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SPECTRAL PROPERTIES OF A CONFORMABLE BOUNDARY VALUE PROBLEM ON TIME SCALES

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Abstract. We study a self-adjoint conformable dynamic equation of second order on an arbitrary time scale $\mathbb{T}$. We state an existence and uniqueness theorem for the solutions of this equation. We prove the conformable Lagrange identity on time scales. Then, we consider a conformable eigenvalue problem generated by the above-mentioned dynamic equation of second order and we examine some of the spectral properties of this boundary value problem.

1. Introduction

We study the self-adjoint conformable dynamic equation of second order

$$Lx = 0, \text{ where } Lx(t) = (px^{\Delta_\alpha})^{\Delta_\alpha}(t) + q(t)x^\sigma(t) \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$. Throughout we assume that $p, q \in C_{rd}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$.

Continuous conformable calculus is a natural extension of the usual calculus and has yielded several articles such as [1–6]. Some follow-up papers related to Sturm-Liouville equations for conformable calculus include [7–9].

Recently, researchers have started to deal with studies relating to conformable calculus on time scales (see [10–16]).

A conformable derivative on time scales was first introduced in [10] by the formula

$$f^{\Delta_\alpha}(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}, & \sigma(t) > t, \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} t^{1-\alpha}, & \sigma(t) = t. \end{cases} \tag{1.2}$$

Note that, if f is $\Delta-$differentiable at a right scattered point $t \in \mathbb{T}^\kappa_{[0,\infty)}$ [17], then f is $\alpha-$ conformable differentiable and for the above definition we have

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$$f^{\Delta_\alpha}(t) = t^{1-\alpha} f^{\Delta}(t) \tag{1.3}$$

where $f^{\Delta}(t) = \dfrac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}$.

The following formula [10, Theorem 4, iv] will be needed in the sequel:

$$f(\sigma(t)) = f(t) + \mu(t) t^{\alpha-1} f^{\Delta_\alpha}(t). \tag{1.4}$$

Assume $f, g : \mathbb{T} \to \mathbb{R}$ are conformable differentiable of order α. Then, if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is conformable differentiable with

$$(fg)^{\Delta_\alpha} = f^{\Delta_\alpha} g + (f o \sigma) g^{\Delta_\alpha} = f^{\Delta_\alpha}(g o \sigma) + f g^{\Delta_\alpha}, \tag{1.5}$$

if f and g are continuous, then $\dfrac{f}{g}$ is conformable differentiable with

$$\left(\frac{f}{g}\right)^{\Delta_\alpha} = \frac{f^{\Delta_\alpha} g - f g^{\Delta_\alpha}}{g(g o \sigma)} \tag{1.6}$$

valid at all points $t \in \mathbb{T}^{\kappa}$ for which $g(t) g(\sigma(t)) \neq 0$.

This paper consists of five sections. After this Introduction part, we will state an existence and uniqueness theorem in Section 2. In Section 3, we will recall the definition of α-Wronskian and we prove the conformable Lagrange identity after defining the Lagrange bracket of two functions. In Section 4, we will consider a conformable boundary value problem and we will investigate some of its spectral properties after proving the Green's theorem. Finally, we conclude the paper in Section 5 by giving some remarks about the current paper.

2. A Conformable Dynamic Equation

In this section, we will investigate the self-adjoint conformable dynamic equation (1.1). First, we will state a theorem concerning the existence and uniqueness of solutions of initial value problems for $Lx(t) = f(t)$. Then, we will suggest a method to construct the solutions of (1.1).

Theorem 2.1. *If $f \in C_{rd}$, $t_0 \in \mathbb{T}$, and x_0, x_0^{α} are given constants, then the initial value problem*

$$Lx(t) = f(t), \ x(t_0) = x_0, \ x^{\Delta_\alpha}(t_0) = x_0^{\alpha} \tag{2.1}$$

has a unique solution that exists on the whole time scale $\mathbb{T}$.

Proof. Define $y(t) = p(t) x^{\Delta_\alpha}(t)$. From here we have

$$x^{\Delta_\alpha}(t) = \frac{y(t)}{p(t)}. \tag{2.2}$$

With the help of (1.1), (1.4) , (1.3) and (2.1) we have

$$\begin{aligned} y^{\Delta_\alpha}(t) &= (px^{\Delta_\alpha})^{\Delta_\alpha}(t) = f(t) - q(t)x^\sigma(t) \\ &= f(t) - q(t)\big(x(t) + \mu(t)t^{\alpha-1}x^{\Delta_\alpha}(t)\big) \\ &= f(t) - q(t)x(t) - q(t)\mu(t)t^{\alpha-1}x^{\Delta_\alpha}(t) \\ &= f(t) - q(t)x(t) - q(t)\mu(t)t^{\alpha-1}\frac{y(t)}{p(t)} \\ &= -q(t)x(t) - \frac{q(t)\mu(t)t^{\alpha-1}}{p(t)}y(t) + f(t). \end{aligned}$$

Let $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. From (1.3) We may compute

$$\begin{aligned} t^{1-\alpha}z^\Delta(t) &= t^{1-\alpha}\begin{bmatrix} x^\Delta(t) \\ y^\Delta(t) \end{bmatrix} = \begin{bmatrix} x^{\Delta_\alpha}(t) \\ y^{\Delta_\alpha}(t) \end{bmatrix} = \begin{bmatrix} \frac{y(t)}{p(t)} \\ f(t) - q(t)x(t) - q(t)\mu(t)t^{\alpha-1}\frac{y(t)}{p(t)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{y(t)}{p(t)} \\ -q(t)x(t) - q(t)\mu(t)t^{\alpha-1}\frac{y(t)}{p(t)} \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{p(t)} \\ -q(t) & \frac{-q(t)\mu(t)t^{\alpha-1}}{p(t)} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}. \end{aligned}$$

This is a system of dynamic equations whose components are rd-continuous. Therefore by Thorem 5.24 [18] we observe that a unique solution exists. □

Now, let us suggest a method to construct solutions of the self-adjoint conformable dynamic equation (1.1). To do this, first, assume that x is a nontrivial solution of (1.1). Then

$$x^2(t) + (px^{\Delta_\alpha})^2(t) \neq 0$$

holds for all $t \in \mathbb{T}$, and for each t, we can find real numbers $\varrho(t) > 0$ and $\varphi(t)$ with $0 \leq \varphi(t) \leq 2\pi$ such that the equations

$$x(t) = \varrho(t)\sin\varphi(t), \tag{2.3}$$

$$p(t)x^{\Delta_\alpha}(t) = \varrho(t)\cos\varphi(t) \tag{2.4}$$

are satisfied. We call (2.3) and (2.4) the Prüfer transformation.

Theorem 2.2. *If x is a nontrivial solution of* (1.1) *and if ϱ and φ are defined by* (2.3) *and* (2.4)*, then the equations*

$$\begin{aligned} \varrho^{\Delta_\alpha} &= \frac{\varrho}{p}\cos\varphi(\sin\varphi)^\sigma - \varrho q\sin\varphi(\cos\varphi)^\sigma - \frac{\varrho\mu q t^{\alpha-1}}{p}\cos\varphi(\cos\varphi)^\sigma \\ &- \varrho(\sin\varphi)^{\Delta_\alpha}(\cos\varphi)^\sigma - \varrho(\cos\varphi)^{\Delta_\alpha}(\cos\varphi)^\sigma, \end{aligned} \tag{2.5}$$

$$\begin{aligned} (\sin\varphi)^{\Delta_\alpha}(\cos\varphi)^\sigma &- (\cos\varphi)^{\Delta_\alpha}(\sin\varphi)^\sigma = \frac{1}{p}\cos\varphi(\cos\varphi)^\sigma \\ &+ q\sin\varphi(\sin\varphi)^\sigma + \frac{\mu q t^{\alpha-1}}{p}\cos\varphi(\sin\varphi)^\sigma \end{aligned} \tag{2.6}$$

are satisfied.

Proof. Using the product rule for (2.3) yields

$$\begin{aligned}\varrho^{\Delta_\alpha}(\sin\varphi)^\sigma + \varrho(\sin\varphi)^{\Delta_\alpha} = (\varrho\sin\varphi)^{\Delta_\alpha} = x^{\Delta_\alpha} &= \frac{1}{p}(px^{\Delta_\alpha}) \\ &= \frac{1}{p}\varrho\cos\varphi,\end{aligned}$$

while doing the same for (2.4) implies

$$\begin{aligned}\varrho^{\Delta_\alpha}(\cos\varphi)^\sigma + \varrho(\cos\varphi)^{\Delta_\alpha} = (\varrho\cos\varphi)^{\Delta_\alpha} &= (px^{\Delta_\alpha})^{\Delta_\alpha} \\ &= -q(x + \mu t^{\alpha-1}x^{\Delta_\alpha}) \\ &= -qx - \frac{\mu q t^{\alpha-1}}{p}(px^{\Delta_\alpha}) \\ &= -q\varrho\sin\varphi - \frac{\mu q t^{\alpha-1}}{p}\varrho\cos\varphi,\end{aligned}$$

where we have also used that x is a solution of (1.1) and got help from equation (1.4). Hence we obtain the two equations

$$\varrho^{\Delta_\alpha}(\sin\varphi)^\sigma + \varrho(\sin\varphi)^{\Delta_\alpha} = \frac{1}{p}\varrho\cos\varphi, \tag{2.7}$$

$$\varrho^{\Delta_\alpha}(\cos\varphi)^\sigma + \varrho(\cos\varphi)^{\Delta_\alpha} = -q\rho\sin\varphi - \frac{\mu q t^{\alpha-1}}{p}\varrho\cos\varphi. \tag{2.8}$$

We now multiply (2.7) by $(\sin\varphi)^\sigma$ and (2.8) by $(\cos\varphi)^\sigma$ and add the resulting equations to obtain (2.5). To verify (2.6), we multiply (2.7) by $(\cos\varphi)^\sigma$ and (2.8) by $-(\sin\varphi)^\sigma$ and add the resulting equations. Dividing the obtained equation by $\varrho > 0$ directly yields (2.6). ☐

Observe that the conformable dynamic equation (2.6) for φ is independent of ϱ. Of course it might be difficult to solve this equation, but once a solution of (2.6) is obtained, the linear conformable dynamic equation (2.5) for ϱ is readily solved.

3. Conformable Lagrange Identity on Time Scales

In this section, we collect some knowledge needed in the rest of this paper. First, we introduce the definition of α-Wronskian and give one of its properties, then we define the Lagrange bracket of two functions and prove the conformable Lagrange identity on time scales.

Definition 3.1. If $x, y : \mathbb{T} \to \mathbb{R}$ are conformable differentiable on $\mathbb{T}^\kappa$, then we define the α-Wronskian of x and y by

$$W_\alpha(x,y)(t) = det\begin{pmatrix} x(t) & y(t) \\ x^{\Delta_\alpha}(t) & y^{\Delta_\alpha}(t)\end{pmatrix}$$

for $t \in \mathbb{T}^\kappa$.

The following lemma will be used in the proof of Theorem 3.4.

Lemma 3.2. *If $x, y : \mathbb{T} \to \mathbb{R}$ are conformable differentiable on $\mathbb{T}^\kappa$, then*

$$W_\alpha(x,y)(t) = det\begin{pmatrix} x^\sigma(t) & y^\sigma(t) \\ x^{\Delta_\alpha}(t) & y^{\Delta_\alpha}(t)\end{pmatrix}$$

holds for $t \in \mathbb{T}^\kappa$.

Proof. For $t \in \mathbb{T}^\kappa$, we have by equation (1.4)

$$\begin{aligned} det\begin{pmatrix} x^\sigma(t) & y^\sigma(t) \\ x^{\Delta_\alpha}(t) & y^{\Delta_\alpha}(t)\end{pmatrix} &= det\begin{pmatrix} x(t)+\mu(t)t^{\alpha-1}x^{\Delta_\alpha}(t) & y(t)+\mu(t)t^{\alpha-1}y^{\Delta_\alpha}(t) \\ x^{\Delta_\alpha}(t) & y^{\Delta_\alpha}(t)\end{pmatrix} \\ &= det\begin{pmatrix} x(t) & y(t) \\ x^{\Delta_\alpha}(t) & y^{\Delta_\alpha}(t)\end{pmatrix} \\ &= W_\alpha(x,y)(t) \end{aligned}$$

which gives us the desired result. □

Now, let us define the Lagrange bracket of two conformable functions, before proving the conformable Lagrange identity.

Definition 3.3. If $x, y : \mathbb{T} \to \mathbb{R}$ are conformable differentiable on $\mathbb{T}^\kappa$, then the Lagrange bracket of x and y is defined by

$$\{x; y\}(t) = p(t)W_\alpha(x, y)(t)$$

for $t \in \mathbb{T}^\kappa$.

Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta_\alpha} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$ is rd-continuous. A function $x \in \mathbb{D}$ is then said to be a solution of (1.1) provided $Lx(t) = 0$ holds for all $t \in \mathbb{T}^{\kappa^2}$. The below-given theorem follows immediately.

Theorem 3.4 (Conformable Lagrange Identity)**.** *If $x, y \in \mathbb{D}$, then*

$$x^\sigma(t)Ly(t) - y^\sigma(t)Lx(t) = \{x; y\}^{\Delta_\alpha}(t)$$

holds for $t \in \mathbb{T}^{\kappa^2}$.

Proof. By the product rule (1.5), we have

$$\begin{aligned} \{x; y\}^{\Delta_\alpha} &= \{xpy^{\Delta_\alpha} - px^{\Delta_\alpha}y\}^{\Delta_\alpha} \\ &= x^\sigma(py^{\Delta_\alpha})^{\Delta_\alpha} + (py^{\Delta_\alpha})x^{\Delta_\alpha} - y^\sigma(px^{\Delta_\alpha})^{\Delta_\alpha} - y^{\Delta_\alpha}(px^{\Delta_\alpha}) \\ &= x^\sigma(py^{\Delta_\alpha})^{\Delta_\alpha} - y^\sigma(px^{\Delta_\alpha})^{\Delta_\alpha} \\ &= x^\sigma(py^{\Delta_\alpha})^{\Delta_\alpha} - y^\sigma(px^{\Delta_\alpha})^{\Delta_\alpha} + qx^\sigma y^\sigma - qx^\sigma y^\sigma \\ &= x^\sigma((py^{\Delta_\alpha})^{\Delta_\alpha} + qy^\sigma) - y^\sigma((px^{\Delta_\alpha})^{\Delta_\alpha} + qx^\sigma) \\ &= x^\sigma Ly - y^\sigma Lx \end{aligned}$$

on $\mathbb{T}^{\kappa^2}$. □

4. A Conformable Boundary Value Problem and Green's Function

In this section, we consider a conformable boundary value problem of the form

$$Lx + \lambda x^\sigma = 0, \; R_a(x) = R_b(x) = 0, \tag{4.1}$$

where $Lx = x^{\Delta_\alpha\Delta_\alpha} + qx^\sigma$ such that $q : \mathbb{T} \to \mathbb{R}$ is rd-continuous, and

$$R_a(x) = \gamma_1 x(\rho(a)) + \gamma_2 x^{\Delta_\alpha}(\rho(a)), \; R_b(x) = \delta_1 x(\rho(b)) + \delta_2 x^{\Delta_\alpha}(\rho(b))$$

such that $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ with $(\gamma_1^2 + \gamma_2^2)(\delta_1^2 + \delta_2^2) \neq 0$ hold.

A number $\lambda \in \mathbb{R}$ is called an eigenvalue of (4.1) provided there exists a nontrivial solution x of the conformable boundary value problem (4.1). Such an x is then called an eigenfunction corresponding to the eigenvalue λ.

We define the inner product of x and y on $[\rho(a), b]$ by

$$\langle x, y\rangle = \int_{\rho(a)}^{b} x(t)y(t)\Delta_\alpha t := \int_{\rho(a)}^{b} x(t)y(t)t^{\alpha-1}\Delta t,$$

and we say that x and y are orthogonal on $[\rho(a), b]$ provided $\langle x, y\rangle = 0$. The norm of x is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

The next theorem follows immediately from Theorem 3.4.

Theorem 4.1 (Green's Theorem)**.** *If $x, y \in \mathbb{D}$, then*

$$\langle x^\sigma, Ly\rangle - \langle y^\sigma, Lx\rangle = \{x; y\}(b) - \{x; y\}(a)$$

holds.

Proof. This results from integrating both sides of the conformable Lagrange identity in Theorem 3.4 on $[a, b]$ and using the definition of the inner product. □

It is easy to see that $W_\alpha(x, y)(\rho(a)) = 0$ if $R_a(x) = R_a(y) = 0$ and $W_\alpha(x, y)(b) = 0$ if $R_b(x) = R_b(y) = 0$. Indeed, from Definiton 3.1 we have

$$\begin{aligned} W_\alpha(x, y)(\rho(a)) &= det\begin{pmatrix} x(\rho(a)) & y(\rho(a)) \\ x^{\Delta_\alpha}(\rho(a)) & y^{\Delta_\alpha}(\rho(a)) \end{pmatrix} \\ &= det\begin{pmatrix} -\frac{\gamma_2}{\gamma_1}x^{\Delta_\alpha}(\rho(a)) & -\frac{\gamma_2}{\gamma_1}y^{\Delta_\alpha}(\rho(a)) \\ x^{\Delta_\alpha}(\rho(a)) & y^{\Delta_\alpha}(\rho(a)) \end{pmatrix} = 0. \end{aligned}$$

The fact that $R_b(x) = R_b(y) = 0$ implies $W_\alpha(x, y)(b) = 0$ follows similarly.

Now, we shall provide a characterization for the eigenvalues of (4.1). For this, we denote the unique solutions (see Theorem 2.1) of the conformable initial value problem

$$Lx + \lambda x^\sigma = 0,\ x(\rho(a)) = \gamma_2,\ x^{\Delta_\alpha}(\rho(a)) = -\gamma_1$$

by $x(\cdot, \lambda)$, where $\lambda \in \mathbb{R}$, and we put $\Lambda(\lambda) = R_b(x(\cdot, \lambda))$. With this notation in mind, we have the following.

Theorem 4.2. *λ is an eigenvalue of* (4.1) *if and only if* $\Lambda(\lambda) = 0$.

Proof. If $R_b(x(\cdot, \lambda)) = 0$, then $x = x(\cdot, \lambda)$ satisfies

$$Lx + \lambda x^\sigma = 0,\ R_a(x) = R_b(x) = 0,$$

i.e., λ is an eigenvalue of (4.1). Conversely, let $\lambda \in \mathbb{R}$ be an eigenvalue of (4.1) with corresponding eigenfunction x. Then because of the unique solvability of the conformable initial value problem (observe $R_a(x) = 0$), the equation $x = cx(\cdot, \lambda)$ holds with $c = \dfrac{\gamma_2 x(\rho(a)) - \gamma_1 x^{\Delta_\alpha}(\rho(a))}{\gamma_1^2 + \gamma_2^2}$. Hence $R_b(x(\cdot, \lambda)) = 0$ which gives us the desired result. □

The proof of Theorem 4.2 also shows that all eigenvalues of (4.1) are simple.

5. Conclusion

In this paper, we deal with a self-adjoint conformable dynamic equation of second order on an arbitrary time scale. We prove an existence and uniqueness theorem for the solutions of this equation and we suggest a method to construct these solutions via Prüfer transformation. Then we prove the conformable Lagrange identity. After that, we derive a conformable boundary value problem which consists of the above-mentioned conformable dynamic equation and boundary conditions. We prove Green's theorem with the help of conformable Lagrange identity and we provide a characterization for the eigenvalues of this conformable boundary value problem. Presented results of this paper are generalizations of some results in [17] via conformable derivative.

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