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On Mappings Of $p-$ Dimensional Surfaces in Euclidean Spaces E_n

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ABSTRACT

In the paper, we study one-to-one mappings of p dimensional surfaces of n dimensional euclidean spaces in $2n$ dimensional euclidean space. In the paper, we use the method of moving frames and exterior forms. The results about conjugacy and orthogonality of nets and conformity of mappings are obtained.

Keywords: *Mappings of n-dimensional surfaces; higher dimensional euclidean spaces; method of moving frame; method of exterior forms.*

AMS Subject Classification (2020): *Primary: 53A07; Secondary: 53A60.*

1. Introduction

Let us consider completely orthogonal Euclidean spaces E_n and $\bar{E}_n$ in Euclidean space E_{2n}, with common point O. Let V_p and $\overline{V}_p$ be smooth surfaces in E_n and $\bar{E}_n$, respectively. We are going to study a differentiable and one to one mapping $T : V_p \to \overline{V}_p$ which transforms a domain $\Omega \subset V_p$ into a domain $\bar{\Omega} \subset \overline{V}_p$. If point x_1 changes in the domain Ω then the point $x_2 = T(x_1)$ describes the domain $\bar{\Omega} \subset \overline{V}_p$, and the point x with position vector $\overrightarrow{x} = \overrightarrow{x_1} + \overrightarrow{x_2}$, where $\overrightarrow{x_1} = \overrightarrow{Ox_1}$ and $\overrightarrow{x_2} = \overrightarrow{Ox_2}$, describes a domain Ω^* of surface V_p^*, which is called a graph of the mapping T [1].

2. Main Results

Let $R_1 = \{x_1, \vec{e}_i, \vec{e}_\alpha\}$, $R_2 = \{x_2, \vec{e}_{n+i}, \vec{e}_{n+\alpha}\}$ $(i, j = \overline{1,p}; \alpha, \beta = \overline{p+1,n})$, be corresponding moving frames in E_n and $\bar{E}_n$, and let $\vec{e}_i \in T_p(x_1)$, $(dT)_{x_1}(\vec{e}_i) = \vec{e}_{n+i} \in T_p(x_2)$, where $T_p(x_1)$,$T_p(x_2)$ are tangent planes of the surfaces V_p and $\overline{V}_p$, respectively, at the corresponding points x_1 and x_2, and $\vec{e}_\alpha$ and $\vec{e}_{n+\alpha}$ form an orthonormal basis for the orthogonal complements of $T_p(x_1)$ and $T_p(x_2)$ in corresponding spaces E_n and $\bar{E}_n$. Derivational formulae for these frames are the following:

$$d\vec{x}_1 = \omega^i \vec{e}_i, d\vec{e}_i = \omega_i^j \vec{e}_j + \omega_i^\alpha \vec{e}_\alpha, d\vec{e}_\alpha = \omega_\alpha^i \vec{e}_i + \omega_\alpha^\beta \vec{e}_\beta \tag{2.1}$$

$$d\vec{x}_2 = \bar{\omega}^i \vec{e}_{n+i}, d\vec{e}_{n+i} = \bar{\omega}_i^j \vec{e}_{n+j} + \bar{\omega}_i^\alpha \vec{e}_{n+\alpha}, d\vec{e}_{n+\alpha} = \omega_\alpha^i \vec{e}_{n+i} + \omega_\alpha^\beta \vec{e}_{n+\beta}. \tag{2.2}$$

At the point $x \in V_p^*$ a frame $R = \{x, \vec{\varepsilon}_i, \vec{\varepsilon}_{p+i}, \vec{\varepsilon}_{p+\alpha}, \vec{\varepsilon}_{n+\alpha}\}$ is formed. Here $\vec{\varepsilon}_i = \vec{e}_i + \vec{e}_{n+i}$, $\vec{\varepsilon}_{p+i} = \vec{e}_i - \eta_{is}\bar{\eta}^{sj}\vec{e}_{n+j}$, $\vec{\varepsilon}_{p+\alpha} = \vec{e}_\alpha$,$\vec{\varepsilon}_{n+\alpha} = \vec{e}_{n+\alpha}$, $\vec{\varepsilon}_i \in T_p(x)$, and $\eta_{ij}, \bar{\eta}_{ij}$ are metrical tensors of the the surfaces V_p and $\overline{V}_p$, respectively. Then metrical tensor of the surface V_p^* will be $g_{ij} = \eta_{ij} + \bar{\eta}_{ij}$. The vectors $\vec{e}_{p+i}, \vec{e}_{p+\alpha}, \vec{e}_{n+\alpha}$ determine a plane $N_{2n-p}(x)$, which is an orthogonal complement of the plane $T(x)$ in the space E_{2n}. Infinitesimal displacement of the frame R are defined by the following equations:

$$d\overline{x} = \theta^i \vec{\varepsilon}_i, \tag{2.3}$$

$$d\vec{\varepsilon}_h = \theta_h^i \vec{\varepsilon}_i + \theta_h^{p+j} \vec{\varepsilon}_{p+j} + \theta_h^{p+\alpha} \vec{\varepsilon}_{p+\alpha} + \theta_h^{n+\alpha} \vec{\varepsilon}_{n+\alpha} (h = i, p+i, p+\alpha, n+\alpha) \tag{2.4}$$

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The frames R_1, R_2 and R are agreed and therefore the following system of differential equations is obtained:

$$\omega^i = \bar{\omega}^i = \theta^i, \omega_i^\alpha = a_{ij}^\alpha \omega^j, a_{ij}^\alpha = a_{ji}^\alpha, \omega^\alpha = 0, \overline{\omega}^\alpha = 0, \tag{2.5}$$

$$\overline{\omega}_i^\alpha = b_{ij}^\alpha \omega^j, b_{ij}^\alpha = b_{ji}^\alpha, \theta^{p+i} = 0, \theta^{p+\alpha} = 0, \theta^{n+\alpha} = 0, \tag{2.6}$$

$$\theta_k^{p+i} = c_{kj}^{p+i}\theta^j, c_{kj}^{p+i} = c_{jk}^{p+i}, \theta_k^{p+\alpha} = c_{kj}^{p+\alpha}\theta^j, c_{kj}^{p+\alpha} = c_{jk}^{p+\alpha}, \tag{2.7}$$

$$\theta_k^{n+\alpha} = c_{kj}^{n+\alpha}\theta^j, c_{kj}^{n+\alpha} = c_{jk}^{n+\alpha}, \omega_i^j = \theta_i^j + \theta_i^{p+j}, \tag{2.8}$$

$$\omega_i^\alpha = \theta_i^{p+\alpha}, \bar{\omega}_i^j = \theta_i^j + \theta_i^{p+k}\eta_{ks}\bar{\eta}^{sj}, \bar{\omega}_i^\alpha = \theta_i^{n+\alpha} \tag{2.9}$$

$$\theta_{n+\alpha}^{p+\beta} = 0, \omega_\alpha^\beta + \omega_\beta^\alpha = 0 (\alpha \neq \beta), \overline{\omega}_\alpha^\beta + \overline{\omega}_\beta^\alpha = 0 (\alpha \neq \beta), \tag{2.10}$$

$$\omega_\alpha^\alpha = 0, \overline{\omega}_\alpha^\alpha = 0. \tag{2.11}$$

If we consider in the plane $T_p(x_2)$ a unit sphere

$$\bar{\eta}_{ij}\overline{u}^i\overline{u}^j = 1, \tag{2.12}$$

then in the transformation K_{af} this sphere is obtained from an ellipsoid of plane $T_p(x_1)$, which is called *the deformation ellipsoid* [2]. Main axes of this ellipsoid are defined from the system of equations

$$(\bar{\eta}_{ij} - \mu\eta_{ij})\omega^j = 0, \tag{2.13}$$

where μ is a root of the equation

$$\det \|\bar{\eta}_{ij} - \mu\eta_{ij}\| = 0. \tag{2.14}$$

System of p linearly independent fields of main directions of the ellipsoids of deformation define an orthogonal net σ_p in, Ω which corresponds to the orthogonal net $\overline{\sigma}_p = T(\sigma_p)$ in the domain $\bar{\Omega} \subset \overline{V}_p$.
In the case when all the semiaxes of the ellipsoid of deformation are different, for all points of the domain under consideration, the nets σ_p and $\overline{\sigma}_p$, are defined uniquely. This transformation should be included into the type $(1, 1, 1, ..., 1)$, based on the classification introduced by Melzi [3]. Bazylev V.T. calls the mappings of the mentioned type as simple [1].
Parameters of the characteristic directions of the mapping are defined through the system of equations

$$(\omega_i^k - \overline{\omega}_i^k)\omega^i = \lambda\omega^k. \tag{2.15}$$

From the equations (2.5),(2.6),(2.7),(2.8),(2.9) we find

$$a_{ij}^\alpha = c_{ij}^{p+\alpha}, b_{ij}^\alpha = c_{ij}^{n+\alpha}. \tag{2.16}$$

The equalities (2.16) show that there are $2(n-p)$ forms within $2n-p$ second quadratical forms of the surface V_p^*, which are transferred from V_p and $\overline{V}_p$ without any change.
Using the equalities (2.16), it is more suitable to denote the quadratical forms of the surface V_p^* as follows:

$$\phi^{p+k} = c_{ij}^{p+k}\omega^i\omega^j, \phi^{p+\alpha} = a_{ij}^\alpha\omega^i\omega^j, \phi^{n+\alpha} = b_{ij}^{n+\alpha}\omega^i\omega^j.$$

The quadratical forms $\phi^{p+\alpha}$, $\phi^{n+\alpha}$ are transferred from V_p and $\overline{V}_p$, respectively.
Suppose that within $2n-p$ quadratical forms ϕ^{p+k}, $\phi^{p+\alpha}$, $\phi^{n+\alpha}$ the forms ϕ^{p+k} are linearly independent. Then we have

$$\phi^{p+\alpha} = \lambda_i^{p+\alpha}\phi^{p+i}, \phi^{n+\alpha} = \lambda_i^{n+\alpha}\phi^{p+i}. \tag{2.17}$$

In the general case each of the surfaces V_p and $\overline{V}_p$ has $\frac{p(p+1)}{2}$ linearly independent quadratic forms. The conditions (2.17) mean that within $n-p$ quadratical asymptotic forms of the surfaces V_p and $\overline{V}_p$, which have in general $\frac{p(p+1)}{2}$ linearly independent quadratic forms, there are only p forms remaining independent, for each.
We can consider in the plane $N_{2n-p}^*(x)$ of the graph of V_p^* the vectors

$$\overrightarrow{c}_{ij} = c_{ij}^{p+k}\overrightarrow{\varepsilon}_{p+k} + \overrightarrow{a}_{ij} + \overrightarrow{b}_{ij} \tag{2.18}$$

$$\overrightarrow{a}_{ij} = a_{ij}^{p+k}\overrightarrow{e}_{p+k}, \overrightarrow{b}_{ij} = b_{ij}^{p+k}\overrightarrow{e}_{n+p+k}. \tag{2.19}$$

By taking into account (2.16), (2.17), from (2.18) we find

$$\overrightarrow{c}_{ij} = c_{ij}^{p+k}\left(\overrightarrow{\varepsilon}_{p+k} + \lambda_k^{n+i}\overrightarrow{\varepsilon}_{2p+i} + \lambda_k^{n+p+i}\overrightarrow{\varepsilon}_{n+p+i}\right). \tag{2.20}$$

Let us denote $\overrightarrow{c}_k = \overrightarrow{\varepsilon}_{p+k} + \lambda_k^{n+i}\overrightarrow{\varepsilon}_{2p+i} + \lambda_k^{n+p+i}\overrightarrow{\varepsilon}_{n+p+i}$. Consequently, the vectors $\overrightarrow{c}_k$ will be basis vectors of the main normal N_p^* of the surface V_p^*. By projecting orthogonally the the main normal N_p^* on E_n and $\bar{E}_n$, we find two planes

$$\tilde{N}_p(x_1) = [x_1, \overrightarrow{a}_i], \tilde{N}_p(x_2) = [x_2, \overrightarrow{b}_i],$$

where $\overrightarrow{a}_i = \overrightarrow{e}_i + \lambda_i^{n+k}\overrightarrow{e}_{p+k}$, $\overrightarrow{b}_i = -\eta_{is}\overline{\eta}^{sj}\overrightarrow{e}_{n+j} + \lambda_i^{n+p+k}\overrightarrow{e}_{n+p+k}$. We can show that the plane $\tilde{N}_p(x_1)$ $\left(\tilde{N}_p(x_2)\right)$ does not share a common line with the plane $T_p(x_1)$ and $N_p(x_1)$ ($T_p(x_2)$ and $N_p(x_2)$).
Let us fix a net Σ_p on the surface V_p. The the set $\overline{\Sigma}_p = T\left(\Sigma_p\right)$ on the surface $\overline{V}_p$ is fixed, too. Let us direct the vectors $\overrightarrow{e}_1, \overrightarrow{e}_2, ..., \overrightarrow{e}_p$ along the tangent lines of the lines of the net Σ_p at the point x_1. Then the vectors $\overrightarrow{e}_{n+1}, \overrightarrow{e}_{n+2}, ..., \overrightarrow{e}_{n+p}$ will be directed along the tangent lines of the lines of the net $\overline{\Sigma}_p$ at the point $x_2 = T(x_1)$. Consequently, linear forms ω_i^j $(i \neq j)$, $\overline{\omega}_i^j$ $(i \neq j)$ will be main that is

$$\omega_i^j = a_{ik}^j\omega^k, \overline{\omega}_i^j = b_{ik}^j\overline{\omega}^k. \tag{2.21}$$

Let us consider $p+1$ dimensional planes $[x, \overrightarrow{a}_i, \overrightarrow{e}_j]$ $\left(j = \overline{1,p}\right)$ in the space E_n. These planes have a common line with the planes $N_p(x_1)$. The directional vectors of these lines can be written as

$$\overrightarrow{\eta}_i = \lambda_i^{n+k}\overrightarrow{e}_{p+k}. \tag{2.22}$$

So, if quadratical forms ϕ^{p+k} of the surface V_p^* are linearly independent then each net Σ_p on the surface V_p is related in an invariant way to p linearly independent vector fields

$$\overrightarrow{\eta}_i = \lambda_i^{n+k}\overrightarrow{e}_{p+k}. \tag{2.23}$$

We will consider here the vector $\overrightarrow{\eta}_{i_0}$ as corresponding to the line ω_{i_0} of the net Σ_p. In an analogous way it is proved that the net $\overline{\Sigma}_p$ is related in an invariant way to linearly independent vector fields

$$\overrightarrow{\overline{\eta}}_i = \overline{\eta}_{is}\eta^{sj}\lambda_i^{n+p+k}\overrightarrow{e}_{n+p+k}. \tag{2.24}$$

The directional vectors of the associated nets Σ_p' and $\overline{\Sigma'}_p$ of the corresponding nets Σ_p and $\overline{\Sigma}_p$, have expansions

$$\vec{E}^i = \eta^{ij}\overrightarrow{e}_j, \overrightarrow{\overline{E}}^i = \overline{\eta}^{ij}\overrightarrow{e}_{n+j}. \tag{2.25}$$

Let us consider the planes $\Pi_{p+1}^{(i)}(x_1) = [x_1, \overrightarrow{a}_j, \vec{E}^i]$ $(i = \overline{1,p}; j = \overline{1,p})$ and their relative position with the plane $N_p(x_1)$. These planes share a common direction. So, the associated net $\overline{\Sigma}_p$ of the net Σ_p generates on $N_p(x_1)$ linearly independent vectors

$$\vec{N}^i = \eta^{ij}\lambda_i^{n+k}\overrightarrow{e}_{p+k}. \tag{2.26}$$

In an analogous way we can show that the planes $\Pi_{p+1}^{(k)}(x_2) = [x_2, \overrightarrow{b}_i, \overline{\eta}^{kj}\overrightarrow{e}_{n+j}]$ $(k = \overline{1,p})$ and $\overline{N}_p(x_2)$ are related to the vector

$$\overrightarrow{\overline{N}}^i = \eta^{is}\lambda_s^{n+p+k}\overrightarrow{e}_{n+p+k}. \tag{2.27}$$

We find that

$$\vec{N}^i = \eta^{ij}\overrightarrow{n}_j, \overrightarrow{\overline{n}}_i = \overline{\eta}_{is}\overrightarrow{\overline{N}}^s, \overrightarrow{\overline{N}}^k = \overline{\eta}^{ki}\overrightarrow{\overline{n}}_i. \tag{2.28}$$

Using the formulae (2.18) and (2.19), we find that the vectors of forced curvature of the lines of the net Σ_p and $\overline{\Sigma}_p$ have the following expansions:

$$\overrightarrow{a}_{ij} = c_{ij}^{p+s}\overrightarrow{n}_s, \overrightarrow{b}_{ij} = c_{ij}^{p+s}\eta_{se}\overline{\eta}^{se}\overrightarrow{\overline{n}}_t. \tag{2.29}$$

The conjugacy conditions of the net $\Sigma_p \subset V_p$ can be expressed by the equalities $\overrightarrow{a}_{ij} = 0$ $(i \neq j)$. Consequently, $c_{ij}^{p+s}\overrightarrow{n}_s = 0$ $(i \neq j)$. Here the vectors $\overrightarrow{n}_s$ are linearly independent, and therefore

$$c_{ij}^{p+s}\overrightarrow{n}_s = 0\,(i \neq j)\,. \tag{2.30}$$

From (2.30) and (2.18), (2.19) we find that $\overrightarrow{b}_{ij} = 0$ $(i \neq j)$, $\overrightarrow{c}_{ij} = 0$ $(i \neq j)$. So, the following theorem holds true.

Theorem 2.1. *If the net $\Sigma_p \subset V_p$ is conjugate then the corresponding nets $\overline{\Sigma}_p = T(\Sigma_p)$ and $\Sigma_p^* \subset V_p^*$ are also conjugate.*

It is known that [4] the conjugate nets Σ'_p and $\overline{\Sigma'}_p$ of the nets Σ_p and $\overline{\Sigma}_p$ will be conjugate if and only if

$$\eta^{it}\eta^{jk}a_{tk}^{p+s} = 0\,(i \neq j)\,, \bar{\eta}^{it}\bar{\eta}^{jk}b_{tk}^{p+s} = 0\,(i \neq j)\,. \tag{2.31}$$

From (2.16)-(2.19) we obtain $\eta^{it}\eta^{jk}c_{tk}^{p+e}\lambda_e^{n+s} = 0$, $\bar{\eta}^{it}\bar{\eta}^{jk}c_{tk}^{p+e}\lambda_e^{n+p+s} = 0$. By taking into account the fact that $\det \|\lambda_e^{n+s}\| \neq 0$, $\det \|\lambda_e^{n+p+s}\| \neq 0$ we obtain

$$\eta^{it}\eta^{jk}c_{tk}^{p+e} = 0, \bar{\eta}^{it}\bar{\eta}^{jk}c_{tk}^{p+e} = 0\,(i \neq j)\,. \tag{2.32}$$

Let the net be conjugate. Then $c_{ij}^{p+s} = 0$ $(i \neq j)$. Using this we obtain from (2.32) that

$$\sum_s \eta^{is}\eta^{js}c_{ss}^{p+e} = 0\,(i \neq j)\,, \sum_s \bar{\eta}^{is}\bar{\eta}^{js}c_{ss}^{p+e} = 0\,(i \neq j)\,, \tag{2.33}$$

where $\det \left\|c_{ss}^{p+l}\right\| \neq 0$. We obtain

$$\eta^{is}\eta^{js} = 0\,(i \neq j)\,, \bar{\eta}^{is}\bar{\eta}^{js} = 0\,(i \neq j)\,. \tag{2.34}$$

Let $i = s$. Then we have $\eta^{ss}\eta^{js} = 0$, $\bar{\eta}^{ss}\bar{\eta}^{js} = 0$ $(j \neq s)$. Consequently,we obtain $\eta^{js} = 0$ $(j \neq s)$, $\bar{\eta}^{js} = 0$ $(j \neq s)$. But this means that the nets Σ_p and $\overline{\Sigma}_p$ are orthogonal. The opposite is obvious. Therefore, we proved the following theorem.

Theorem 2.2. *The conjugate net Σ_p (or $\overline{\Sigma}_p = T(\Sigma_p)$) is a basis for the mapping T if and only if the nets Σ_p and $\overline{\Sigma}_p = T(\Sigma_p)$ are orthogonal.*

Let us consider the case when the associated nets Σ'_p and $\overline{\Sigma'}_p$, of the nets Σ_p and $\overline{\Sigma}_p = T(\Sigma_p)$, correspond in the mapping T. By demanding $\vec{E}^k \overset{(dT)_{x_1}}{\longrightarrow} \overrightarrow{\overline{E}}^k$ and using $\omega^i = \bar{\omega}^i$, we obtain $\bar{\eta}^{ij} = k\eta^{ij}$, and from this we obtain $\eta_{ij} = \bar{k}\bar{\eta}_{ij}$. Comsequently, the mapping T is conformal. The opposite is also true. Let the mapping T be conformal that is $\bar{\eta}_{ij} = \alpha\eta_{ij}$. By multipying both sides of this equality by η^{is} and then finding sum for i we obtain $\bar{\eta}_{ij}\eta^{is} = \alpha\eta_{ij}\eta^{is} = \alpha\delta_j^s$. Then we multiply both sides of the last equality by $\bar{\eta}^{je}$ and find their sum for j. We obtain $\bar{\eta}^{je}\bar{\eta}_{ij}\eta^{is} = \alpha\delta_j^s\bar{\eta}^{je} = \alpha\bar{\eta}^{se}$ that is $\overline{\Sigma'}_p = T\left(\Sigma'_p\right)$. So, the following theorem holds true.

Theorem 2.3. *$\overline{\Sigma'}_p = T\left(\Sigma'_p\right)$ if and only if the mapping T is conformal.*

Note: Here we assume that $\overline{\Sigma}_p = T(\Sigma_p)$.
Remark: The author studied in his previous paper similar problems for 4 dimensional spaces [5].

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