the target lies in any point of the track of h_2, then $t_2 = 2(b_1 + b_2) + \pi\left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right)$.

IF the target lies in any point of the track of h_3, then $t_2 = 2(b_1 + b_2 + b_3) + \pi\left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3}\right)$, and so on.

However, each sector is divided into an equal small sectors $l_k,\ k=1,\ 2,\ ...,\ n,$ where these sectors make a set of an equal cones have the same vertex (0,0) as in **Figure 2**. But the searchers S_1 and S_2 can cover a tracks with width $a_i - a_{i-1}$ and $b_i - b_{i-1},$ so that each one can cover an equal small areas from cones in the track number *i*. The cones is determined by a set of lines with equations $x = m_k y = \tan\theta\ y,$ where $\theta = \theta_k - \theta_{k-1},\ k=1,\ 2,\ ...,\ n,$ where this set of equations make a set of an equal small areas, by which we mean for the moment that the searcher searches for every thing from his position, and nothing beyond that. So that, to evaluate the expected value of the time for the searchers to detect the target, we use the polar coordinates with $x = r\cos\theta$ and $y = r\sin\theta,$ $r: a_{i-1} \to a_i,\ i=1,\ 2,\ ...,$ in the right part and $r: b_{i-1} \to b_i,\ i=1,\ 2,\ ...,$ in the left part, $\theta: \theta_{k-1} \to \theta_k,\ k=1,\ 2,\ ...,\ n,$ where $a_0 = r_0 = 0,\ \theta_0 = 0.$ The searchers search the sectors and its tracks in anti clockwise. Hence

$$
\begin{aligned}
E(t(\psi)) = &\left(2a_1 + \frac{\pi}{\omega_1}\right).\left[\int_0^{b_1}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_0^{b_1}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ \left(2(a_1 + a_2) + \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\right).\left[\int_{b_1}^{b_2}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_{b_1}^{b_2}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ \left(2(a_1 + a_2 + a_3) + \pi\left(\frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3}\right)\right).\left[\int_{b_2}^{b_3}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_{b_2}^{b_3}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ ... \\
&+ \left(2b_1 + \frac{\pi}{\Gamma_1}\right).\left[\int_0^{a_1}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_0^{a_1}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ \left(2(b_1 + b_2) + \pi\left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right)\right).\left[\int_{a_1}^{a_2}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_{a_1}^{a_2}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ \left(2(b_1 + b_2 + b_3) + \pi\left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3}\right)\right).\left[\int_{a_2}^{a_3}\int_0^{\theta_1} g(r,\theta) r dr d\theta + ... + \int_{a_2}^{a_3}\int_{\theta_{n-1}}^{\theta_n} g(r,\theta) r dr d\theta\right] \\
&+ ...
\end{aligned}
$$

and so on, then

$$
E(t(\psi)) = \left(2a_1 + \frac{\pi}{\omega_1}\right).\left[\sum_{k=1}^{n}\int_0^{b_1}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta) r dr d\theta\right]
$$

$$+\left(2(a_1+a_2)+\pi\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\right).\left[\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right]$$

$$+\left(2(a_1+a_2+a_3)+\pi\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)\right).\left[\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right]$$

$$+\ldots$$

$$+\left(2b_1+\frac{\pi}{\Gamma_1}\right).\left[\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right]$$

$$+\left(2(b_1+b_2)+\pi\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}\right)\right).\left[\sum_{k=1}^{n}\int_{a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right]$$

$$+\left(2(b_1+b_2+b_3)+\pi\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}+\frac{1}{\Gamma_3}\right)\right).\left[\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right]$$

$$+\ldots$$

$$=\left(2a_1+\frac{\pi}{\omega_1}\right).\left[\sum_{k=1}^{n}\int_{0}^{b_1}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\left(2a_2+\frac{\pi}{\omega_2}\right).\left[\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_3}^{b_4}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\left(2a_3+\frac{\pi}{\omega_3}\right).\left[\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_3}^{b_4}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{b_4}^{b_5}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\ldots$$

$$+\left(2b_1+\frac{\pi}{\Gamma_1}\right).\left[\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\left(2b_2+\frac{\pi}{\Gamma_2}\right).\left[\sum_{k=1}^{n}\int_{a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\left(2b_3+\frac{\pi}{\Gamma_3}\right).\left[\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\sum_{k=1}^{n}\int_{a_4}^{a_5}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta+\ldots\right]$$

$$+\ldots$$

$$= \sum_{i=1}^{\infty}\left[\left(2a_i + \frac{\pi}{\omega_i}\right)\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right) + \left(2b_i + \frac{\pi}{\Gamma_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right]$$

Corollary 1: In the case of $a_i = b_i,\ i = 1,\ 2,\ ...,$ (i.e. the target has symmetric distribution), the expected value of the time for the searchers to return to the point (0,0) after one of them has detected it is given by:

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(4a_i + \frac{2\pi}{\omega_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right].$$

2.1. Special Cases

Case I. If the width in the right part is fixed (i.e. $a_i - a_{i-1} = a$), then $a_1 = a,\ a_2 = 2a,\ a_3 = 3a,\ ...,$ in (1) we get:

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(2ia + \frac{\pi}{\omega_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right.$$
$$\left. + \left(2b_i + \frac{\pi}{\Gamma_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{(s-1)a}^{sa}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right].$$

Case II. If the width in the left part is fixed (i.e. and $b_i - b_{i-1} = b$), then $b_1 = b,\ b_2 = 2b,\ b_3 = 3b,\ ...,$ in (1) we get:

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(2a_i + \frac{\pi}{\omega_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{(s-1)b}^{sb}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right.$$
$$\left. + \left(2ib + \frac{\pi}{\Gamma_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right].$$

Case III. If the width in the two parts are fixed (i.e. $a_i - a_{i-1} = a$ and $b_i - b_{i-1} = b$), then $a_1 = a,\ a_2 = 2a,\ a_3 = 3a,\ ...,$ and $b_1 = b,\ b_2 = 2b,\ b_3 = 3b,\ ...,$ in (1) we get:

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(2ia + \frac{\pi}{\omega_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{(s-1)b}^{sb}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right.$$
$$\left. + \left(2ib + \frac{\pi}{\Gamma_i}\right).\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{(s-1)a}^{sa}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right].$$

3. Optimal Search Plan

The goal of the searching strategy could be minimize the expected time to detect the target.

Definition 1 *Let* $\psi^* \in \Psi$ *be a search plan, then* ψ^* *is an optimal search plan, if* $E(t(\psi^*)) = \inf \{E(t(\psi)),\ \psi \in \Psi\}$.

ψ^* will be an optimal search path if the sequences $a = \{a_i,\ i \geq 0\}$ and $b = \{b_i,\ i \geq 0\}$ are optimal sequences, i.e. $a^* = \{a_i^*,\ i \geq 0\}$ and $b^* = \{b_i^*,\ i \geq 0\}$. The problem is to find which values of the turning points a_i and b_i are optimal for a given target distribution function. There is obviously some similarity between this problem and the well known Linear Search Problem which had been studied before, see [13, 14, 15, 16, 17]. In that problem a single searcher, starting at zero and moving with speed one, aims to minimize the expected value of some function of the time taken to find an object hidden according to a known distribution on the line. An optimizing searcher goes to successively increasing distances in alternating directions until the object had found. In our problem, the searchers wish to find the optimal search paths to search the sectors and its tracks.

The search path ψ will be an optimal search path if the sequences $a = \{a_i,\ i \geq 0\}$ and $b = \{b_i,\ i \geq 0\}$ are optimal sequences, so we can assumed the certain conditions (necessary) on underlying distribution under which, there exists a search path ψ^* from class Ψ such that $E(t(\psi^*)) = \inf \{E(t(\psi)),\ \psi \in \Psi\}$.

As it can be seen that the search path depends on two unknown factors. Those are the target distribution W and the search path ψ which depend on $a = \{a_i,\ i \geq 0\}$ and $b = \{b_i,\ i \geq 0\}$ used by the searchers in the right and the left parts, respectively. Let us assume, from now on, that the target distribution is known. Nevertheless we still facing a difficult optimization problem. Because this problem has an infinite number of variables; that is $a = \{a_i,\ i \geq 0\},\ b = \{b_i,\ i \geq 0\}$ and ω_i's , Γ_i's which they are also depended on a_i's, b_i's.

The following recursions gives a necessary conditions for a strategy to be optimal with respect to bivariate Balakrishnan skew-normal distribution.

3.1. The Case of Position Given by a Bivariate Balakrishnan Skew-Normal Distribution

If we assume (from now on), that the target has a bivariate Balakrishnan skew-normal distribution with parameters $\lambda_1,\ \lambda_2$ and $\lambda_1 \neq \lambda_2$, see [18]. And, we consider, the surface of the region be a standard Eculidean 2-space E, with points designated by ordered pairs (x, y). This is a reasonable assumption for small areas about the target's reported position. In this coordinate system, the target's reported position is (0,0). A two-dimensional random variables (X,Y) have the bivariate Balakrishnan skew-normal distribution if it has the following density:

$$w_m(x,y;\lambda_1,\lambda_2,\rho)=c_m(\lambda_1,\lambda_2,\rho)\Phi^m(\lambda_1 x+\lambda_2 y)\phi(x,y,\rho), \quad \text{for} \quad (X,Y)\in E. \tag{2}$$

where $\phi(x,y,\rho)$ is the density of $N_2(0,0,1,1,\rho)$ and

$$c_m(\lambda_1,\lambda_2,\rho)=\frac{1}{E\left[\Phi^m(\lambda_1 x+\lambda_2 y)\right]}. \tag{3}$$

Whatever, $\lambda_1 x+\lambda_2 y \sim N(0,\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2\rho)$, we can calculate $E\left[\Phi^m(\lambda_1 x+\lambda_2 y)\right]$ by using orthant probability (Lemma 1 see [18]). But, if (X,Y) give the target's actual position then X is independent of Y. Then, $\rho=0$, where ρ is the correlation coefficient of X and Y. So that, (2) becomes

$$w_m(x,y;\lambda_1,\lambda_2)=c_m(\lambda_1,\lambda_2)\Phi^m(\lambda_1 x+\lambda_2 y)\phi(x,y), \quad \text{for} \quad (X,Y)\in E, \tag{4}$$

such that

$$c_m(\lambda_1,\lambda_2)=\frac{1}{b_m(\lambda_1,\lambda_2)}=\frac{1}{E\left[\Phi^m(\lambda_1 x+\lambda_2 y)\right]}=\frac{1}{\int_{-\infty}^{\infty}\int_{-\infty}^{-\infty}\Phi^m(\lambda_1 x+\lambda_2 y)\phi(x,y)dxdy}$$

$$=\frac{1}{E\left[\Phi^m\sqrt{\lambda_1^2+\lambda_2^2}U\right]},$$

where $U\sim N(0,1)$ and by using orthant probability, see [19], with $\rho=0$, one can find that: $b_1(\lambda_1,\lambda_2)=\frac{1}{2}$, $b_2(\lambda_1,\lambda_2)=\frac{1}{4}$, $b_3(\lambda_1,\lambda_2)=\frac{1}{8}$ and so on,

$$\phi(x,y)=\frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(x^2+y^2\right)\right];$$

and

$$\Phi^m(\lambda_1 x+\lambda_2 y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda_1 x+\lambda_2 y}\exp\left[-\frac{1}{2}\left(z^2\right)\right]dz, \text{ where } \lambda_1 x+\lambda_2 y\sim N(0,1),$$

then

$$\Phi^m(\lambda_1 x+\lambda_2 y)=\frac{1}{2}\left(1+erf\left(\frac{\lambda_1 x+\lambda_2 y}{\sqrt{2}}\right)\right).$$

Without loss of generality, let $m=1$ then $b_1(\lambda_1,\lambda_2)=\frac{1}{2}$. Thus (4) becomes :

$$w(x,y;\lambda_1,\lambda_2)=\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 x+\lambda_2 y}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}\left(x^2+y^2\right)\right]. \tag{5}$$

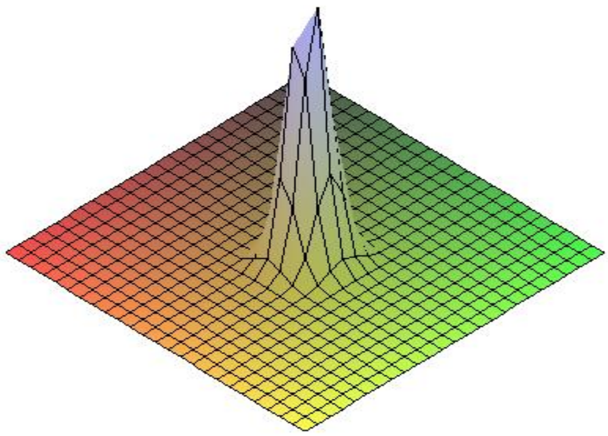

Figure 3 The Density Of The Bivariate Balakrishnan Skew-Normal Distribution For $\lambda_1 = 1,\ \lambda_2 = 2.$

Then the structure of the search path becomes easy and even simple as we shall see below.

Definition 2 If ψ is a search path from class Ψ such that the derivative of $E(t(\psi))$ with respect to a and b does exist and all partial derivatives of $E(t(\psi))$ with respect to the a_i's and b_i's are vanish, then ψ is said to be a critical search path (C.S.P) from class Ψ.

Remark 1 We infer that if $E(t(\psi))$ is differentiable on Ψ then the set of critical search paths from Ψ will contain all of the relative minimal and relative maximal search paths. Of course this set may also contain search paths at which Ψ does not have relative minimal or maximal search paths. In addition the function Ψ may have relative extremum at a search path from Ψ at which the derivative of $E(t(\psi))$ with respect to a and b does not exist or $E(t(\psi))$ may have a relative extremum at a search path which is not an interior point from Ψ.

If ψ is a (C.S.P) from class Ψ, then $\frac{\partial E(t(\psi))}{\partial a_i}$ and $\frac{\partial E(t(\psi))}{\partial b_i}$ are exist for all pertinent values of i, and then

$$\frac{\partial E(t(\psi))}{\partial a_i} = \frac{\partial E(t(\psi))}{\partial b_i} = 0, \qquad i \geq 0 \tag{6}$$

Theorem 2 Let (X, Y) be two independent random variables have a bivariate Balakrishnan skew-normal distribution with joint density function $w(x, y; \lambda_1, \lambda_2)$ as in (5), then a_i's and b_i's of a (C.S.P) $\psi \in \Psi$ are given by the following relations with $(a_0 = b_0 = 0)$:

$$b_i = \left[12\pi^{3/2} \left[P - \sum_{k=1}^{n} \int_{0}^{b_{i-2}} \int_{\theta_{k-1}}^{\theta_k} \frac{1}{2\pi} \left(1 + erf\left(\frac{\lambda_1 r \cos\theta + \lambda_2 r \sin\theta}{\sqrt{2}} \right) \right) . \exp\left[-\frac{1}{2} r^2 \right] r dr d\theta \right] \right] \Big/$$
$$\Bigg[\left(a_{i-1} \exp\left(-\frac{1}{2} a_{i-1}^2 \right) \right) \times \left(\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_{i-1} - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k)a_{i-1} \right.$$
$$- \lambda_2^2 \lambda_1 \sqrt{2}\sin(\theta_k)a_{i-1}^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_{i-1}^3\cos^2(\theta_k)$$
$$- \lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2 a_{i-1}^3 - 6\lambda_2\sqrt{2}\cos(\theta_k)a_{i-1}$$
$$+ \lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2 a_{i-1}^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_{i-1}$$
$$\left. + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_{i-1}^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_{i-1}^3\cos^2(\theta_{k-1}) \right) \Bigg], \quad i > 1. \tag{7}$$

and

$$a_i = \left[12\pi^{3/2} \left[P - \sum_{k=1}^{n} \int_{0}^{a_{i-2}} \int_{\theta_{k-1}}^{\theta_k} \frac{1}{2\pi} \left(1 + erf\left(\frac{\lambda_1 r \cos\theta + \lambda_2 r \sin\theta}{\sqrt{2}} \right) \right) . \exp\left[-\frac{1}{2} r^2 \right] r dr d\theta \right] \right] \Big/$$
$$\Bigg[\left(b_{i-1} \exp\left(-\frac{1}{2} b_{i-1}^2 \right) \right) \times \left(\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1})b_{i-1} - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k)b_{i-1} \right.$$
$$- \lambda_2^2 \lambda_1 \sqrt{2}\sin(\theta_k)b_{i-1}^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)b_{i-1}^3\cos^2(\theta_k)$$
$$- \lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2 b_{i-1}^3 - 6\lambda_2\sqrt{2}\cos(\theta_k)b_{i-1}$$
$$+ \lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2 b_{i-1}^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1})b_{i-1}$$
$$\left. + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})b_{i-1}^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})b_{i-1}^3\cos^2(\theta_{k-1}) \right) \Bigg], \quad i > 1. \tag{8}$$

Proof: From (1) we get:

$$E(t(\psi)) = \sum_{i=1}^{\infty} \left[\left(2a_i + \frac{\pi}{\omega_i} \right) . \left(\sum_{s=i}^{\infty} \sum_{k=1}^{n} \int_{b_{s-1}}^{b_s} \int_{\theta_{k-1}}^{\theta_k} g(r,\theta) r dr d\theta \right) + \left(2b_i + \frac{\pi}{\Gamma_i} \right) . \left(\sum_{s=i}^{\infty} \sum_{k=1}^{n} \int_{a_{s-1}}^{a_s} \int_{\theta_{k-1}}^{\theta_k} g(r,\theta) r dr d\theta \right) \right].$$

Since $v_i = \Gamma_i r_i$ and speed v_i was "*regular speed*" on any sector so if we take $v_i = \beta = const.$ we will obtain the "*angular velocity*" in any sector from $\Gamma_i = \frac{\beta}{r_i}$, $i = 1, 2, 3, \ldots$, also in the right part $\omega_i = \frac{\beta}{r_i}$, $i = 1, 2, 3, \ldots$, then

$$E(t(\psi)) = \left(2+\frac{\pi}{\beta}\right).(a_1).\left[\sum_{k=1}^{n}\int_{0}^{b_1}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$\left.+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\right]$$

$$+\left(2+\frac{\pi}{\beta}\right).(a_2).\left[\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$\left.+\sum_{k=1}^{n}\int_{b_3}^{b_4}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\right]$$

$+\ldots$

$$+\left(2+\frac{\pi}{\beta}\right).(b_1).\left[\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$\left.+\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\right]$$

$$+\left(2+\frac{\pi}{\beta}\right).(b_2).\left[\sum_{k=1}^{n}\int_{a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$\left.+\sum_{k=1}^{n}\int_{a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\right]$$

$+\ldots$

By differentiate with respect to a_1, then we get:

$$\frac{\partial E(t(\psi))}{\partial a_1}=\left(2+\frac{\pi}{\beta}\right).\Bigg[\sum_{k=1}^{n}\int_{0}^{b_1}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$+\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$

$$+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\Bigg]$$

$$+\left(2+\frac{\pi}{\beta}\right).(b_2).\left[-\frac{a_1\exp\left(-\frac{1}{2}a_1^2\right)}{12\pi^{3/2}}\right]\times$$

$$\Big[\sum_{k=1}^{n}6\sqrt{\pi}\theta_k+6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_1-6\sqrt{\pi}\theta_{k-1}+6\lambda_1\sqrt{2}\sin(\theta_k)a_1$$

$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3\cos^2(\theta_k)$$

$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_1^3-6\lambda_2\sqrt{2}\cos(\theta_k)a_1$$

$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_1^3-6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1$$

$$+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3\cos^2(\theta_{k-1})\Big]$$

$$=0,$$

since the target had unsymmetric distribution, then

$$\sum_{i=1}^{\infty}\sum_{k=1}^{n}\int_{b_{i-1}}^{b_i}\int_{\theta_{k-1}}^{\theta_k}g(r,\theta)rdrd\theta=1-\sum_{i=1}^{\infty}\sum_{k=1}^{n}\int_{a_{i-1}}^{a_i}\int_{\theta_{k-1}}^{\theta_k}g(r,\theta)rdrd\theta.$$

Thus, if we consider, the probability of detecting the target in the left part is $P,$ then the probability of detecting it in the right part is $1-P,$ which leads to

$$P=(b_2).\left[-\frac{a_1\exp\left(-\frac{1}{2}a_1^2\right)}{12\pi^{3/2}}\right]\times$$

$$\Big[\sum_{k=1}^{n}6\sqrt{\pi}\theta_k+6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_1-6\sqrt{\pi}\theta_{k-1}+6\lambda_1\sqrt{2}\sin(\theta_k)a_1$$

$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3\cos^2(\theta_k)$$

$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_1^3-6\lambda_2\sqrt{2}\cos(\theta_k)a_1$$

$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_1^3-6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1$$

$$+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3\cos^2(\theta_{k-1})\Big]$$

then

$$b_2 = \left[12\pi^{3/2}P\right] \Big/$$

$$\left[a_1 \exp\left(-\frac{1}{2}a_1^2\right)\right] \times \left[\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_1 - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k)a_1 \right.$$
$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_1^3\cos^2(\theta_k)$$
$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_1^3 - 6\lambda_2\sqrt{2}\cos(\theta_k)a_1$$
$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_1^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1$$
$$\left. +\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_1^3\cos^2(\theta_{k-1})\right]$$

By differentiate with respect to a_2, then we get:

$$\frac{\partial E(t(\psi))}{\partial a_2} = \left(2+\frac{\pi}{\beta}\right).\left[\sum_{k=1}^{n}\int_{b_1}^{b_2}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$
$$+\sum_{k=1}^{n}\int_{b_2}^{b_3}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta$$
$$\left.+\sum_{k=1}^{n}\int_{b_3}^{b_4}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta+\ldots\right]$$

$$+\left(2+\frac{\pi}{\beta}\right).(b_3).\left[-\frac{a_2\exp\left(-\frac{1}{2}a_2^2\right)}{12\pi^{3/2}}\right]\times$$

$$\left[\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_2 - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k)a_2 \right.$$
$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3\cos^2(\theta_k)$$
$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_2^3 - 6\lambda_2\sqrt{2}\cos(\theta_k)a_2$$
$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_2^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2$$
$$\left. +\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3\cos^2(\theta_{k-1})\right]$$

$$= 0,$$

thus

$$\left(2+\frac{\pi}{\beta}\right)\cdot\left[P-\sum_{k=1}^{n}\int_{0}^{b_1}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right)\cdot\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right.$$
$$+\left(2+\frac{\pi}{\beta}\right)\cdot(b_3)\cdot\left[-\frac{a_2\exp\left(-\frac{1}{2}a_2^2\right)}{12\pi^{3/2}}\right]\times$$
$$\left[\sum_{k=1}^{n}6\sqrt{\pi}\theta_k+6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_2-6\sqrt{\pi}\theta_{k-1}+6\lambda_1\sqrt{2}\sin(\theta_k)a_2\right.$$
$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3\cos^2(\theta_k)$$
$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_2^3-6\lambda_2\sqrt{2}\cos(\theta_k)a_2$$
$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_2^3-6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2$$
$$\left.+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3\cos^2(\theta_{k-1})\right]$$
$$=0,$$

which leads to

$$b_3=\left[12\pi^{3/2}\left[P-\sum_{k=1}^{n}\int_{0}^{b_1}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1+erf\left(\frac{\lambda_1 r\cos\theta+\lambda_2 r\sin\theta}{\sqrt{2}}\right)\right)\cdot\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right]\right]\Big/$$
$$\left[\left(a_2\exp\left(-\frac{1}{2}a_2^2\right)\right)\times\left(\sum_{k=1}^{n}6\sqrt{\pi}\theta_k+6\lambda_2\sqrt{2}\cos(\theta_{k-1})a_2-6\sqrt{\pi}\theta_{k-1}+6\lambda_1\sqrt{2}\sin(\theta_k)a_2\right.\right.$$
$$-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k)a_2^3\cos^2(\theta_k)$$
$$-\lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2a_2^3-6\lambda_2\sqrt{2}\cos(\theta_k)a_2$$
$$+\lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2a_2^3-6\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2$$
$$\left.\left.+\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3-\lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1})a_2^3\cos^2(\theta_{k-1})\right)\right],$$

Similarly,

$$b_4 = \left[12\pi^{3/2} \left[P - \sum_{k=1}^{n} \int_0^{b_2} \int_{\theta_{k-1}}^{\theta_k} \frac{1}{2\pi} \left(1 + erf\left(\frac{\lambda_1 r\cos\theta + \lambda_2 r\sin\theta}{\sqrt{2}} \right) \right) . \exp\left[-\frac{1}{2} r^2 \right] r dr d\theta \right] \right] \Bigg/$$

$$\left[\left(a_3 \exp\left(-\frac{1}{2} a_3^2 \right) \right) \times \left(\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1}) a_3 - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k) a_3 \right. \right.$$
$$- \lambda_2^2 \lambda_1 \sqrt{2}\sin(\theta_k) a_3^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k) a_3^3 \cos^2(\theta_k)$$
$$- \lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2 a_3^3 - 6\lambda_2\sqrt{2}\cos(\theta_k) a_3$$
$$+ \lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2 a_3^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_3$$
$$\left. \left. + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_3^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_3^3 \cos^2(\theta_{k-1}) \right) \right],$$

And so on we can get:

$$b_i = \left[12\pi^{3/2} \left[P - \sum_{k=1}^{n} \int_0^{b_{i-2}} \int_{\theta_{k-1}}^{\theta_k} \frac{1}{2\pi} \left(1 + erf\left(\frac{\lambda_1 r\cos\theta + \lambda_2 r\sin\theta}{\sqrt{2}} \right) \right) . \exp\left[-\frac{1}{2} r^2 \right] r dr d\theta \right] \right] \Bigg/$$

$$\left[\left(a_{i-1} \exp\left(-\frac{1}{2} a_{i-1}^2 \right) \right) \times \left(\sum_{k=1}^{n} 6\sqrt{\pi}\theta_k + 6\lambda_2\sqrt{2}\cos(\theta_{k-1}) a_{i-1} - 6\sqrt{\pi}\theta_{k-1} + 6\lambda_1\sqrt{2}\sin(\theta_k) a_{i-1} \right. \right.$$
$$- \lambda_2^2 \lambda_1 \sqrt{2}\sin(\theta_k) a_{i-1}^3 + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_k) a_{i-1}^3 \cos^2(\theta_k)$$
$$- \lambda_2\sqrt{2}\cos^3(\theta_{k-1})\lambda_1^2 a_{i-1}^3 - 6\lambda_2\sqrt{2}\cos(\theta_k) a_{i-1}$$
$$+ \lambda_2\sqrt{2}\cos^3(\theta_k)\lambda_1^2 a_{i-1}^3 - 6\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_{i-1}$$
$$\left. \left. + \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_{i-1}^3 - \lambda_2^2\lambda_1\sqrt{2}\sin(\theta_{k-1}) a_{i-1}^3 \cos^2(\theta_{k-1}) \right) \right],$$

By the same method, we can prove (8).

The searchers search inside the tracks with width $a_i - a_{i-1}$ and $b_i - b_{i-1}$ in the right and the left parts of $y-$axis respectively, then by choosing many values of $a_i,\ i = 1,\ 2,\ 3,\ \ldots,$ we can find $b_{i+1},\ i = 1,\ 2,\ 3,\ \ldots,$ where the above theorem is true for all values of $a_i,\ i = 2,\ 3,\ \ldots,$ and vice versa. So, we need to satisfy the codition $a_i \geq a_{i-1}$ and $b_i \geq b_{i-1}$ along the searching process and this is called the optimal case otherwise we stop the process and reject the values of a_i or $b_i,\ i = 1,\ 2,\ 3,\ \ldots.$

4. Algorithm

The solutions presented in the form of algorithm written in the style of computer program. We construct an algorithm to calculate the minimum expected value of the time of detection after choosing the optimal values of the radiuses a_i and b_i, $i = 1,\ 2,\ 3,\ \ldots.$. All these values obtained from generation method.

The steps of the algorithm can be summarized as in the following:

Step 1. Input the values of:

$P \equiv$ the probability of detecting the target in the left part,

$z \equiv +ve$ integer number (number of generation),

$q \equiv$ the counter which used in this process, and take the initial value 1,

$n \equiv$ the number of angles which done during the searchers search any sector and its track as in **Figure 2**,

$\lambda_1, \lambda_2 \equiv$ the parameters of *bivariate Balakrishnan skew-normal distribution* and $\lambda_1 \neq \lambda_2$,

$a_1 \equiv$ the first distance which the searcher S_1 go it through $-ve$ part of $y-$axis,

$b_1 \equiv$ the first distance which the searcher S_2 go it through $+ve$ part of $y-$axis;

$v_i \equiv$ "*regular speed*" on any sector.

Step 2. Compute $\theta = \theta_k - \theta_{k-1}$, where the searchers can cover an equal small areas from cones in the track number i;

Step 3. If $q \leq z$; Generate a_i and b_i, $i = 1, 2, 3, ...,$ original solutions randomly from (7) and (8);

Step 4. Test the condition $a_i \geq a_{i-1}$ and $b_i \geq b_{i-1}$, If it is satisfied, then go to **Step 5**, Else where stop the process and then go to **Step 7**;

Step 5. If the target is not detected, Put $q = q + 1$ and go to **Step 3**, Else where go to **Step 6**.

Step 6. Compute $E(t(\psi))$ from (1) and then go to **Step 8**;

Step 7. Reinput another values of a_1 and b_1 and repeat the above **Steps** from **3** to **6** again until the target is detected;

Step 8. End (Stop).

5. Numerical Example

To illustrate the operation of the algorithm. The following simple numerical example is considered for the sake of illustration, we have assumed an a priori knowledge of the target.

If the probability of the target in the left part is $P = 0.6$ then the probability of it in the other part is $1 - P = 0.4$. Then, we can take $a_1 < b_1$. Let $a_1 = 0.5$, $b_1 = 0.6$, $\lambda_1 = 27$, $\lambda_2 = 28$, $n = 20$ and let $\theta = \theta_k - \theta_{k-1} = \frac{\pi}{20}$, so that $\theta_k = k\vartheta = \frac{k\pi}{20}$, $\theta_{k-1} = (k-1)\vartheta = \frac{(k-1)\pi}{20}$, then we can get a_{i+1}, b_{i+1}, $i = 1, 2, 3, ...$ from (7) and (8) respectively, that minimize the expected value of the time to detect the target and satisfy the condition $a_i \geq a_{i-1}$ and $b_i \geq b_{i-1}$ along the searching process. We consider the positive values for calculating $a_{i+1},\ b_{i+1},\ i = 3,\ 4,\ 5,\ ...$ after substuting in (7) and (8) respectively as in the following: Then,

Step 1. Input the values of:

$$P = 0.6,\quad z = 50,\quad q = 1,\quad n = 20,\quad \lambda_1 = 27,\quad \lambda_2 = 28,$$
$$a_1 = 0.5,\quad b_1 = 0.6,\quad v_i = \beta;$$

Step 2. Such that $\theta_k = k\vartheta = \frac{k\pi}{20}$ and $\theta_{k-1} = (k-1)\vartheta = \frac{(k-1)\pi}{20}$ then $\theta = \theta_k - \theta_{k-1} = \frac{\pi}{20}$, where the searchers can cover an equal small areas from cones in the track number i;

Step 3. Put $q = 1$ and $q \leq 5$; Generate a_2 and b_2 original solutions randomly from (7) and (8), respectively as follows: from (8), we get: the solution is $\{[b_2 = 1.982170372]\}$, and for calculating a_2, we use (7) and we get: the solution is $\{[a_2 = 0.5855043261]\}$,

Step 4. The condition $a_2 \geq a_1$ and $b_2 \geq b_1$ is satisfied, then go to **Step 5**;

Step 5. Suppose the target is not detected, then Put $q = q + 1$ and go to **Step 6**;

Step 6. Such that $q = 2$ and $q \leq 5$; Generate a_3 and b_3 from (7) and (8), respectively as in **Step 3**, we get $a_3 = 0.6453403178$ and $b_3 = 17.39328954$;

Step 7. The condition $a_3 \geq a_2$ and $b_3 \geq b_2$ is satisfied, then go to **Step 8**;

Step 8. Consider the target is detected, then Compute $E(t(\psi))$ from (1) as follows:

$$E(t(\psi)) = \sum_{i=1}^{2}\left[\left(2a_i + \frac{\pi}{\omega_i}\right)\left(\sum_{s=i}^{2}\sum_{k=1}^{20}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1 + erf\left(\frac{27r\cos\theta + 28r\sin\theta}{\sqrt{2}}\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right)\right.$$
$$\left.+\left(2b_i + \frac{\pi}{\Gamma_i}\right)\left(\sum_{s=i}^{2}\sum_{k=1}^{20}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k}\frac{1}{2\pi}\left(1 + erf\left(\frac{27r\cos\theta + 28r\sin\theta}{\sqrt{2}}\right)\right)\right).\exp\left[-\frac{1}{2}r^2\right]rdrd\theta\right].$$

substuite with $v_i = \beta = const.$, we can obtain the "*angular velocity*" in any sector in the left part from $\Gamma_i = \frac{\beta}{b_i},\ i = 1,\ 2,$ also in the right part $\omega_i = \frac{\beta}{a_i},\ i = 1,\ 2.$

After calculate $E(t(\psi))$, go to **Step 10**;

If the condition $a_i \geq a_{i-1}$ and $b_i \geq b_{i-1}$, is not satisfied in **Step 4** or in **Step 7**, then the process must stop and go to **Step 9**;

Step 9. Reinput another values of a_1 and b_1 and repeat the above **Steps** from **3** to **6** again until the target is detected;

Step 10. End (Stop).

We can apply this algorithm for $i = 3, 4, 5, ...,$ to obtain $a_{i+1}, b_{i+1},$ such that the condition $a_{i+1} \geq a_i$ and $b_{i+1} \geq b_i,$ must be satisfied to make the process continues. If this condition is not satisfied before the target is found, then we must stop the process and take another values of a_1 and b_1.

6. Conclusion and Future Research

A coordinated search technique for a lost target has been presented. The necessary conditions has been given to make the search plan be optimal. Also, we have developed a dynamic programming algorithm that provides minimum expected value of the time.

The proposed model will be extendible to the multiple searcher case by considering the combinations of multiple lost targets on the plane.

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