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# Theoretical Analysis of the Weibull Alpha Power Inverted Exponential Distribution: Properties and Applications

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## Highlights

- The article focuses on a class of inverted exponential distribution.
- The proposed distribution aims at proposing a better flexible model for lifetime data.
- The performance is validated by application of real life data in existing literature.
- Results indicate that the new model competes favourably well with other distributions.

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**Abstract**

This article proposed a Weibull-Alpha Power Inverted Exponential (WAPIE) distribution for lifetime processes. Statistical properties of this distribution such as survival, hazard, reversed hazard, cumulative, odd functions, kurtosis, quantiles, skewness, order statistics and the entropies were derived. Parameters of this family of distribution were also obtained by maximum likelihood method. The behaviour of the estimators was studied through simulation. The behavior of the new developed distribution was further examined through real life data. The WAPIE distribution competes favourably well with other distributions.

## 1. INTRODUCTION

Lifetime distributions have received several attentions over the years. Thus, its interest has grown over time. Researchers in distribution theory do this either by introducing a new parameter to make the distribution of interest more flexible or possibly produce a new family of distribution [1]. The Weibull distribution was proposed by a famous statistician called Weibull in 1951[2]. This Weibull distribution has a wide range of application in modelling lifetime processes, failure time process, mechanical, electrical system and even in machine learning. [3] proposed the generalized odd Weibull generated family of distributions, [4] developed the generalized exponential distribution and the failure time data were modeled by Lehmann alternatives in [5], Kumaraswamy-inverse exponential distribution was proposed in [6], the properties of the exponentiated generalized inverted exponential distribution was examined in [7], with the transmuted inverse exponential extensively developed in [8], the exponentiated generalized-G Poisson family of distributions was proposed in [9], the generalized transmuted-G family of distributions was proposed in [10], [11] introduced a new family of distributions, [12] proposed the exponentiated generalized class of distributions, [13] proposed the Kumaraswamy Weibull and proposed the McDonald Weibull model in [14], [15] proposed the Burr X generator of distributions and proposed the beta Weibull-G family in [16], The transmuted Topp-Leone G was proposed in [17], [18] proposed the generalization of the inverse exponential Distribution, with the Lomax distribution in [19] and [20] proposed a class or family of distribution called sum of exponentially distributed random variables. This class of exponential distribution plays important role for a process with continuous memory-less random processes with a constant failure rate which is almost impossible in real life cases. Hence, to account for this disadvantage

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[21] introduced the inverted exponential (*IE*) distribution with an inverted bathtub failure rate which was further studied and examined by [22-28] proposed the truncated-exponential skewsymmetric distributions and [29] proposed the alpha power Weibull distribution.

The Weibull-G family of distribution was proposed by [30] with a cumulative distribution function given as:

$$F(t)=\int_0^{\frac{G(t)}{1-G(t)}}\varphi\beta t^{\beta-1}e^{-\varphi t^{\beta-1}}dt \qquad \varphi,\ \beta>0. \tag{1}$$

Where $\alpha$ and $\beta$ are the shape and scale parameters respectively. The pdf in Equation (1) can be explicitly expressed as

$$F(t)=1-\exp\left\{-\varphi\left[\frac{G(t)}{1-G(t)}\right]^{\beta}\right\} \qquad \varphi,\beta>0. \tag{2}$$

The corresponding (pdf) of the Weibull G-family of distribution is given as

$$f(t)=\varphi\beta g(t)\frac{[G(t)]^{\beta-1}}{[1-G(t)]^{\beta+1}}\exp\left\{-\varphi\left[\frac{G(t)}{1-G(t)}\right]^{\beta}\right\} \qquad \varphi,\beta>0. \tag{3}$$

The inverted exponential $(IE)$ distribution has an inverted bathtub hazard rate function with the probability density function (pdf) given in [31] as

$$f_{IE}(t)=\frac{\lambda}{t}\exp\left(-\frac{\lambda}{t}\right) \qquad t>0,\quad \lambda>0 \tag{4}$$

where $\lambda$ is the parameter of the $(IE)$. The cumulative distribution function (cdf) is given as

$$F_{IE}(t)=\exp\left(-\frac{\lambda}{t}\right) \qquad t>0,\quad \lambda>0. \tag{5}$$

[31] proposed the alpha power inverted exponential distribution $(APIE)$ with pdf given as

$$f_{APIE}(t)=\frac{\lambda\log\alpha}{t^2(\alpha-1)}\exp\left(-\frac{\lambda}{t}\right)\alpha^{\exp\left(-\frac{\lambda}{t}\right)} \qquad for\ \alpha>0,\quad \alpha\neq 1. \tag{6}$$

The cumulative distribution function of the $(APIE)$ is given as

$$F_{APIE}(t)=\frac{\alpha^{\exp\left(-\frac{\lambda}{t}\right)}-1}{\alpha-1}, \qquad \alpha>0,\quad \alpha\neq 1\ . \tag{7}$$

Motivated by the studies based on the results obtained from the literature research such as the alpha power inverted exponential (APIE), inverted exponential (IE) and the alpha power transmuted (APT) distributions, the Weibull-Alpha Power Inverted Exponential Distribution (WAPIE) four parameters distribution is proposed using Weibull distribution characterizations. Its major characteristic is that two more shape parameters are added to make it more flexible. A comprehensive statistical property of the

WAPIE model is also provided for better view of its applications. This model aims to attract wider range of application in machine learning, medicine, engineering and other related areas.

In this article, the Weibull-Alpha Power Inverted Exponential Distribution (WAPIE) four parameters distribution is developed and proposed as motivated by the alpha power inverted exponential (APIE), inverted exponential (IE) and the alpha power transmuted (APT) distributions method using the Weibull characterizations.

## 2. THE WEIBULL ALPHA POWER INVERTED EXPONENTIAL DISTRIBUTION

The Weibull alpha power inverted exponential distribution is a member of the alpha power exponential distribution with the following probability density function

$$f_{WAPIE}(x)=\varphi\beta\frac{\lambda\log\alpha}{x^{2}(\alpha-1)}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\cdot\frac{\left[\dfrac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]^{\beta-1}}{\left[\dfrac{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-1}\right]^{\beta+1}}\exp\left\{-\varphi\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right]^{\beta}\right\} \tag{8}$$

$$\varphi,\alpha,\beta,\lambda>0;\alpha\neq 1.$$

The associated cumulative distribution function is given as

$$F_{WAPIE}(x)=1-\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)\qquad \varphi,\alpha,\beta,\lambda>0;\quad \alpha\neq 1. \tag{9}$$

Figure 1 is the plot of the pdf of the WAPIE distribution for different values of parameters.

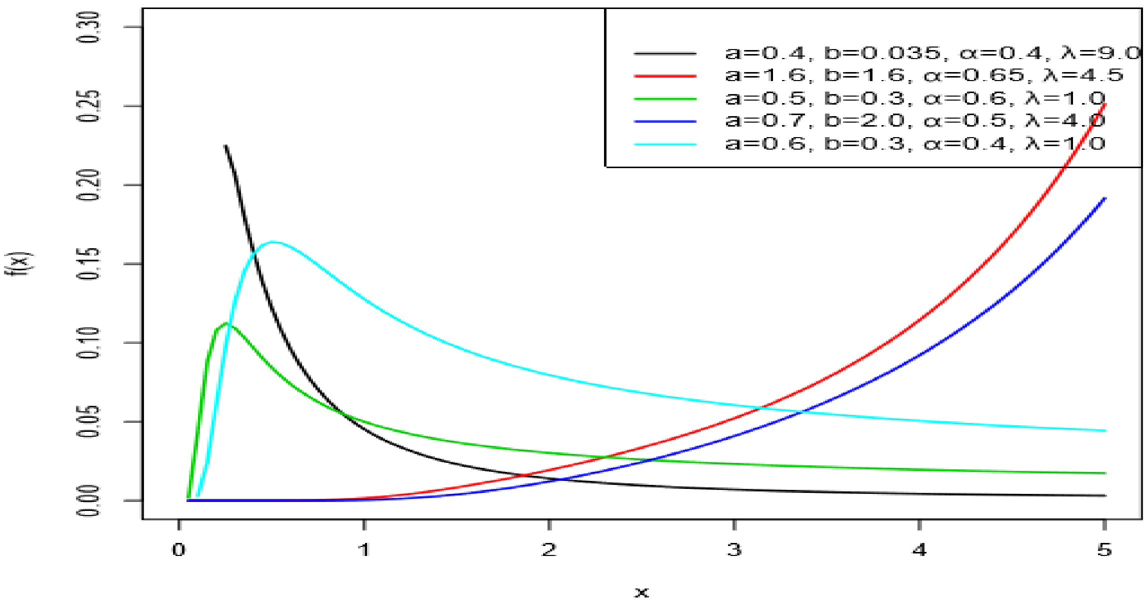


***Figure 1.*** *The pdf of the WAPIE distribution with different parameter values*

**Remark 1.**

*The shape of the pdf of the WAPIE distribution could be bathtub or skewed depending on the values of the parameters.*

Figure 2 is the plot of the cdf of the WAPIE distribution for different values of parameters. The shape of the cdf of the WAPIE distribution could be bathtub depending on the values of the parameters

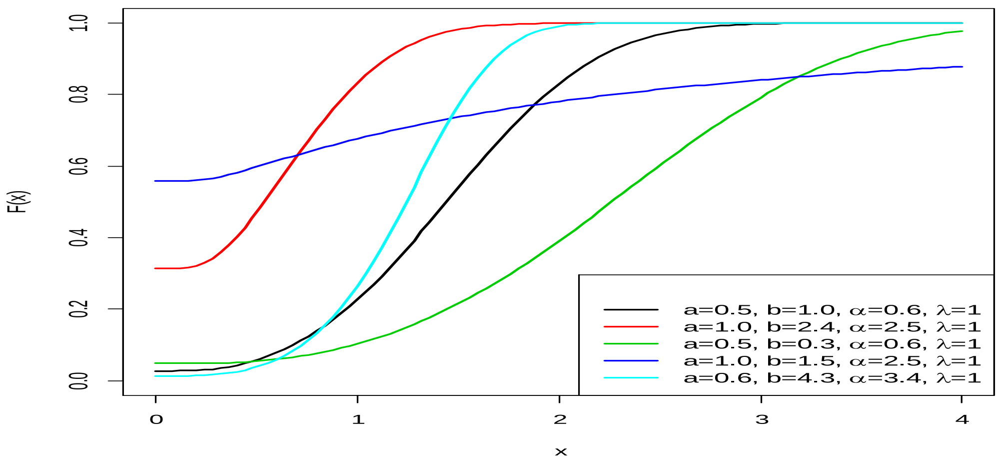


***Figure 2.*** *The cdf of the WAPIE distribution with different parameters*

**Remark 2.**

*The shape of the cdf of the WAPIE distribution is increasing depending on the values of the parameters.*

### 2.1. Parameter Estimation for the Weibull Alpha Power Inverted Exponential Distribution Formulation

Let $x_1, x_{2,}, \cdots, x_n$ be random variable obtained from a population with weilbull alpha power inverted exponential distribution. Then, the log-likelihood of weilbull alpha power inverted exponential distribution for vector $\Theta = (\varphi, \beta, \alpha, \lambda)^T$ can be represented as $\ell_n(x_i, \Theta)$ as

$$\ell_n(x_i,\Theta) = n\log\varphi + n\log\beta + \sum_{i=1}^{n}\log g(x_i) + (\beta-1)\sum_{i=1}^{n}\log G(x_i) - (\beta+1)\sum_{i=1}^{n}\log(1-G(x_i)) - \varphi\sum_{i=1}^{n}\left(\frac{G(x_i)}{1-G(x_i)}\right)^{\beta}. \quad (10)$$

Let $p = \dfrac{\partial\sum_{i=1}^{n}\left(\frac{G(x_i)}{1-G(x_i)}\right)^{\beta}}{\partial\beta}$, $r_i = \dfrac{\partial\sum_{i=1}^{n}\log g(x_i)}{\partial\alpha}$, $z_i = \dfrac{\partial\sum_{i=1}^{n}\log G(x_i)}{\partial\alpha}$, $k_i = \dfrac{\partial\sum_{i=1}^{n}\log(1-G(x_i))}{\partial\alpha}$,

$w_i = \dfrac{\partial\sum_{i=1}^{n}\left(\frac{G(x_i)}{1-G(x_i)}\right)^{\beta}}{\partial\alpha}$, $t_i = \dfrac{\partial\sum_{i=1}^{n}\left(\frac{G(x_i)}{1-G(x_i)}\right)^{\beta}}{\partial\lambda}$, $m_i = \dfrac{\partial\sum_{i=1}^{n}\log g(x_i)}{\partial\lambda}$, $s_i = \dfrac{\partial\sum_{i=1}^{n}\log G(x_i)}{\partial\lambda}$,

$q_i = \dfrac{\partial\sum_{i=1}^{n}\log(1-G(x_i))}{\partial\lambda}$.

Taking partial derivative of Equation (10) with respect to the parameters, we have

$$U_{\varphi}=\frac{n}{\varphi}-\sum_{i=1}^{n}\left(\frac{G(x_i)}{1-G(x_i)}\right)^{\beta} \tag{11}$$

$$U_{\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log G(x_i)-\sum_{i=1}^{n}\log(1-G(x_i))-\varphi\, p \tag{12}$$

$$U_{\alpha}=r_i+(\beta-1)z_i-(\beta+1)k_i-\varphi w_i\,. \tag{13}$$

$$U_{i}=m_i+(\beta-1)s_i-(\beta+1)q_i-\varphi t_i\,. \tag{14}$$

Thus, setting $U_{\alpha}= U_{\beta}= U_{\lambda} =U_{\alpha} = 0$. The solution to the nonlinear equation for the parameters can be obtained using R software, MATLAB, and MAPLE. Thus, yield the maximum likelihood estimate $\hat{\Theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\alpha})$.

## 3. SOME STATISTICAL PROPERTIES OF THE WEIBULL ALPHA POWER INVERTED EXPONENTIAL DISTRIBUTION FORMULATION

In this section, we study some statistical properties of the WAPIE distribution. It comprise reliability analysis, hazard rate function, cumulative hazard rate function, reserved hazard function, odds function, quantile function, moments, and order statistics.

### 3.1. Reliability Analysis

The survival function of the WAPIE distribution for variable $X > 0$ is given as

$$S_{WAPIE}(x)=\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right), \qquad for\ \varphi>0,\ \alpha\neq 1,\ \beta,\lambda>0\ . \tag{15}$$

### 3.2. Hazard Rate Function

The Hazard rate function of the WAPIE is given as

$$H_{WAPIE}(x)=\varphi\beta\frac{\lambda\log\alpha}{x^2(\alpha-1)}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\left(\frac{\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right)^{\beta-1}}{\left(\frac{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-1}\right)^{\beta+1}}\right), \quad \varphi,\ \beta,\ \lambda>0,\quad \alpha\neq 1. \tag{16}$$

### 3.3. Cumulative Hazard rate Function

The cumulative hazard function (CH) of the WAPIE distribution is given as

$$CH_{WAPIE}(x)=\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}, \qquad \varphi,\beta,\ \lambda>0,\ \alpha\neq 1. \tag{17}$$

### 3.4. Reversed Hazard Function

The WAPIE distribution has a reversed hazard function rate (RH) given as

$$RH_{WAPIE}(x)=\frac{\varphi\beta\lambda\log\alpha\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right)^{\beta-1}\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)}{x^2(\alpha-1)\left(\frac{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-1}\right)^{\beta+1}\left(1-\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)\right)},$$

$\varphi,\ \beta,\ \lambda>0\ ,\alpha\neq 1.$ (18)

### 3.5. Odds Function

The odds function (O) of the WAPIE distribution is given as

$$O_{WAPIE}(x)=\exp\left(\varphi\left(\frac{1-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)-1 \qquad \varphi,\ \beta,\ \lambda>0,\ \alpha\neq 1 \qquad . \tag{19}$$

### 3.6. Quantile and Median Function

The Quantile function of the WAPIE distribution is derived from the equation

$$Q(u)=F^{-1}(x). \tag{20}$$

Therefore, the quantile function of the Weibull-Alpha power inverted exponential distribution is given as

$$Q(u)=-\lambda\left[\log\left[(\log\alpha)^{-1}\log\left((\alpha-1)\left(\left(1+\left(-\varphi^{-1}\log(1-u)\right)^{\frac{1}{\beta}}\right)^{-1}\left(-\varphi^{-1}\log(1-u)\right)\right)+1\right)\right]\right]^{-1} \tag{21}$$

where u~ uniform $[0,1]$.

Using the WAPIE distribution, random numbers $X$ generated from the WAPIE distribution is given by

$$x=-\lambda\left[\log\left[(\log\alpha)^{-1}\log\left((\alpha-1)\left(\left(1+\left(-\varphi^{-1}\log(1-u)\right)^{\frac{1}{\beta}}\right)^{-1}\left(-\varphi^{-1}\log(1-u)\right)\right)+1\right)\right]\right]^{-1}. \tag{22}$$

The median of the WAPIE distribution can be obtained by substituting $u=\frac{1}{2}$ in Equation (22) as

$$Median=-\lambda\left[\log\left[(\log\alpha)^{-1}\log\left((\alpha-1)\left(\left(1+\left(-\varphi^{-1}\log\left(\frac{1}{2}\right)\right)^{\frac{1}{\beta}}\right)^{-1}\left(-\varphi^{-1}\log\left(\frac{1}{2}\right)\right)\right)+1\right)\right]\right]^{-1} \tag{23}$$

Then, the 25$^{th}$ percentile and the 75$^{th}$ percentile are given by Equations (24) and (25) respectively

$$Q_1 = -\lambda\left[\log\left[(\log\alpha)^{-1}\log\left((\alpha-1)\left(\left(1+\left(-\varphi^{-1}\log(0.75)\right)^{\frac{1}{\beta}}\right)^{-1}\left(-\varphi^{-1}\log(0.75)\right)\right)+1\right)\right]\right]^{-1} \tag{24}$$

$$Q_3 = -\lambda\left[\log\left[(\log\alpha)^{-1}\log\left((\alpha-1)\left(\left(1+\left(-\varphi^{-1}\log(0.25)\right)^{\frac{1}{\beta}}\right)^{-1}\left(-\varphi^{-1}\log(0.25)\right)\right)+1\right)\right]\right]^{-1}. \tag{25}$$

The Bowley's formula for finding the coefficient of skewness is given as

$$S_k(B) = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}. \tag{26}$$

The Moor's formula for coefficient of kurtosis is given as

$$K_k(u) = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{x_{0.75} - x_{0.25}}. \tag{27}$$

### 3.7. The $r^{th}$ Moments

The $r^{th}$ moment of the WAPIE distribution is given as

$$\mu_r' = E(X^r) = \int_0^\infty \frac{x^r \varphi\beta \dfrac{\lambda\log\alpha}{x^2(\alpha-1)}\exp\left(-\dfrac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)} \cdot \left[\dfrac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]^{\beta-1}}{\left[\dfrac{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-1}\right]^{\beta+1}} \times \exp\left\{-\varphi\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right]^{\beta}\right\} dx. \tag{28}$$

### 3.8. The Probability Weighted Moments

This is a class of moments used to derived estimators of the quantiles and parameters of the WAPIE distribution expressed in inverse form. Thus, for a random variable $X$ , the probability weighted moments denoted by

$$\tau_{r,s} = E\left(X^r F_{WAPIE}(x)^s\right) = \int_{-\infty}^{\infty} x^r f_{WAPIE}(x) F_{WAPIE}(x)^s \, dx. \tag{29}$$

### 3.9. Distribution of Order Statistics

Suppose $x_1, x_2, \cdots, x_n$ be a random sample drawn from an infinite population with a pdf of the WAPIE distribution. Then the pdf of the $k^{th}$ order statistics is given by

$$g_k(x)=\frac{n!}{(k-1)!(n-k)!}\left[1-\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)\right]^{k-1}\left[\exp\left(-\varphi\left(\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}\right)^{\beta}\right)\right]^{n-k-1}$$

$$\times\varphi\beta\frac{\lambda\log\alpha}{x^2(\alpha-1)}\exp\left(-\frac{\lambda}{x}\right)\alpha^{\exp\left(-\frac{\lambda}{x}\right)}.\frac{\left[\frac{\alpha^{\exp\left(-\frac{\lambda}{x}\right)}-1}{\alpha-1}\right]^{\beta-1}}{\left[\frac{\alpha-\alpha^{\exp\left(-\frac{\lambda}{x}\right)}}{\alpha-1}\right]^{\beta+1}}. \tag{30}$$

We obtain the minimum and maximum order statistics when $k=1$ and $k=n$ respectively.

## 4. SIMULATION STUDY

A simulation is carried out to test the efficiency of the weibull alpha power inverted exponential distribution. Table 1 shows the simulation for different values of parameters for skewness, kurtosis, median, 25th and 75$^{th}$ percent of the WAPIE distribution. Increase parameters decreases the skewness and kurtosis but increases the percentile

***Table 1.*** *A simulation study of the weibull alpha power inverted exponential distribution*

| Parameters | | | | Skewness | Kurtosis | Median | 25$^{th}$ percent | 75$^{th}$percent |
|---|---|---|---|---|---|---|---|---|
| $\varphi$ | β | λ | α | | | | | |
| 0.500 | 0.500 | 0.500 | 2.000 | 0.5560 | 1.6147 | 1.5834 | 0.4393 | 5.5928 |
| 1.000 | 0.500 | 0.500 | 3.000 | 0.5160 | 1.4852 | 0.6355 | 0.2446 | 1.8598 |
| 0.500 | 1.000 | 1.000 | 7.000 | 0.2490 | 0.5722 | 3.8509 | 1.9345 | 7.0381 |
| 1.000 | 1.000 | 1.000 | 20.00 | 0.2305 | 0.5247 | 3.1113 | 1.6891 | 5.3854 |
| 2.000 | 2.000 | 2.000 | 50.00 | 0.0554 | 0.1163 | 7.1088 | 5.2480 | 9.1882 |
| 1.500 | 1.500 | 3.000 | 2.000 | 0.1186 | 0.2539 | 3.8471 | 2.6432 | 5.3750 |
| 2.000 | 1.500 | 1.500 | 3.000 | 0.1153 | 0.2465 | 1.9412 | 1.3385 | 2.7008 |
| 2.500 | 2.000 | 1.500 | 7.000 | 0.0570 | 0.1200 | 2.7222 | 2.0146 | 3.5154 |
| 2.500 | 2.500 | 2.000 | 20.00 | 0.0219 | 0.0473 | 5.5825 | 4.3502 | 6.8701 |
| 3.000 | 3.000 | 3.000 | 50.00 | -0.0017 | -5.78e-05 | 10.987 | 8.9361 | 13.030 |
| 1.500 | 1.500 | 5.000 | 2.000 | 0.1186 | 0.2540 | 6.4118 | 4.4054 | 8.9583 |
| 2.000 | 1.500 | 5.000 | 3.000 | 0.1153 | 0.2465 | 6.4705 | 4.4618 | 9.0027 |
| 2.500 | 2.000 | 5.000 | 7.000 | 0.0570 | 0.1200 | 9.0740 | 6.7151 | 11.718 |
| 2.500 | 2.500 | 5.000 | 20.00 | 0.0219 | 0.0473 | 13.956 | 10.875 | 17.175 |
| 3.000 | 3.000 | 5.000 | 50.00 | -0.0017 | -5.78e-05 | 18.311 | 14.894 | 21.717 |
| 1.500 | 1.500 | 7.000 | 2.000 | 0.1186 | 0.2540 | 8.9764 | 6.1676 | 12.542 |
| 2.000 | 1.500 | 7.000 | 3.000 | 0.1153 | 0.2465 | 9.0587 | 6.2465 | 12.604 |
| 2.500 | 2.000 | 7.000 | 7.000 | 0.0570 | 0.1200 | 12.704 | 9.4013 | 16.405 |
| 2.500 | 2.500 | 7.000 | 20.00 | 0.0219 | 0.0473 | 19.537 | 15.226 | 24.045 |
| 3.000 | 3.000 | 7.000 | 50.00 | -0.0017 | -5.78e-05 | 25.636 | 20.851 | 30.404 |

## 5. APPLICATION

To examine the performance of the WAPIE model with other competing distributions the gas fiber and carbon data real-life datasets were used. We considered the Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn Information

Criteria (HQIC), The Anderson Darling (A) statistic, Cramer-von Mises statistic (W), Kolmogorov Smirnov (KS) statistic, Log-likelihood and the P value to compare the fits of the WAPIE model to other competing models Kumaraswamy Lomax (KULOMAX), Kumaraswamy Exponential (KUEX), Kumaraswamy-Alpha Power Inverted Exponential (KAPIE), Kumaraswamy Burx11 (KUBUXI), Kumaraswamy Inverse Exponential (KUIEX), Alpha Power Inverted Exponential (APIE) and Exponential (EX)distributions.

### 5.1. First Set of Glass Fiber Data

Datasets were collected for 1.5 cm strengths of glass fibres data at the UK National Physical Laboratory and was used to test the performance of the WAPIE distribution as used by [30, 32-36]. The observations are as follows:

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24

The performance of the WAPIE distribution compared to other distribution is shown in Table 2 and 3.

***Table 2**. The goodness of fit statistics for the glass fiber's data*

| MODEL | AIC | CAIC | BIC | HQIC | A | W |
|---|---|---|---|---|---|---|
| WAPIE | 39.55332 | 40.24298 | 48.12586 | 42.92494 | 0.2668984 | 1.461937 |
| KULOMAX | 43.67962 | 44.36927 | 52.25216 | 47.05124 | 0.3530559 | 1.933592 |
| KUEX | 40.89735 | 41.30413 | 47.32676 | 43.42607 | 0.3352229 | 1.835432 |
| KAPIE | 52.71052 | 53.40017 | 61.28306 | 56.08214 | 0.5063777 | 2.770684 |
| KUBUXI | 47.19761 | 47.88727 | 55.77015 | 50.56923 | 0.4179052 | 2.289465 |
| KUIEX | 50.12111 | 50.52789 | 56.55051 | 52.64983 | 0.4813895 | 2.632409 |
| APIE | 196.3253 | 196.5253 | 200.6116 | 198.0111 | 0.777503 | 4.238456 |

***Table 3.** MLEs of parameters WAPIE distribution for the glass fibers*

| PARAMETERS | VALUES | St.D | Inf. 95% CI | Sup. 95% CI |
|---|---|---|---|---|
| $\varphi$ | 0.005860867 | 0.003039024 | -9.551092e-05 | 0.01181724 |
| $\beta$ | 4.979656013 | 0.484188807 | 4.030663e+00 | 5.92864864 |
| $\lambda$ | 0.365468737 | 0.140473092 | 9.014654e-02 | 0.64079094 |
| $\alpha$ | 2.035789436 | 1.976538211 | -1.83815e+00 | 5.90973314 |

From the result it shows that the WAPIE distribution has the smallest AIC, CAIC, BIC, HQIC, A and W when compared to KAPIE, KUBUXI, KULOMAX, KUEX, KUIEX and APIE Distribution. The measure of the test statistics is shown in Table 4 below.

***Table 4.** Measure of test statistics collections for the glass fibers[37]*

| TEST STATISTICS | KS Statistic | KS p-value | log-likelihood |
|---|---|---|---|
| WAPIE | 0.1646002 | 0.06583302 | 15.77666 |
| KAPIE | 0.2100631 | 0.007698024 | 22.35526 |
| KUBUXI | 0.2017095 | 0.01187427 | 19.59881 |
| KULOMAX | 0.1818623 | 0.03098763 | 17.83981 |
| KUEX | 0.1769054 | 0.03877044 | 17.44868 |
| KUIEX | 0.1992272 | 0.0134607 | 22.06056 |
| APIE | 0.4645605 | 3.099632e-12 | 96.16265 |

### 5.2. Second Set of Carbon Data

Our second set of data is from [37]. It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). The dataset are as follow:

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

Tables 5 and 6 show the goodness-of-fit and the performance rating of the WAPIE distribution using several test statistics for the carbon fibers dataset.

**Table 5**. *The goodness of fit statistics for the carbon data*

| MODEL | AIC | CAIC | BIC | HQIC | A | W |
|---|---|---|---|---|---|---|
| WAPIE | 290.8659 | 291.287 | 301.2866 | 295.0834 | 0.07010233 | 0.4242744 |
| KULOMAX | 290.9681 | 291.3891 | 301.3888 | 295.1855 | 0.0842262 | 0.4531522 |
| KUEX | 288.7576 | 289.0076 | 296.5731 | 291.9207 | 0.07768531 | 0.4303957 |
| KAPIE | 294.6224 | 295.0435 | 305.0431 | 298.8398 | 0.1338752 | 0.6933353 |
| KUBUXI | 292.8652 | 293.2863 | 303.2859 | 297.0827 | 0.1183604 | 0.6086712 |
| KUIEX | 308.4821 | 308.7321 | 316.2976 | 311.6452 | 0.3207067 | 1.745378 |
| APIE | 422.3312 | 422.455 | 427.5416 | 424.44 | 0.37259 | 2.042718 |

***Table 6.*** *MLEs of parameters WAPIE distribution for the carbon data*

| PARAMETERS | VALUES | St.D | Inf. 95% CI | Sup. 95% CI |
|---|---|---|---|---|
| $\varphi$ | 0.03428428 | 0.08370638 | -0.1297772 | 0.1983458 |
| $\beta$ | 2.53280436 | 0.25368799 | 2.0355850 | 3.0300237 |
| $\lambda$ | 0.39871544 | 0.57995511 | -0.7379757 | 1.5354066 |
| $\alpha$ | 3.61479397 | 14.43971884 | -24.6865349 | 31.9161229 |

The measure of the test statistics is shown in Table 7 below.

***Table 7.*** *Measure of test statistics collections for the carbon data*

| TEST STATISTICS | KS Statistic | KS p-value | Log-likelihood |
|---|---|---|---|
| WAPIE | 0.06283299 | 0.8247003 | 141.433 |
| KAPIE | 0.09093334 | 0.3799776 | 143.3112 |
| KUBUXI | 0.08533937 | 0.4601788 | 142.4326 |
| KULOMAX | 0.07543761 | 0.6198049 | 141.484 |
| KUEX | 0.07046968 | 0.7034028 | 141.3788 |
| KUIEX | 0.130143 | 0.06758742 | 151.241 |
| APIE | 0.3503104 | 4.384659e-11 | 209.1656 |

## 6. CONCLUSION

The concept of the WAPIE distribution has been well defined and studied. The mathematical expression for the probability density function (pdf) and cumulative distribution function (cdf) were examined. We also derived some of the statistical properties of the WAPIE distribution including survival function, hazard function, reversed hazard function, odds function, order statistics, cumulative hazard rate function, quantile and median function. Then the parameter estimation was obtained using the maximum likelihood estimation (MLE) approach. When applied to data the WAPIE distribution better than the KAPIE, KUBUXI, KUEX, KULOMAX, KUIEX, and APIE distributions.

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors

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