Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (3) (2021), 710 – 720
DOI : 10.15672/hujms.695858

Research Article

Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving a modified sigmoid activation function

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Abstract

The aim of this paper is to introduce some special families of holomorphic and Sălăgean type bi-univalent functions by making use of Horadam polynomials involving the modified sigmoid activation function $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$ in the open unit disc $\mathfrak{D}$. We investigate the upper bounds on initial coefficients for functions of the form $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j$, in these newly introduced special families and also discuss the Fekete-Szegö problem. Some interesting consequences of the results established here are also indicated.

Mathematics Subject Classification (2020). Primary 11B39, 30C45, 33C45; Secondary 30C50, 33C05

Keywords. holomorphic function, bi-univalent function, Fekete - Szegö inequality, Horadam polynomials, modified sigmoid function

1. Introduction and preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Let $\mathcal{A}$ be the family of normalized functions that have the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the collection of all members of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. It is well- known (see[6]) that every function $g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $z \in \mathfrak{D}$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2 d_3 + d_4)\omega^4 + \dots . \tag{1.2}$$

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Received: 28.02.2020; Accepted: 28.10.2020

A member g of $\mathcal{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both g and g^{-1} are univalent in $\mathfrak{D}$. We denote the family of bi-univalent functions having the form (1.1), by $\sum$. For detailed investigations of the family $\sum$ and various subfamilies of the family $\sum$, one can see the works of [3–5, 13, 16].

We recall the principle of subordination between two holomorphic functions $g(z)$ and $f(z)$ in $\mathfrak{D}$. It is known that $g(z)$ is subordinate to $f(z)$, written as $g(z) \prec f(z)$, $z \in \mathfrak{D}$, if there is a $\psi(z)$ holomorphic in $\mathfrak{D}$, such that $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathfrak{D}$, such that $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. In particular, if f is univalent in $\mathfrak{D}$, $g(z) \prec f(z) \Leftrightarrow g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Recently, Hörzum and Koçer [12] (See also [11]) examined the Horadam polynomials $h_j(x, a, b; p, q)$ (or briefly $h_j(x)$). It is defined by the recurrence relation

$$h_j(x) = pxh_{j-1}(x) + qh_{j-2}(x), \quad h_1(x) = a, \ h_2(x) = bx, \tag{1.3}$$

where $j \in \mathbb{N} - \{1, 2\}$, $x \in \mathbb{R}, a, b, p$ and q are real constants. It is very clear from (1.3) that $h_3(x) = pbx^2 + qa$. The generating function of the Horadam polynomials $h_j(x)$ is as given below (see [12]):

$$\mathcal{G}(x, z) := \sum_{j=1}^{\infty} h_j(x) z^{j-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}, \tag{1.4}$$

where $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$, that is $x \neq \Re(z)$.

Few particular cases of Horadam polynomials $h_j(x, a, b; p, q)$ are:
i) $h_j(x, 1, 1; 1, 1) = F_j(x)$, the Fibonacci polynomials, *ii)* $h_j(x, 2, 1; 1, 1) = L_j(x)$, the Lucas polynomials, *iii)* $h_j(x, 1, 2; 2, 1) = P_j(x)$, the Pell polynomials, *iv)* $h_j(x, 2, 2; 2, 1) = Q_j(x)$, the Pell-Lucas polynomials, *v)* $h_j(x, 1, 1; 2, -1) = T_j(x)$, the first kind Chebyshev polynomials and *vi)* $h_j(x, 1, 2; 2, -1) = U_j(x)$, the second kind Chebyshev polynomials.

In the literature, estimates on $|d_2|$, $|d_3|$ and celebrated Fekete- Szegö inequality were found for bi-univalent functions associated with certain polynomials like the second kind Chebyshev polynomials and Horadam polynomials. We also note that the above polynomials and other special polynomials are potentially important in engineering, mathematical, statistical and physical sciences. More details associated with these polynomials can be found in [9–11, 15, 19]. Additional informations about Fekete-Szegö problem associated with Haradam polynomials are available with the works of [2] and [18].

Let $\mathcal{A}_\phi$ denote the family of functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \frac{2}{1 + e^{-s}} d_j z^j = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s) = \frac{2}{1+e^{-s}}, s \geq 0$, is a modified sigmoid function. Clearly $\phi(0) = 1$ and hence $\mathcal{A}_1 := \mathcal{A}$ (see [7]). For $g_\phi \in \mathcal{A}_\phi$, $k \in \mathbb{N} \cup \{0\}$, Sălăgean type differential operator $D^k : \mathcal{A}_\phi \to \mathcal{A}_\phi$, is defined by

$$D^0 g_\phi(z) = g_\phi(z),\ D^1 g_\phi(z) = zg'_\phi(z),\ ...,\ D^k g_\phi(z) = D(D^{k-1} g_\phi(z)), z \in \mathfrak{D}.$$

It is easy to see that if $g_\phi(z) = z + \sum\limits_{j=2}^{\infty} \phi(s) d_j z^j \in \mathcal{A}_\phi$, $z \in \mathfrak{D}$, then

$$D^k g_\phi(z) = z + \sum_{j=2}^{\infty} j^k \phi(s) d_j z^j.$$

When $\phi(s) = 1$, we have the Sălăgean differential operator [14].

Inspired by recent trends on bi-univalent functions, we define the following special families of $\sum$ by making use of the Horadam polynomials $h_j(x)$, which are given by the recurrence relation (1.3) and the generating function (1.4).

Definition 1.1. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$, $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\mu \geq \gamma$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}, s \geq 0$, if

$$\frac{z(D^k g_\phi(z))' + \mu z^2 (D^k g_\phi(z))''}{(1-\gamma)D^k g_\phi(z) + \gamma z (D^k g_\phi(z))'} \prec \mathcal{G}(x,z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$\frac{\omega(D^k f_\phi(\omega))' + \mu \omega^2 (D^k f_\phi(\omega))''}{(1-\gamma)D^k f_\phi(\omega) + \gamma \omega (D^k f_\phi(\omega))'} \prec \mathcal{G}(x,\omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

It is interesting to note that the special values of γ and μ lead the family $\mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$ to the following various subfamilies:

1. For $\gamma = \mu = \frac{1}{2}$, we get the family $\mathscr{K}_{\sum}(x,k,\phi(s)) = \mathfrak{S}_{\sum}(x,\frac{1}{2},\frac{1}{2},k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying
$\frac{(z^2(D^k g_\phi(z))')'}{(zD^k g_\phi(z))'} \prec \mathcal{G}(x,z) + 1 - a$ and $\frac{(\omega^2(D^k f_\phi(\omega))')'}{(\omega D^k f_\phi(\omega))'} \prec \mathcal{G}(x,\omega) + 1 - a,\ z,\ \omega \in \mathfrak{D}$,
where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

2. When $\gamma = 0, \mu = \frac{1}{2}$, we obtain the family $\mathscr{J}_{\sum}(x,k,\phi(s)) = \mathfrak{S}_{\sum}(x,0,\frac{1}{2},k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying
$\frac{(z^2(D^k g_\phi(z))')'}{2\,D^k g_\phi(z)} \prec \mathcal{G}(x,z) + 1 - a$ and $\frac{(\omega^2(D^k f_\phi(\omega))')'}{2\,D^k f_\phi(\omega)} \prec \mathcal{G}(x,\omega) + 1 - a,\ z,\ \omega \in \mathfrak{D}$,
where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

3. On putting $\gamma = \frac{1}{2}, \mu = 1$, we have the family $\mathscr{L}_{\sum}(x,k,\phi(s)) = \mathfrak{S}_{\sum}(x,\frac{1}{2},1,k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$\frac{2z(z(D^k g_\phi(z))')'}{(zD^k g_\phi(z))'} \prec \mathcal{G}(x,z) + 1 - a,\ z \in \mathfrak{D} \text{ and } \frac{2\omega(\omega(D^k f_\phi(\omega))')'}{(\omega D^k f_\phi(\omega))'} \prec \mathcal{G}(x,\omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

4. On taking $\gamma = 0$, we get the family $\mathscr{P}_{\sum}(x,\mu,k,\phi(s)) = \mathfrak{S}_{\sum}(x,0,\mu,k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$\left(\frac{z(D^k g_\phi(z))'}{D^k g_\phi(z)}\right)\left(1 + \mu\frac{z(D^k g_\phi(z))''}{(D^k g_\phi(z))'}\right) \prec \mathcal{G}(x,z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$\left(\frac{\omega(D^k f_\phi(\omega))'}{D^k f_\phi(\omega)}\right)\left(1 + \mu\frac{\omega(D^k f_\phi(\omega))''}{(D^k f_\phi(\omega))'}\right) \prec \mathcal{G}(x,\omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

Definition 1.2. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{L}_{\sum}(x,\gamma,\mu,k,\phi(s))$, $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\mu \geq \gamma$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}, s \geq 0$, if

$$\frac{z(D^k g_\phi(z))' + \mu z^2 (D^k g_\phi(z))''}{(1-\gamma)z + \gamma z (D^k g_\phi(z))'} \prec \mathcal{G}(x,z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$\frac{\omega(D^k f_\phi(\omega))' + \mu \omega^2 (D^k f_\phi(\omega))''}{(1-\gamma)\omega + \gamma \omega (D^k f_\phi(\omega))'} \prec \mathcal{G}(x,\omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

It is easy to observe that the special values of γ lead the family $\mathfrak{L}_{\sum}(x, \gamma, \mu, k, \phi(s))$ to the following various subfamilies:

1. For $\gamma = 0$, we get the family $\mathfrak{K}_{\sum}(x, \mu, k, \phi(s)) = \mathfrak{L}_{\sum}(x, 0, \mu, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$(D^k g_\phi(z))' + \mu z(D^k g_\phi(z))'' \prec \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$(D^k f_\phi(\omega))' + \mu\omega(D^k f_\phi(\omega))'' \prec \mathcal{G}(x, \omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

2. When $\gamma = 1$, we have the family $\mathfrak{J}_{\sum}(x, \mu, k, \phi(s)) = \mathfrak{L}_{\sum}(x, 1, \mu, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$1 + \mu\left(\frac{z(D^k g_\phi(z))''}{(D^k g_\phi(z))'}\right) \prec \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$1 + \mu\left(\frac{\omega(D^k f_\phi(\omega))''}{(D^k f_\phi(\omega))'}\right) \prec \mathcal{G}(x, \omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

Definition 1.3. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{B}_{\sum}(x, \xi, \tau, k, \phi(s))$, $\xi \geq 1$, $\tau \geq 1$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, if

$$\frac{(1-\xi) + \xi[(z(D^k g_\phi(z))')']^\tau}{(D^k g_\phi(z))'} \prec \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$\frac{(1-\xi) + \xi[(\omega(D^k f_\phi(\omega))')']^\tau}{(D^k f_\phi(\omega))'} \prec \mathcal{G}(x, \omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

Note that the particular values of ξ and τ lead the family $\mathfrak{B}_{\sum}(x, \xi, \tau, k, \phi(s))$ to the following two subfamilies:

1. When $\tau = 1$, we have the family $\mathscr{M}_{\sum}(x, \xi, k, \phi(s)) = \mathfrak{B}_{\sum}(x, \xi, 1, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$(1-\xi)\frac{1}{(D^k g_\phi(z))'} + \xi\left(1 + \frac{z(D^k g_\phi(z))''}{(D^k g_\phi(z))'}\right) \prec \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D}$$

and

$$(1-\xi)\frac{1}{(D^k f_\phi(\omega))'} + \xi\left(1 + \frac{\omega(D^k f_\phi(\omega))''}{(D^k f_\phi(\omega))'}\right) \prec \mathcal{G}(x, \omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

2. For $\xi = 1$, we have the family $\mathfrak{N}_{\sum}(x, \tau, k, \phi(s)) = \mathfrak{B}_{\sum}(x, 1, \tau, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1.1) satisfying

$$\frac{[(z(D^k g_\phi(z))')']^\tau}{(D^k g_\phi(z))'} \prec \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D} \text{ and } \frac{[(\omega(D^k f_\phi(\omega))')']^\tau}{(D^k f_\phi(\omega))'} \prec \mathcal{G}(x, \omega) + 1 - a,\ \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), a, b, p and q are as in (1.3) and $\mathcal{G}$ is as in (1.4).

For functions of the form (1.1) belonging to these newly introduced families $\mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$, $\mathfrak{L}_{\sum}(x,\gamma,\mu,k,\phi(s))$ and $\mathfrak{B}_{\sum}(x,\xi,\tau,k,\phi(s))$, we derive the estimates for the coefficients $|d_2|$ and $|d_3|$ and also consider the celebrated Fekete- Szegö problem [8] in Section 2.

2. Coefficient estimates and Fekete-Szegö inequality

We obtain coefficient estimates in the following theorem for functions in $\mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$.

Theorem 2.1. *Let $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}$, $s \ge 0$. If $g(z)$ of the form (1.1) is in $\mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$, then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|((1-\gamma)(1-\gamma+2\mu)+2\mu)(bx)^2-(1-\gamma+2\mu)^2(pbx^2+qa)|}}, \tag{2.1}$$

$$|d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{(bx)^2}{(1-\gamma+2\mu)^2} + \frac{|bx|}{2(1-\gamma+3\mu)}\right] \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|bx|}{2(3^k\phi(s))(1-\gamma+3\mu)} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \le J \\ \frac{|bx|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|((1-\gamma)(1-\gamma+2\mu)+2\mu)(bx)^2-(1-\gamma+2\mu)^2(pbx^2+qa)|} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \ge J, \end{cases} \tag{2.3}$$

where

$$J = \frac{1}{2(1-\gamma+3\mu)}\left|(1-\gamma)(1-\gamma+2\mu)+2\mu-(1-\gamma+2\mu)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Proof. Let $g(z) \in \mathfrak{S}_{\sum}(x,\gamma,\mu,k,\phi(s))$. Then, for two holomorphic functions $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m}(0) = \mathfrak{n}(0) = 0$, $|\mathfrak{m}(z)| < 1$ and $|\mathfrak{n}(\omega)| < 1$, $z,\omega \in \mathfrak{D}$, and using Definition 1.1, we can write

$$\frac{z(D^kg_\phi(z))' + \mu z^2(D^kg_\phi(z))''}{(1-\gamma)D^kg_\phi(z) + \gamma z(D^kg_\phi(z))'} = \mathcal{G}(x,\mathfrak{m}(z)) + 1 - a,$$

and

$$\frac{\omega(D^kf_\phi(\omega))' + \mu\omega^2(D^kf_\phi(\omega))''}{(1-\gamma)D^kf_\phi(\omega) + \gamma\omega(D^kf_\phi(\omega))'} = \mathcal{G}(x,\mathfrak{n}((\omega)) + 1 - a.$$

Or, equivalently

$$\frac{z(D^kg_\phi(z))' + \mu z^2(D^kg_\phi(z))''}{(1-\gamma)D^kg_\phi(z) + \gamma z(D^kg_\phi(z))'} = 1 + h_1(x) - a + h_2(x)\mathfrak{m}(z) + h_3(x)(\mathfrak{m}(z))^2 + \ldots \tag{2.4}$$

and

$$\frac{\omega(D^kf_\phi(\omega))' + \mu\omega^2(D^kf_\phi(\omega))''}{(1-\gamma)D^kf_\phi(\omega) + \gamma\omega(D^kf_\phi(\omega))'} = 1 + h_1(x) - a + h_2(x)\mathfrak{n}(\omega) + h_3(x)(\mathfrak{n}(\omega))^2 + \ldots. \tag{2.5}$$

From (2.4) and (2.5), in view of (1.3), we obtain

$$\frac{z(D^kg_\phi(z))' + \mu z^2(D^kg_\phi(z))''}{(1-\gamma)D^kg_\phi(z) + \gamma z(D^kg_\phi(z))'} = 1 + h_2(x)\mathfrak{m}_1z + [h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2]z^2 + \ldots \tag{2.6}$$

and

$$\frac{\omega(D^kf_\phi(\omega))' + \mu\omega^2(D^kf_\phi(\omega))''}{(1-\gamma)D^kf_\phi(\omega) + \gamma\omega(D^kf_\phi(\omega))'} = 1 + h_2(x)\mathfrak{n}_1\omega + [h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2]\omega^2 + \ldots. \tag{2.7}$$

It is well known that if $|\mathfrak{m}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1, \quad z \in \mathfrak{D}$ and $|\mathfrak{n}(\omega)| = |\mathfrak{n}_1 \omega + \mathfrak{n}_2 \omega^2 + \mathfrak{n}_3 \omega^3 + ...| < 1, \quad \omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leq 1 \text{ and } |\mathfrak{n}_i| \leq 1 \quad (i \in \mathbb{N}). \tag{2.8}$$

Comparing the corresponding coefficients in (2.6) and (2.7), we have

$$2^k \phi(s)(1-\gamma+2\mu)d_2 = h_2(x)\mathfrak{m}_1 \tag{2.9}$$

$$2(3^k\phi(s))(1-\gamma+3\mu)d_3 - 2^{2k}\phi^2(s)(1+\gamma)(1-\gamma+2\mu)d_2^2 = h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2 \tag{2.10}$$

$$-2^k\phi(s)(1-\gamma+2\mu)d_2 = h_2(x)\mathfrak{n}_1 \tag{2.11}$$

$$-2(3^k\phi(s))(1-\gamma+3\mu)d_3 + 2^{2k}\phi^2(s)(\gamma^2-(4+2\mu)\gamma+(3+10\mu))d_2^2 = h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2. \tag{2.12}$$

From (2.9) and (2.11), we can easily see that

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \tag{2.13}$$

and also

$$2^{2k+1}\phi^2(s)(1-\gamma+2\mu)^2 d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(h_2(x))^2. \tag{2.14}$$

If we add (2.10) and (2.12), then we obtain

$$2^{2k+1}\phi^2(s)((1-\gamma)(1-\gamma+2\mu)+2\mu)d_2^2 = h_2(x)(\mathfrak{m}_2+\mathfrak{n}_2) + h_3(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.15}$$

Substituting the value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (2.14) in (2.15), we get

$$d_2^2 = \frac{(h_2(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2^{2k+1}\phi^2(s)\left[((1-\gamma)(1-\gamma+2\mu)+2\mu)(h_2(x))^2 - (1-\gamma+2\mu)^2 h_3(x)\right]}, \tag{2.16}$$

which yields (2.1) on using (2.8).

Using (2.13) in the subtraction of (2.12) from (2.10), we obtain

$$d_3 = \frac{2^{2k}\phi(s)}{3^k}d_2^2 + \frac{h_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{4(3^k\phi(s))(1-\gamma+3\mu)}. \tag{2.17}$$

Then in view of (2.14), (2.17) becomes

$$d_3 = \frac{(h_2(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(3^k\phi(s))(1-\gamma+2\mu)^2} + \frac{h_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{4(3^k\phi(s))(1-\gamma+3\mu)},$$

which yields (2.2) on using (2.8).

From (2.16) and (2.17), for $\delta \in \mathbb{R}$, we get

$$|d_3 - \delta d_2^2| =$$

$$|h_2(x)|\left|\left(T(\delta,x) + \frac{1}{4(3^k\phi(s))(1-\gamma+3\mu)}\right)\mathfrak{m}_2 + \left(T(\delta,x) - \frac{1}{4(3^k\phi(s))(1-\gamma+3\mu)}\right)\mathfrak{n}_2\right|,$$

where

$$T(\delta,x) = \frac{\left(\frac{2^{2k}\phi(s)}{3^k} - \delta\right)(h_2(x))^2}{2^{2k+1}\phi^2(s)\left[((1-\gamma)(1-\gamma+2\mu)+2\mu)(h_2(x))^2 - (1-\gamma+2\mu)^2 h_3(x)\right]}.$$

In view of (1.3), we conclude that

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|h_2(x)|}{2(3^k\phi(s))(1-\gamma+3\mu)} & ; 0 \leq |T(\delta,x)| \leq \frac{1}{4(3^k\phi(s))(1-\gamma+3\mu)} \\ 2|h_2(x)||T(\delta,x)| & ; |T(\delta,x)| \geq \frac{1}{4(3^k\phi(s))(1-\gamma+3\mu)}, \end{cases}$$

which yields (2.3). This evidently completes the proof of Theorem 2.1. □

Remark 2.2. The results obtained in Theorem 2.1 coincide with Corollary 1 and Corollary 3 obtained in [15], for $\mu = 0$, $\gamma = 0$, $k = 0$ and $\phi(s) = 1$.

In the following theorem, we find coefficient estimates for functions in $\mathfrak{L}_{\sum}(x,\gamma,\mu,k,\phi(s))$.

Theorem 2.3. *Let $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}, s \ge 0$. If $g(z)$ of the form (1.1) is in $\mathfrak{L}_{\sum}(x, \gamma, \mu, k, \phi(s))$, then*

$$|d_2| \le \frac{|b(x)|\sqrt{|b(x)|}}{2^k\phi(s)\sqrt{|(4\gamma^2 - (7+4\mu)\gamma + 3(1+2\mu))(bx)^2 - 4\vartheta^2(pbx^2+qa)|}}, \tag{2.18}$$

$$|d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{b^2x^2}{4\vartheta^2} + \frac{|b(x)|}{3(\vartheta+\mu)}\right] \tag{2.19}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|b(x)|}{3^{k+1}\phi(s)(\vartheta+\mu)} & ; |1 - \frac{3^k\delta}{2^{2k}\phi(s)}| \le M \\ \frac{|b(x)|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu))(bx)^2-4\vartheta^2(pbx^2+qa)|} & ; |1 - \frac{3^k\delta}{2^{2k}\phi(s)}| \ge M, \end{cases} \tag{2.20}$$

where

$$M = \frac{1}{3(\vartheta+\mu)}\left|(4\gamma^2 - (7+4\mu)\gamma + 3(1+2\mu)) - 4\vartheta^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \quad \text{and} \quad \vartheta = 1 - \gamma + \mu. \tag{2.21}$$

Proof. Let $g(z) \in \mathfrak{L}_{\sum}(x, \gamma, \mu, k, \phi(s))$. Then, for two holomorphic functions $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m}(0) = \mathfrak{n}(0) = 0$, $|\mathfrak{m}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1$, $z \in \mathfrak{D}$, $|\mathfrak{n}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1$, $\omega \in \mathfrak{D}$, and using Definition 1.2, we can write

$$\frac{z(D^k g_\phi(z))' + \mu z^2(D^k g_\phi(z))''}{(1-\gamma)z + \gamma z(D^k g_\phi(z))'} = \mathcal{G}(x, \mathfrak{m}(z)) + 1 - a \tag{2.22}$$

and

$$\frac{\omega(D^k f_\phi(\omega))' + \mu\omega^2(D^k f_\phi(\omega))''}{(1-\gamma)\omega + \gamma\omega(D^k f_\phi(\omega))'} = \mathcal{G}(x, \mathfrak{n}(\omega)) + 1 - a. \tag{2.23}$$

Following (2.4), (2.5), (2.6), and (2.7) in the proof of Theorem 2.1, one gets the following in view of (2.22) and (2.23):

$$2^{k+1}\phi(s)\vartheta d_2 = h_2(x)\mathfrak{m}_1 \tag{2.24}$$

$$3^{k+1}\phi(s)(\vartheta+\mu)d_3 - 2^{2k+2}\phi^2(s)\vartheta\gamma d_2^2 = h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2 \tag{2.25}$$

$$-2^{k+1}\phi(s)\vartheta\, d_2 = h_2(x)\mathfrak{n}_1 \tag{2.26}$$

$$-3^{k+1}\phi(s)(\vartheta+\mu)d_3 + 2^{2k+1}\phi^2(s)[2\gamma^2 - (5+2\mu)\gamma + 3(1+2\mu)]d_2^2 = h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2, \tag{2.27}$$

where ϑ is as in (2.21).

The results (2.18)-(2.20) of this theorem now follow from (2.24)-(2.27) by applying the procedure as in Theorem 2.1 with respect to (2.9)-(2.12). □

Remark 2.4. The results obtained in Theorem 2.3 conincide with results obtained in [1, Theorem 2.2] for $\mu = 0$ and $\gamma = 0$, $k = 0$ and $\phi(s) = 1$.

Remark 2.5. The results obtained in Theorem 2.3 conincide with Theorem 2.1 of [17] for $k = 0$ and $\phi(s) = 1$.

Theorem 2.6. *Let $\xi \ge 1$, $\tau \ge 1$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}, s \ge 0$. If $g(z)$ of the form (1.1) is in $\mathfrak{B}_{\sum}(x, \xi, \tau, k, \phi(s))$, then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|(8\xi\tau^2 - 7\xi\tau + 1)(bx)^2 - 4(2\xi\tau - 1)^2(pbx^2+qa)|}}, \tag{2.28}$$

$$|d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{(bx)^2}{4(2\xi\tau-1)^2} + \frac{|bx|}{3(3\xi\tau-1)}\right] \tag{2.29}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3^{k+1}\phi(s)(3\xi\tau-1)} & ; \quad |1 - \frac{3^k\delta}{2^{2k}\phi(s)}| \leq \Omega \\ \frac{\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right||bx|^3}{3^k\phi(s)|(8\xi\tau^2-7\xi\tau+1)(bx)^2-4(2\xi\tau-1)^2(pbx^2+qa)|} & ; \quad |1 - \frac{3^k\delta}{2^{2k}\phi(s)}| \geq \Omega, \end{cases} \tag{2.30}$$

where $\Omega = \frac{1}{(3\xi\tau-1)}\left|(8\xi\tau^2 - 7\xi\tau + 1) - 4(2\xi\tau - 1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Proof. Let $g(z) \in \mathfrak{B}_{\sum}(x, \xi, \tau, k, \phi(s))$. Then, for some analytic functions $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m}(0) = \mathfrak{n}(0) = 0$, $|\mathfrak{m}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1$, $z \in \mathfrak{D}$, $|\mathfrak{n}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1$, $\omega \in \mathfrak{D}$, and using Definition 1.3, we can write

$$\frac{(1-\xi) + \xi[(z(D^k g_\phi(z))')']^\tau}{(D^k g_\phi(z))'} = \mathcal{G}(x, \mathfrak{m}(z)) + 1 - a,\ z \in \mathfrak{D} \tag{2.31}$$

and

$$\frac{(1-\xi) + \xi[(\omega(D^k f_\phi(\omega))')']^\tau}{(D^k f_\phi(\omega))'} = \mathcal{G}(x, \mathfrak{n}(\omega)) + 1 - a,\ \omega \in \mathfrak{D}. \tag{2.32}$$

Following (2.4), (2.5), (2.6), and (2.7) in the proof of Theorem 2.1, one gets the following in view of (2.31) and (2.32):

$$(2\xi\tau - 1)2^{k+1}\phi(s)d_2 = h_2(x)\mathfrak{m}_1 \tag{2.33}$$

$$2^{2k+2}\phi^2(s)(2\xi\tau^2 - 4\xi\tau + 1)d_2^2 + 3^{k+1}\phi(s)(3\xi\tau - 1)d_3 = h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2 \tag{2.34}$$

$$-(2\xi\tau - 1)2^{k+1}\phi(s)d_2 = h_2(x)\mathfrak{n}_1 \tag{2.35}$$

$$2^{2k+1}\phi^2(s)(4\xi\tau^2 + \xi\tau - 1)d_2^2 - 3^{k+1}\phi(s)(3\xi\tau - 1)d_3 = h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2. \tag{2.36}$$

The results (2.28)-(2.30) of this theorem now follow from (2.33)-(2.36) by applying the procedure as in Theorem 2.1 with respect to (2.9)-(2.12). □

Remark 2.7. The results obtained in Theorem 2.6 conincide with Theorem 2.2 of [17] when $k = 0$ and $\phi(s) = 1$.

In the next section, we present some interesting consequences of our main results.

3. Corollaries and consequences

Corollary 3.1. *Let* $g(z)$ *be in the family* $\mathscr{K}_{\sum}(x, k, \phi(s))$. *Then*

$$|d_2| \leq \frac{2|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|7(bx)^2 - 9(pbx^2 + qa)|}},\ |d_3| \leq \frac{1}{3^k\phi(s)}\left[\frac{4b^2x^2}{9} + \frac{|bx|}{4}\right]$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{4(3^k\phi(s))} & ; \left|1 - \frac{3^k\delta}{2^{2k}\phi(s)}\right| \leq \frac{1}{16}\left|7 - 9\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \\ \frac{2|bx|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|7(bx)^2-9(pbx^2+qa)|} & ; \left|1 - \frac{3^k\delta}{2^{2k}\phi(s)}\right| \geq \frac{1}{16}\left|7 - 9\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \end{cases}$$

Corollary 3.2. *Let* $g(z)$ *be in the family* $\mathscr{J}_{\sum}(x, k, \phi(s))$. *Then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|3(bx)^2 - 4(px^2 + qa)|}},\ |d_3| \leq \frac{1}{3^k\phi(s)}\left[\frac{b^2x^2}{4} + \frac{|bx|}{5}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{5(3^k\phi(s))} & ; \left|1 - \frac{3^k\delta}{2^{2k}\phi(s)}\right| \leq \frac{1}{5}\left|3 - 4\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \\ \frac{|bx|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|3(bx)^2-4(pbx^2+qa)|} & ; \left|1 - \frac{3^k\delta}{2^{2k}\phi(s)}\right| \geq \frac{1}{5}\left|3 - 4\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \end{cases}$$

Corollary 3.3. *Let $g(z)$ be in the family $\mathscr{L}_{\sum}(x,k,\phi(s))$. Then*

$$|d_2| \le \frac{2|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|13(bx)^2-25(pbx^2+qa)|}},\ |d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{4b^2x^2}{25}+\frac{|bx|}{7}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|bx|}{7(3^k\phi(s))} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \le \frac{1}{28}\left|13-25\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \\ \frac{2|bx|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|13(bx)^2-25(pbx^2+qa)|} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \ge \frac{1}{28}\left|13-25\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \end{cases}$$

Corollary 3.4. *Let $g(z)$ be in the family $\mathscr{P}_{\sum}(x,\mu,k,\phi(s))$. Then*
$|d_2| \le \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{\left[|(1+4\mu)(bx)^2-(1+2\mu)^2(pbx^2+qa)|\right]}},\ |d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{b^2x^2}{(1+2\mu)^2}+\frac{|bx|}{2(1+3\mu)}\right]$
and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|bx|}{2(1+3\mu)3^k\phi(s)} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \le J_1 \\ \frac{|bx|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)[(1+4\mu)(bx)^2-(1+2\mu)^2(pbx^2+qa)]} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \ge J_1, \end{cases}$$

where $J_1 = \frac{1}{2(1+3\mu)}\left|(1+4\mu)-(1+2\mu)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Corollary 3.5. *Let $g(z)$ be in the family $\mathfrak{K}_{\sum}(x,\mu,k,\phi(s))$. Then*
$|d_2| \le \frac{|b(x)|\sqrt{|b(x)|}}{2^k\phi(s)\sqrt{|3(1+2\mu)(bx)^2-4(1+\mu)^2(pbx^2+qa)|}},\ |d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{b^2x^2}{4(1+\mu)^2}+\frac{|b(x)|}{3(1+2\mu)}\right]$
and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|b(x)|}{3^{k+1}\phi(s)(1+2\mu)} & ; |1-\frac{3^k\delta}{2^{2k}\phi(s)}| \le M_1 \\ \frac{|b(x)|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{3^k\phi(s)|3(1+2\mu)(bx)^2-4(1+\mu)^2(pbx^2+qa)|} & ; |1-\frac{3^k\delta}{2^{2k}\phi(s)}| \ge M_1, \end{cases}$$

where $M_1 = \frac{1}{3(1+2\mu)}\left|3(1+2\mu)-4(1+\mu)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Corollary 3.6. *Let $g(z)$ be in the family $\mathfrak{I}_{\sum}(x,\mu,k,\phi(s))$. Then*
$|d_2| \le \frac{|b(x)|\sqrt{|b(x)|}}{2^k\phi(s)\sqrt{2\mu|(bx)^2-2\mu(pbx^2+qa)|}},\quad |d_3| \le \frac{1}{3^k(2\mu\phi(s))}\left[\frac{b^2x^2}{2\mu}+\frac{|b(x)|}{3}\right]$
and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|b(x)|}{2\mu(3^{k+1}\phi(s))} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \le \frac{1}{3}\left|1-2\mu\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \\ \frac{|b(x)|^3\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right|}{\mu(3^k\phi(s))|(bx)^2-2\mu(pbx^2+qa)|} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \ge \frac{1}{3}\left|1-2\mu\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \end{cases}$$

Corollary 3.7. *Let $g(z)$ be in the family $\mathscr{M}_{\sum}(x,\xi,k,\phi(s))$. Then*
$|d_2| \le \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|(\xi+1)(bx)^2-4(2\xi-1)^2(pbx^2+qa)|}}, |d_3| \le \frac{1}{3^k\phi(s)}\left[\frac{(bx)^2}{4(2\xi-1)^2}+\frac{|bx|}{3(3\xi-1)}\right]$
and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|b(x)|}{3^{k+1}\phi(s)(3\xi-1)} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \le \Omega_1 \\ \frac{2\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right||bx|^3}{3^k\phi(s)|(\xi+1)(bx)^2-4(2\xi-1)^2(pbx^2+qa)|} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \ge \Omega_1, \end{cases}$$

where $\Omega_1 = \frac{1}{(3\xi-1)}\left|(\xi+1)-4(2\xi-1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Corollary 3.8. *Let $g(z)$ be in the family $\mathfrak{N}_{\sum}(x,\tau,k,\phi(s))$. Then*
$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{2^k\phi(s)\sqrt{|(8\tau^2-7\tau+1)(bx)^2-4(2\tau-1)^2(pbx^2+qa)|}}$, $|d_3| \leq \frac{1}{3^k\phi(s)}\left[\frac{(bx)^2}{4(2\tau-1)^2}+\frac{|bx|}{3(3\tau-1)}\right]$
and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3^{k+1}\phi(s)(3\tau-1)} & ; \ \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \leq \Omega_2 \\ \frac{\left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right||bx|^3}{3^k\phi(s)|(8\tau^2-7\tau+1)(bx)^2-4(2\tau-1)^2(pbx^2+qa)|} & ; \left|1-\frac{3^k\delta}{2^{2k}\phi(s)}\right| \geq \Omega_2, \end{cases}$$

where $\Omega_2 = \frac{1}{(3\tau-1)}\left|(8\tau^2-7\tau+1)-4(2\tau-1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Acknowledgment. The authors would like to thank the referee(s) and editor for their constructive advices, valuable comments and suggestions which improved basically the final version of this work.

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