



## Slash Maxwell Distribution: Definition, Modified Maximum Likelihood Estimation and Applications

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### Highlights

- A new distribution is studied.
- Parameters of the distribution are estimated using the MML methodology.
- Explicit forms of the MML estimators are provided.

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### Abstract

In this study slash Maxwell (SM) distribution, defined as a ratio of a Maxwell random variate to a power of an independent uniform random variate, is introduced. Its stochastic representation and some distributional properties such as moments, skewness and kurtosis measures are provided. The maximum likelihood (ML) method is used for estimating the unknown parameters. However, closed forms of the ML estimators cannot be obtained since the likelihood equations include nonlinear functions of the unknown parameters. We therefore use Tiku's (1967,1968) modified maximum likelihood (MML) methodology which allows to obtain explicit forms of the estimators. Some asymptotic properties of the MML estimators are derived. A Monte-Carlo simulation study is also carried out to compare the performances of the ML and MML estimators. Two data sets taken from the literature are modelled using the SM distribution in application part of the study.

## 1. INTRODUCTION

Slash distribution is defined as

$$Z = \frac{Y}{U^{1/q}}$$

where the random variables  $Y$  and  $U$  are independent and have the standard normal  $N(0,1)$  and uniform  $U(0,1)$  distributions, respectively. Here,  $q > 0$  is the shape parameter which controls the kurtosis of the distribution. It is clear from this representation that the slash distribution is an extension of normal distribution. Indeed, the slash distribution has heavier tails than the normal distribution, see for example Rogers and Tukey [1], Mosteller and Tukey [2].

First aim of this study is to introduce a new distribution obtained using slashing methodology. Indeed, most of the new distributions existing in the literature are obtained by extending well-known distributions [3-6]. Some of the new extended/generalized distributions are also obtained via slashing methodology using different baseline distributions. For example, Gomez et al. [7] propose slash-elliptical distributions which are obtained as an extension of the symmetric distributions. Olivares-Pacheco et al. [8] introduce slash-Weibull distribution. Olmos et al. [9,10] present slash half normal and slash generalized half normal distributions, respectively. Genc [11] obtains a skew extension of the slash distribution is using beta-normal distribution. Genc et al. [12] suggest beta Moyal-slash distribution. Korkmaz [13] proposes gamma-slash

distribution based on gamma-normal distribution. Gomez et al. [14] introduce a slash Gumbel (SG) distribution as a heavy tailed alternative to the Gumbel distribution.

In this study, we consider Maxwell distribution as baseline distribution and propose a slash Maxwell (SM) distribution. Therefore, we first give brief information about the Maxwell distribution. The Maxwell distribution is proposed to model speeds of molecules in thermal equilibrium. Its probability density function (pdf) is given as follows:

$$f_Y(y; \sigma) = \frac{4}{\sigma \Gamma\left(\frac{1}{2}\right)} \left(\frac{y}{\sigma}\right)^2 \exp\left\{-\left(\frac{y}{\sigma}\right)^2\right\}; \quad y > 0, \quad \sigma > 0 \quad (1)$$

where  $\sigma$  stands for the scale parameter and  $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$  is the gamma function. The Maxwell distribution was firstly used for modeling lifetime data by Tyagi and Battacharya [15, 16] in the context of statistics. It has also many applications in different areas of science. For example, Bayes estimators of the scale parameter of the Maxwell distribution under various different loss functions are considered by Dey and Maiti [17]. The maximum likelihood (ML) estimators of the location and the scale parameters of the mixture of the Maxwell distribution under Type I censoring are considered by Kazmi et al. [18]. Li [19] uses the Minimax, the Bayesian and the ML methods to estimate the scale parameter of the Maxwell distribution. The Bayesian method is used to estimate the loss and the risk function for the scale parameter by Fan [20]. Arslan et al. [21, 22] obtain the modified maximum likelihood (MML) estimators for the location and scale parameter of the Maxwell distribution.

Although there is a great interest on the Maxwell distribution in the literature, it may be inadequate for modelling data sets having higher kurtosis values. We therefore introduce the SM distribution which has heavier tails than the Maxwell distribution and thus it is flexible enough to accommodate the greater kurtosis values; see Acitas et al. [23]. The new distribution is obtained as the ratio of a Maxwell variate to a power of an independent uniform variate. The ML method is used to compute the estimates of distribution parameters. However, the ML estimates cannot be obtained explicitly since likelihood equations include nonlinear functions of the parameters. Thus, we use Tiku's [24, 25] MML methodology which allows to obtain the closed forms of the estimators. This is the second aim of the study. We therefore believe that current study has the following contributions to the related literature: (i) A new distribution called as SM is introduced as an alternative to the existing distributions for modelling data sets having skewness and/or excess kurtosis and (ii) Closed forms of the estimators for the location and scale parameters of the SM distribution are obtained using the MML methodology.

Independently, Iriarte et al. [26] also obtain the SM distribution in the context of slashed generalized Rayleigh (SGR) distribution. Essentially, the SGR reduces to the SM distribution for certain values of the parameters. It should be noted that in this study the SM distribution has location, scale and shape parameters. We focus on the MML estimation of the location and scale parameters and the shape parameter is estimated using profile likelihood method. On the other hand, Iriarte et al. [26] consider the SGR distribution including one scale and two shape parameters.

The rest of the paper is organized as follows. The SM distribution and its properties are presented in Section 2. Section 3 includes the MML method for estimating the unknown parameters of the SM distribution. Two data sets taken from the literature are analyzed in Section 4 and the paper is finalized with a conclusion section.

## 2. THE SM DISTRIBUTION: DENSITY FUNCTION AND PROPERTIES

In this section, the SM distribution is derived as a scale mixture extension of the Maxwell distribution. Some basic properties of the SM distribution are also presented. We consider two parameter SM distribution through the subsections 2.1 – 2.4 for the sake of brevity. Location-scale case of the SM distribution is given in subsection 2.5.

## 2.1. Stochastic Representation

The SM distribution is stochastically represented as follows:

$$Z = \frac{Y}{U^{1/q}}, \quad q > 0 \quad (2)$$

where  $Y$  and  $U$  are independent random variables having the Maxwell( $\sigma$ ) and  $U(0,1)$  distributions, respectively. Hereinafter, the random variable  $Z$  having the SM distribution will be denoted by  $Z \sim \text{SM}(\sigma, q)$ .

## 2.2. The pdf and Its Properties

The pdf of the SM distribution is given in Theorem 1.

**Theorem 1.** Let  $Z \sim \text{SM}(\sigma, q)$ , then the random variable  $Z$  has the following pdf

$$f_Z(z; \sigma, q) = \frac{2q}{\Gamma(1/2)\sigma} \Gamma\left(\frac{q+3}{2}\right) \left(\frac{z}{\sigma}\right)^{-(1+q)} G\left(z^2; \frac{q+3}{2}, \sigma^2\right), \quad z > 0, \quad \sigma > 0, \quad q > 0 \quad (3)$$

where  $\sigma$  and  $q$  are the scale and shape parameters, respectively, and  $G(\cdot)$  is the cumulative distribution function (cdf) of gamma distribution defined by

$$G(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^x t^{\alpha-1} \exp\left(-\frac{t}{\beta}\right) dt.$$

*Proof.* We use the stochastic representation given in Equation (2) and Jacobian transformation to complete the proof.

Consider the following transformations

$$\left. \begin{array}{l} Z = \frac{Y}{U^{1/q}} \\ W = U^{1/q} \end{array} \right\} \Rightarrow \left. \begin{array}{l} Y = ZW \\ U = W^q \end{array} \right\} \Rightarrow J = \begin{vmatrix} \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \\ \frac{\partial U}{\partial Z} & \frac{\partial U}{\partial W} \end{vmatrix} = \begin{vmatrix} W & Z \\ 0 & qW^{q-1} \end{vmatrix} = qW^q$$

where  $J$  is the Jacobian,  $0 < w < 1$  and  $z > 0$ . Then, the joint pdf of  $Z$  and  $W$  is obtained as follows:

$$\begin{aligned} f_{Z,W}(z, w) &= |J| f_{Y,U}(y(z, w), u(z, w)) \\ &= qw^q f_Y(zw) f_U(w^q) \\ &= \frac{4q}{\sigma\Gamma(\frac{1}{2})} w^q \frac{z^2 w^2}{\sigma^2} \exp\left(-\frac{z^2 w^2}{\sigma^2}\right). \end{aligned}$$

Taking integration with respect to the variable  $u$ , we obtain the marginal pdf of  $Z$  given by

$$f_Z(z) = \int_0^1 qw^q f_Y(zw) f_U(w^q) dw = \frac{4q}{\sigma\Gamma(\frac{1}{2})} \int_0^1 w^q \frac{z^2 w^2}{\sigma^2} \exp\left(-\frac{z^2 w^2}{\sigma^2}\right) dw.$$

After using the transformation  $t = z^2 w^2$ , we have

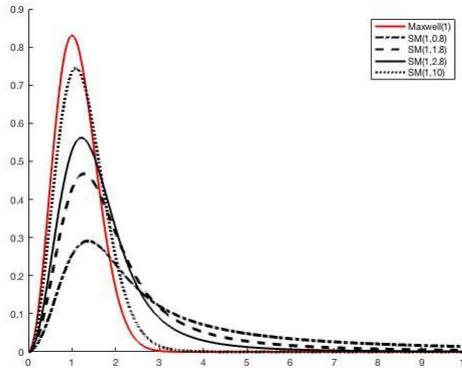
$$f_Z(z) = \frac{2q}{\sigma^3 \Gamma(\frac{1}{2}) z^{q+1}} \int_0^{z^2} t^{\frac{q+3}{2}-1} \exp\left(-\frac{t}{\sigma^2}\right) dt, \quad z > 0$$

where

$$\Gamma\left(\frac{q+3}{2}\right) \sigma^{q+3} G\left(z^2; \frac{q+3}{2}, \sigma^2\right) = \int_0^{z^2} t^{\frac{q+3}{2}-1} \exp\left(-\frac{t}{\sigma^2}\right) dt. \quad (4)$$

It is known that the Equation (4) is essentially cdf of the gamma distribution. Finally, pdf of random variable  $Z$  is obtained as given in Equation (3).

In Figure 1, shape of the SM distribution is illustrated for certain values of the shape parameter  $q$ . It should be noted that the shape parameter  $q$  controls kurtosis of the SM distribution.



**Figure 1.** The pdf plots of the SM distribution for certain values of  $q$  ( $\sigma = 1$ )

It is clear from Figure 1 that the SM tends to the Maxwell distribution as  $q$  gets larger. In other words, the Maxwell distribution is a limiting distribution for the SM distribution when the slashing parameter  $q$  goes to infinity.

After giving the definition of the SM distribution, we now provide the mixing property in Theorem 2. The importance of this theorem is that the SM distribution can be represented as a scale mixture of the Maxwell( $\sigma$ ) and  $U(0,1)$  distributions.

**Theorem 2.** Let  $Z|U = u \sim \text{Maxwell}(u^{-1/q}\sigma)$  and  $U \sim U(0,1)$ , then  $Z \sim \text{SM}(\sigma, q)$ .

*Proof.* The pdf of  $Z$  can be obtained from the following integral

$$f_Z(z; \sigma, q) = \int_0^1 f_{Z|U}(z|u) f_U(u) du$$

$$= \frac{4}{\sigma \Gamma(1/2)} \int_0^1 u^{1/q} \left( \frac{u^{1/q} z}{\sigma} \right)^2 \exp \left\{ - \left( \frac{u^{1/q} z}{\sigma} \right)^2 \right\} du$$

The proof is completed after the following transformation:  $t = (zu^{1/q})^2$ .

### 2.3. Moments

The moments of the SM distribution are obtained using the stochastic representation given in Equation (2). Therefore, following Lemma should be considered before obtaining the moments.

**Lemma 1.** Let  $Y \sim \text{Maxwell}(\sigma)$  and  $U \sim U(0,1)$  be independent random variables, then

$$E[Y^r] = \frac{2}{\sqrt{\pi}} \sigma^r \Gamma\left(\frac{r+3}{2}\right) \quad \text{and} \quad E\left[U^{-\frac{r}{q}}\right] = \frac{q}{q-r}, \quad q > r$$

respectively.

We now give the moments of the SM distribution in Theorem 3.

**Theorem 3.** Let  $Z \sim \text{SM}(\sigma, q)$ , then  $r$ -th non-central moment of the SM distribution is formulated by

$$E[Z^r] = \frac{2}{\sqrt{\pi}} \sigma^r \Gamma\left(\frac{r+3}{2}\right) \frac{q}{q-r}, \quad q > r. \tag{5}$$

*Proof.* The proof follows from the stochastic representation given in Equation (2) and Lemma 1.

$$E[Z^r] = E \left[ Y^r U^{-\frac{r}{q}} \right] = E[Y^r] E \left[ U^{-\frac{r}{q}} \right] = \frac{2}{\sqrt{\pi}} \sigma^r \Gamma \left( \frac{r+3}{2} \right) \frac{q}{q-r} \quad , \quad q > r.$$

Variance (Var), skewness ( $\sqrt{\beta_1}$ ) and kurtosis ( $\beta_2$ ) measures of the SM distribution are formulated using Equation (5) as follows:

$$Var(Z) = \frac{2q}{\sqrt{\pi}} \left[ \frac{\Gamma \left( \frac{5}{2} \right)}{q-2} - \frac{\frac{2q}{\sqrt{\pi}}}{(q-1)^2} \right] \sigma^2, \sqrt{\beta_1} = \frac{\sqrt{2\sqrt{\pi}} \frac{1}{q-3} - \frac{3\Gamma \left( \frac{5}{2} \right) q^2}{(q-2)(q-1)} + \frac{4}{\pi} \frac{q^2}{(q-1)^3}}{\left[ \frac{\Gamma \left( \frac{5}{2} \right)}{q-2} - \frac{\frac{2q}{\sqrt{\pi}}}{q-1} \right]^{\frac{3}{2}}}$$

and

$$\beta_2 = \frac{\frac{\sqrt{\pi}}{2q} \left[ \frac{\Gamma \left( \frac{7}{2} \right)}{q-4} - \frac{16q}{\sqrt{\pi}(q-3)(q-1)} + \frac{12\Gamma \left( \frac{5}{2} \right) q^2}{\pi(q-2)(q-1)^2} - \frac{24q^3}{(\sqrt{\pi})^2 q^4} \right]}{\left[ \frac{\Gamma \left( \frac{5}{2} \right)}{q-2} - \frac{\frac{2q}{\sqrt{\pi}}}{q-1} \right]^2},$$

respectively.

In Table 1, the values of the skewness and kurtosis measures of the SM distribution are tabulated for certain values of the shape parameter  $q$  to gain insight about shape of the distribution.

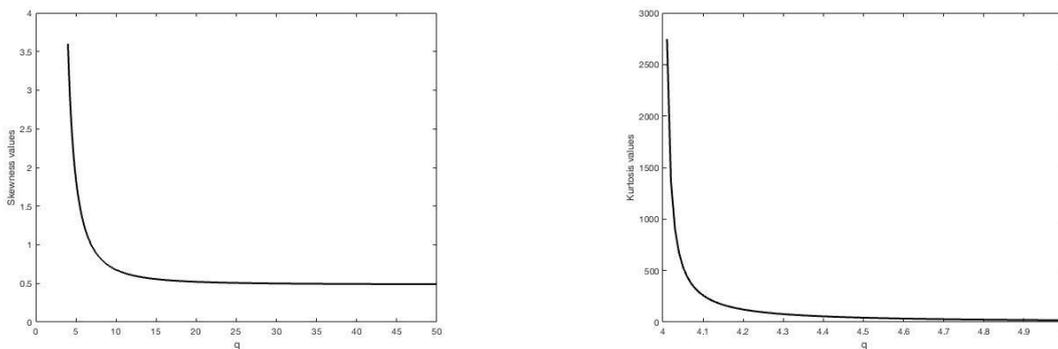
**Table 1.** The values of the skewness and kurtosis measures of the SM distribution for certain values of  $q$

$q$	5	6	7	8	9	10	20	50	100
$\sqrt{\beta_1}$	1.8514	1.2605	0.9832	0.8307	0.7381	0.6779	0.5220	0.4906	0.4868
$\beta_2$	18.7474	8.4771	5.7898	4.6989	4.1535	3.8440	3.2117	3.1203	3.1109

It is clear from Table 1 that the SM distribution has higher skewness and kurtosis values for small values of the shape parameter  $q$ . Indeed,  $\sqrt{\beta_1}$  and  $\beta_2$  are decreasing functions of  $q$ ; see Figure 2 in which the skewness and kurtosis measures are plotted. We also compute the limits of  $\sqrt{\beta_1}$  and  $\beta_2$  when  $q$  goes to infinity as follows:

$$\lim_{q \rightarrow \infty} \sqrt{\beta_1} = 0.4857 \quad \text{and} \quad \lim_{q \rightarrow \infty} \beta_2 = 3.1082.$$

These values are very similar to those of the Maxwell distribution. Therefore, the Maxwell distribution is a limiting case of the SM distribution. It should be noted that  $\sigma$  is taken to be 1 in Table 1 and Figure 2 without loss of generality.



(a) Skewness

(b) Kurtosis

**Figure 2.** The plots of the skewness and kurtosis measures of the SM distribution for certain values of  $q$

## 2.4. Data Generation

Random numbers having the SM distribution can be generated using Equation (2) in which the stochastic representation of the SM distribution is provided. The following steps can be followed for data generation from the SM distribution.

**Step 1.** Generate  $w$  from  $U(0,1)$  distribution to obtain random number from the Maxwell distribution using the following inverse transformation:

$$y = F_Y^{-1}(w) = \sqrt{\Gamma^{-1}(w; 3/2, \sigma)}$$

where  $F_Y^{-1}$  is the inverse of cdf of the Maxwell distribution and  $\Gamma^{-1}$  is the inverse incomplete gamma function.

**Step 2.** Generate  $u$  from the  $U(0,1)$  distribution to obtain random number from the SM distribution using the following equality:

$$z = \frac{y}{u^{1/q}} .$$

## 2.5. Location - Scale case

The SM distribution is represented in the location-scale form as follows:

$$f_X(x; \mu, \sigma, q) = \frac{2q}{\Gamma(1/2)\sigma} \Gamma\left(\frac{q+3}{2}\right) \left(\frac{x-\mu}{\sigma}\right)^{-(1+q)} G\left((x-\mu)^2/\sigma^2; \frac{q+3}{2}, 1\right), \quad x > \mu \quad (6)$$

where  $\mu \in \mathbb{R}$  is the location parameter. Random variable  $X$  having the pdf given in (6) is shortly denoted by  $X \sim SM(\mu, \sigma, q)$ .

## 3. MODIFIED MAXIMUM LIKELIHOOD ESTIMATION

In this section, the MML estimators of the location and scale parameters of the SM distribution are obtained. It is known that the estimation of the shape parameter along with the other parameters yields unreliable results when sample size is not large enough, see for example Bowman and Shenton [27], Kantar and Senoglu [28]. We therefore estimate the shape parameter  $q$  by using the methodology known as profile likelihood. The details of the profile likelihood methodology will be provided in Section 4.

The log-likelihood ( $\ln L$ ) function of the SM distribution is expressed as follows:

$$\begin{aligned} \ln L = & n \ln(2q) - n \ln \left[ \Gamma\left(\frac{1}{2}\right) \right] + n \ln \left[ \Gamma\left(\frac{q+3}{2}\right) \right] - (q+1) \sum_{i=1}^n \ln z_i - n \ln \sigma \\ & + \sum_{i=1}^n \ln \left[ G\left(z_i^2; \frac{q+3}{2}, 1\right) \right] \end{aligned} \quad (7)$$

where  $z_i = (x_i - \mu)/\sigma$  ( $i = 1, 2, \dots, n$ ). After taking derivatives of the  $\ln L$  function with respect to the parameters  $\mu$  and  $\sigma$  and setting them equal to 0, following likelihood equations are obtained:

$$\frac{\partial \ln L}{\partial \mu} = \frac{q+1}{\sigma} \sum_{i=1}^n h_1(z_i) - \frac{2}{\sigma} \sum_{i=1}^n h_2(z_i) = 0 \quad (8)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{q+1}{\sigma} \sum_{i=1}^n z_i h_1(z_i) - \frac{2}{\sigma} \sum_{i=1}^n z_i h_2(z_i) = 0, \quad (9)$$

respectively. Here,

$$h_1(z_i) = z_i^{-1}, \quad h_2(z_i) = \frac{z_i g\left(z_i^2; \frac{q+3}{2}, 1\right)}{G\left(z_i^2; \frac{q+3}{2}, 1\right)},$$

$g(\cdot)$  denotes pdf of the *Gamma*  $\left(\frac{q+3}{2}, 1\right)$  distribution. It is clear that closed form estimators of the location and scale parameters cannot be obtained since the likelihood equations given in the Equation (8) and Equation (9) include nonlinear functions of these parameters. Therefore, numerical methods should be performed. However, using numerical methods causes various problems such as (i) non-convergence of iterations (ii) convergence to multiple roots and (iii) convergence to wrong root, see e.g. Puthenpura and Sinha [29] and Vaughan [30].

In this study, Tiku's [24,25] MML methodology is used to avoid the mentioned computational difficulties. The MML methodology not only gives explicit forms of the estimators but also is asymptotically equivalent to the ML methodology. There are three steps to obtain the MML estimators of the location parameter  $\mu$  and scale parameter  $\sigma$ :

**Step 1.** Standardized observations are ordered in ascending way, i.e.  $z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(n)}$ .

**Step 2.** The ordered standardized observations are incorporated into the likelihood equations since complete sums are invariant to ordering, i.e.,  $\sum_{i=1}^n f(z_i) = \sum_{i=1}^n f(z_{(i)})$ .

**Step 3.**  $h_1(\cdot)$  and  $h_2(\cdot)$  functions are linearized about the expected values of the ordered standardized observations, i.e.  $t_{(i)} = E(z_{(i)})$ , using the first two terms of Taylor series expansion:

$$h_1(z_{(i)}) \cong \alpha_{1i} - \beta_{1i}z_{(i)}, \quad \text{and} \quad h_2(z_{(i)}) \cong \alpha_{2i} - \beta_{2i}z_{(i)}; \quad (i = 1, \dots, n). \quad (10)$$

After incorporating the linearized versions of  $h_1(\cdot)$  and  $h_2(\cdot)$  functions into the likelihood equations, following modified likelihood equations are obtained:

$$\frac{\partial \ln L^*}{\partial \mu} = \frac{q+1}{\sigma} \sum_{i=1}^n (\alpha_{1i} - \beta_{1i}z_{(i)}) - \frac{2}{\sigma} \sum_{i=1}^n (\alpha_{2i} + \beta_{2i}z_{(i)}) = 0, \quad (11)$$

$$\frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma} + \frac{q+1}{\sigma} \sum_{i=1}^n z_{(i)}(\alpha_{1i} - \beta_{1i}z_{(i)}) - \frac{2}{\sigma} \sum_{i=1}^n z_{(i)}(\alpha_{2i} + \beta_{2i}z_{(i)}) = 0. \quad (12)$$

Here, the  $\ln L^*$  stands for modified log-likelihood function. Solutions of the Equations in (11)-(12) are the MML estimators and formulated by

$$\hat{\mu}_{MML} = \bar{x}_{[.]} - \frac{\Delta}{m} \hat{\sigma}_{MML} \quad \text{and} \quad \hat{\sigma}_{MML} = \frac{B + \sqrt{B^2 - 4nC}}{2\sqrt{n(n-1)}} \quad (13)$$

where

$$\bar{x}_{[.]} = \sum_{i=1}^n \delta_i x_{(i)} / m, \quad m = \sum_{i=1}^n \delta_i, \quad \delta_i = [(q+1)\beta_{1i} + 2\beta_{2i}], \quad \beta_{1i} = t_{(i)}^{-2},$$

$$\beta_{2i} = \left[ \frac{g\left(t_{(i)}^2; \frac{q+3}{2}, 1\right) + 2t_{(i)}^2 g'\left(t_{(i)}^2; \frac{q+3}{2}, 1\right) G\left(t_{(i)}^2; \frac{q+3}{2}, 1\right) - 2t_{(i)}^2 g^2\left(t_{(i)}^2; \frac{q+3}{2}, 1\right)}{G^2\left(t_{(i)}^2; \frac{q+3}{2}, 1\right)} \right],$$

$$\gamma_i = [(q+1)\alpha_{1i} - 2\alpha_{2i}], \quad \Delta = \sum_{i=1}^n \gamma_i, \quad \alpha_{1i} = 2t_{(i)}^{-1}, \quad \alpha_{2i} = h_2(t_{(i)}) - \beta_{2i}t_{(i)}$$

$$B = \sum_{i=1}^n \gamma_i (x_{(i)} - \bar{x}_{[.]}) \quad \text{and} \quad C = \sum_{i=1}^n \delta_i (x_{(i)} - \bar{x}_{[.]})^2.$$

It should be noted that  $t_{(i)}$  values cannot be obtained exactly. We therefore use their approximate values using the following equality:

$$t_{(i)} = F^{-1}\left(\frac{i}{n+1}\right), \quad i = 1, 2, \dots, n$$

where  $F^{-1}(\cdot)$  is the inverse of cdf of the SM distribution.

**Remark.** The original denominator of  $\hat{\sigma}_{MML}$  is  $2n$ , however it is replaced by  $2\sqrt{n(n-1)}$  for bias correction.

It should be noted that the MML estimator of the location parameter may be greater than the smallest order statistics  $x_{(1)}$ . In case of this situation, it is replaced by  $x_{(1)} - 10^{-4}$ , see for example, Kantar and Senoglu [28], Singh and Sharma [31].

The advantage of the MML estimators is that they do not require any numerical calculations since they are functions of the sample observations and have closed forms. Furthermore, the MML estimators are asymptotically equivalent to the ML estimators.

The asymptotic distributions of  $\hat{\mu}_{MML}$  and  $\hat{\sigma}_{MML}$  are provided in Theorem 4 – 5, respectively.

**Theorem 4.**  $\hat{\mu}_{MML}$  is normally distributed with mean  $\mu$  and variance  $\sigma^2/m$  for  $n \rightarrow \infty$ .

*Proof.* The proof is done based on the following fact: The likelihood and modified likelihood equations, i.e. Equations (8) and (11), are asymptotically equivalent. Furthermore, the  $\partial \ln L^* / \partial \mu$  can be written as

$$\begin{aligned} \frac{\partial \ln L^*}{\partial \mu} &= \frac{m}{\sigma^2} \left[ \left( \bar{x}_{[.]} - \frac{\Delta}{m} \hat{\sigma}_{MML} \right) - \mu \right] \\ &= \frac{m}{\sigma^2} (\hat{\mu}_{MML} - \mu). \end{aligned}$$

See Kendall and Stuart [32].  $\hat{\mu}_{MML}$  is normally distributed since  $E(\partial \ln L^* / \partial \mu^r) = 0$  for all  $r \geq 3$ , see Bartlett [33].

**Theorem 5.**  $\hat{\sigma}_{MML}$  conditional on  $\mu$  known,  $n\hat{\sigma}_{MML}^2/\sigma^2$  is asymptotically chi-square distributed with  $n$  degrees of freedom.

*Proof.* This follows from the fact that  $B_0/\sqrt{C_0} \cong 0$  and thus

$$\frac{\partial \ln L^*}{\partial \sigma} = \frac{n}{\sigma^3} \left( \frac{C_0}{n} - \sigma^2 \right)$$

where  $B_0$  and  $C_0$  are same as the  $B$  and  $C$ , respectively. See for example Tiku [34] and Senoglu [35] for further information.

### 3.1. Simulation Study

In this subsection, we conduct a Monte-Carlo (MC) simulation study to compare the performances of the ML and MML estimators of the parameters of the SM distribution. The ML estimates of the parameters of the SM distribution are obtained using “fminsearch” function available in the optimization toolbox of MATLAB software. The MML estimators are directly obtained using Equation (13). The simulation setup is considered as follows. Without loss generality, true values of  $\mu$  and  $\sigma$  are taken to be 0 and 1, respectively. Three different values of the shape parameter of the SM distribution are considered, i.e.  $q = 0.8, 1.8$  and  $2.8$  for illustrative purposes. The sample sizes are taken to be  $n = 25, 50$  and  $100$ . Mean, variance and

mean squared errors (MSEs) of the estimators are computed based on 1000 MC runs. The deficiencies (DEFs) of the estimators having the following formula

$$DEF = MSE(\hat{\mu}) + MSE(\hat{\sigma})$$

are also considered to compare the performances of the estimators. As it is clear from its formulation, the DEF is a measure of joint efficiency, see for example Kantar and Senoglu [28]. Results of the simulation study are tabulated in Table 2.

**Table 2.** The simulated mean, variance and MSEs of the ML and MML estimators of the location and scale parameters of the SM distribution

q	n	Parameter	ML				MML			
			Mean	Variance	MSE	DEF	Mean	Variance	MSE	DEF
0.8	25	$\mu$	-0.0039	0.1123	0.1123	0.3529	0.2625	0.0618	0.1307	0.2253
		$\sigma$	1.1686	0.2121	0.2406		0.7936	0.0520	0.0946	
	50	$\mu$	0.0006	0.0407	0.0407	0.1066	0.1736	0.0293	0.0594	0.1111
		$\sigma$	1.0816	0.0592	0.0659		0.8460	0.0280	0.0517	
	100	$\mu$	0.0037	0.0204	0.0204	0.041	0.1121	0.0155	0.0281	0.0533
		$\sigma$	1.0427	0.0259	0.0277		0.8951	0.0142	0.0252	
1.8	25	$\mu$	0.0411	0.0608	0.0625	0.1929	0.1056	0.0432	0.0544	0.0953
		$\sigma$	1.0634	0.1263	0.1304		0.9436	0.0377	0.0409	
	50	$\mu$	0.0199	0.0276	0.0280	0.0745	0.0644	0.0221	0.0262	0.0473
		$\sigma$	1.0203	0.0461	0.0465		0.9559	0.0191	0.0211	
	100	$\mu$	0.0153	0.0117	0.0119	0.0308	0.0447	0.0106	0.0126	0.0244
		$\sigma$	1.0033	0.0189	0.0189		0.9660	0.0106	0.0118	
2.8	25	$\mu$	0.0379	0.0467	0.0482	0.1569	0.0669	0.0366	0.0411	0.0743
		$\sigma$	1.0558	0.1056	0.1087		0.9691	0.0322	0.0332	
	50	$\mu$	0.0331	0.0220	0.0231	0.0679	0.0550	0.0182	0.0212	0.0387
		$\sigma$	1.0087	0.0448	0.0448		0.9629	0.0162	0.0175	
	100	$\mu$	0.0206	0.0099	0.0103	0.0286	0.0360	0.0092	0.0105	0.0195
		$\sigma$	0.9987	0.0183	0.0183		0.9762	0.0085	0.0090	

Following conclusions can be drawn from Table 2. The ML estimators of  $\mu$  is almost unbiased for all values of the shape parameter and sample size. The ML estimator of  $\sigma$  has negligible biases except  $q = 0.8$  and  $n = 25$ . On the other hand, the MML estimators of  $\mu$  and  $\sigma$  have larger bias values for small values of  $q$  and  $n$ . As the sample size increases the bias of the MML estimator of  $\mu$  decreases as expected. It should be noticed that the scale parameter is underestimated by the MML method. The variances of the MML estimators are smaller than those of the ML estimator for all considered cases. The ML estimator of  $\mu$  is more preferable to the MML estimator for  $q = 0.8$  and  $n = 25, 50, 100$  in terms of MSE criterion. However, the MML estimator of  $\mu$  gains efficiency for the remaining cases and the MSEs of the ML and MML estimators become more or less the same for larger values of the sample size. The MML estimator of  $\sigma$  has smaller MSE values compared to its ML counterpart. This is because of the fact that the variance of the MML estimator of  $\sigma$  is smaller than that of the ML estimator. The DEF values also indicate that the MML estimators are more preferable to the ML estimators since the DEF values of the MML estimators are much more smaller than those of the ML estimators for majority of the considered cases. Overall, the MML estimators of  $\mu$  and  $\sigma$  are more efficient than the corresponding ML estimators even for small sample size ( $n = 25$ ). These results show that the MML estimators are not only computationally straightforward but also as efficient as or better than the ML estimators. We therefore use the MML method for estimating the location and the scale parameters of the SM distribution in the rest of paper.

#### 4. APPLICATION

In this section, we use the SM distribution to model two data sets taken from Gomez et al. [14] in which the SG distribution is used for modelling purposes. The SG distribution includes the location ( $\mu$ ), scale ( $\sigma$ ) and shape parameter ( $q$ ). Therefore, it is compatible with the SM distribution. We refer to Gomez et al. [14] for further details about the SG distribution.

The SGR distribution is also considered as an alternative to the SM distribution. As indicated in the Introduction, the SM is a submodel of the SGR distribution. It should be noticed that the SGR distribution has one scale ( $\sigma$ ) and two shape parameters ( $\alpha$  and  $q$ ) in Iriarte et al. [26]. However, in this part of the study, we consider the SGR distribution with an additional location parameter to make a precise comparison.

##### 4.1. Maximum Monthly Wind Speed Data

Maximum monthly wind speed data set consists of 246 observations which are measured monthly from January, 1984 to December, 2005 in Palm Beach, Florida (USA). The corresponding data set and further information about it are available in Gomez et al. [14].

Gomez et al. [14] propose to model this data using the SG distribution since its kurtosis value is large. Different from Gomez et al. [14], we use the SM distribution to model maximum wind speed data in this study. Since the MML method is employed for estimating the parameters of the SM distribution, the plausible value of the shape parameter  $q$  should be identified using the methodology known as profile likelihood. It is an efficient methodology to find the plausible value of the shape parameter  $q$ ; see e.g. Acitas et al. [36]. The steps of the profile likelihood procedure are given as follows:

**Step 1.** For a given value of the shape parameter  $q$  calculate the MML estimates of  $\mu$  and  $\sigma$ .

**Step 2.** Calculate the  $\ln L$  value based on these estimates, i.e.  $\widehat{\ln L}$ , using the following equation:

$$\widehat{\ln L} = n \ln(2q) - n \ln \left[ \Gamma \left( \frac{1}{2} \right) \right] + n \ln \left[ \Gamma \left( \frac{q+3}{2} \right) \right] - (q+1) \sum_{i=1}^n \ln \hat{z}_i - n \ln \hat{\sigma} + \sum_{i=1}^n \ln \left[ G \left( \hat{z}_i^2; \frac{q+3}{2}, 1 \right) \right]$$

where  $\hat{z}_i = \frac{x_i - \hat{\mu}_{MML}}{\hat{\sigma}_{MML}}$ ,  $i = 1, 2, \dots, n$ .

**Step 3.** Repeat step 1 and 2 for several values of the shape parameter  $q$ .

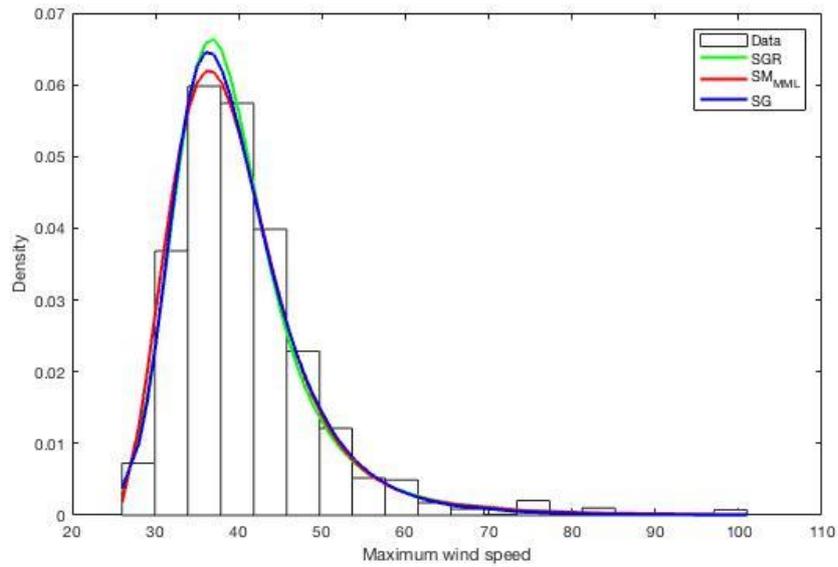
**Step 4.**  $q$  value maximizing the  $\widehat{\ln L}$  among the others is taken as a plausible estimate of  $q$ .

The plausible value of  $q$  is identified as 3.65 at the end of this procedure. Using  $\hat{q} = 3.65$  the MML estimates of the  $\mu$  and  $\sigma$  are obtained as given in the first row of Table 3. In the remaining rows of Table 3, the ML estimates of the parameters of the SG and SGR distributions are also provided. These ML estimates are obtained using “fminsearch” function. The  $\ln L$  and Akaike information criterion (AIC) values along with the Kolmogorov-Smirnov (KS) test statistic and associated  $p$  – values are also computed and tabulated in Table 3. The smaller AIC and KS test statistic values imply the better fitting performance. It is clear from Table 3 that the SM distribution is more preferable than the other considered slash distributions in terms of AIC values. However, the SGR distribution is the best according to the KS test statistic. See also Figure 3 in which the histogram and fitted densities based on the SM, SG and SGR distributions are plotted. It is obvious from Figure 3 that the SM distribution provides a better fitting performance to the maximum wind speed data. On the other hand, the SG and SGR distributions present overfitting.

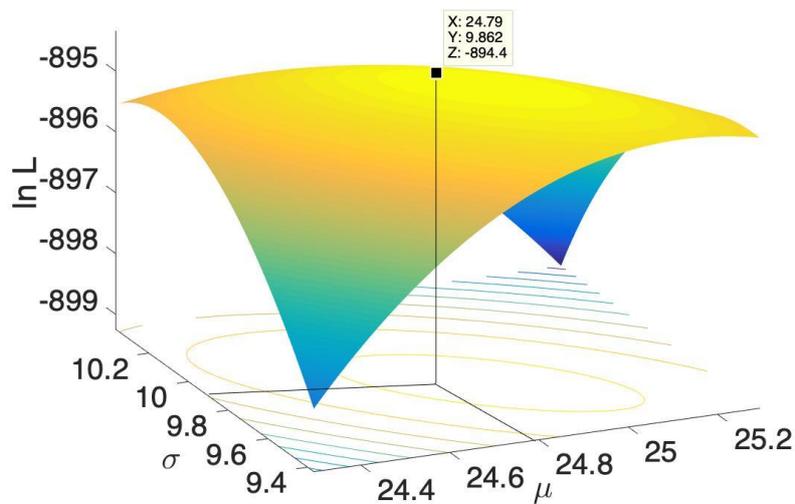
**Table 3.** The estimates of the parameters of the SG, SM and SGR distributions for maximum wind speed data

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{q}$	$\hat{\alpha}$	$\ln L$	AIC	KS	$p$ -value
SM	24.7940	9.8666	3.6500	-	-894.3740	1794.7479	0.0724	0.1192
SG	36.3460	4.5770	4.1220	-	-897.0335	1800.0670	0.0690	0.1545
SGR	11.2314	0.0148	5.0819	7.2315	-894.0409	1796.0819	0.0625	0.2435

Finally, surface plot of the  $\ln L$  of the SM distribution for the maximum wind speed data is given in Figure 4 to check that the  $\ln L$  function attains its maximum at the MML estimates. It is clear that the MML estimates are maximizers of the  $\ln L$  function.



**Figure 3.** The histogram and the fitted densities for maximum wind speed data



**Figure 4.** The loglikelihood surface of the SM distribution for the maximum wind speed data

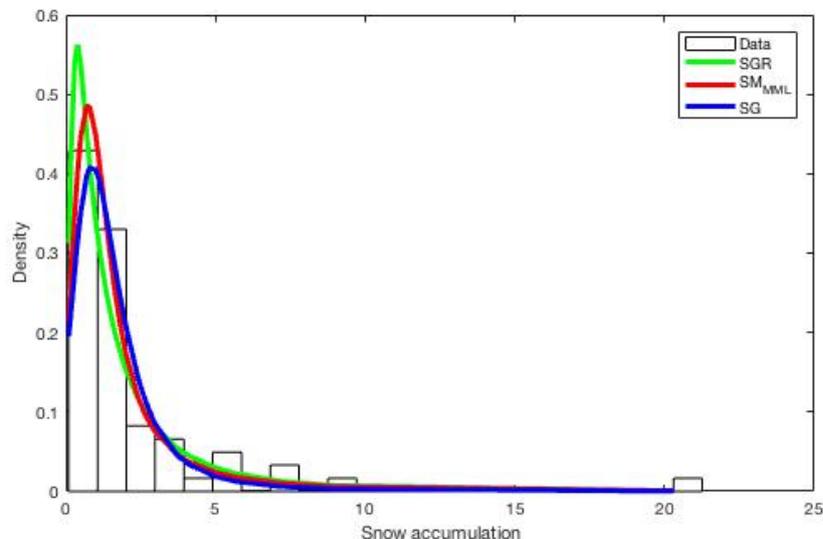
#### 4.1. Snow Accumulation Data

The snow accumulation data including 60 observations is obtained from the Raleigh– Durham airport, North Carolina, for the time period 1948–2000. Full data set can be found in Gomez et al. [14]. In this study, the SM distribution along with the SG and SGR distributions are used for modelling purposes. Since the MML methodology is used for estimating the parameters of the SM distribution, we should first identify the shape parameter  $q$ . The plausible value of  $q$  is obtained as 1.44 using the profile likelihood method explained in the previous subsection. The parameter estimates,  $\ln L$ , AIC value, KS test statistic and associated  $p$  –values are tabulated in Table 4.

**Table 4.** The estimates of the parameters of the SG, SM and SGR distributions for snow accumulation data

	$\hat{\mu}$	$\hat{\sigma}$	$\hat{q}$	$\hat{\alpha}$	$\ln L$	AIC	KS	$p$ –value
SM	-0.3761	0.8575	1.4400	-	-110.9157	227.8314	0.0959	0.5747
SG	0.8760	0.5570	1.6370	-	-116.4960	238.9920	0.1105	0.3964
SGR	-2.6652	8.4871	2.2977	65.9196	-110.0324	228.0648	0.1396	0.1557

Results show that the modelling performance of the SM distribution is substantially better than its rivals in terms of the AIC and KS test statistics values. See also Figure 5 in which the fitted densities are plotted. It is clear that the SM distribution provides a better fitting. It should be noted that the SM is a submodel of the SGR distribution thus the  $\ln L$ , and AIC values are close to each other. However, the SM has superiority over SGR in terms of KS test statistic value.



**Figure 5.** The histogram and the fitted densities for snow accumulation data

## 6. CONCLUSION

Slashing methodology is frequently used in the statistical literature to obtain heavy-tailed distributions so that resulting distribution is flexible enough to model the excess kurtosis. We therefore introduce the SM distribution. The parameters of the SM distribution are estimated using the MML methodology which is computationally straightforward. We conduct a MC simulation study to compare the efficiencies of the ML and MML estimators. Results show that the MML estimators are more preferable. The SM distribution is also used to model the maximum wind speed and snow accumulation data sets taken from the literature. Results and fitted densities demonstrate that the SM distribution has a satisfactory modelling performance.

It should be mentioned that the ML method can also be used for estimating the parameters of the SM distribution instead of the MML method in the applications. It is not surprising that the ML and MML

estimates and the corresponding AIC values and KS test statistics will be close to each other. Since the MML estimators are obtained employing a wise modification on the ML method, the AIC values obtained from the ML method may be less than those of the MML method. This is because of the fact that the ML estimates are obtained numerically. However, using numerical methods can be problematic in some cases. For example, choosing a wrong initial value may lead to wrong or non-convergence of iterations. On the other hand, computation of the MML estimates are straightforward and easy since no iterations are required. Therefore, the MML estimators can be used as an alternative to the ML estimators if the focus is computational ease as well as the efficiency.

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## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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