Lower Bound Eigenvalue Problems of the Compact Riemannian Spin−Submanifold Dirac Operator

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Abstract

In this paper, we construct two modified spinorial Levi−Civita connection based on the Energy−Momentum tensor and its trace to bring an optimal lower bound to the eigenvalues of the compact Riemannian Spin−submanifold Dirac operator in terms of the the Energy−Momentum tensor and its trace. Then, we extend these estimates in terms of the Yamabe number and the area of the submanifold under the conformal change of the metric.

Keywords: Spin and $Spin^c$ geometry, Dirac operator, Estimation of eigenvalues.

Kompakt Riemannian Spin−Alt manifold Dirac Operatörünün Alt Sınır Özdeğer Problemleri

Öz

Bu makalede, kompakt Riemannian Spin altmanifold Dirac operatörünün özdeğerlerine Energy−Momentum tensörü ve onun izine bağlı olarak optimal bir alt sınır getirmek için spinorial Levi−Civita konneksiyonunu Energy−Momentum tensörü ve onun izine bağlı olarak deforme ettik. Daha sonra bu tahminleri konformal değişim metriği altında Yamabe sayısı ve altmanifoldun alanına bağlı olarak genişlettik.

Anahtar Kelimeler: Spin and $Spin^c$ geometry, Dirac operator, Estimation of eigenvalues

1. Introduction

Obtaining information about the topology and geometry of the manifolds is the main research topic for the mathematicians. In doing so, many formulas have been developed that naturally comes from the structure of the manifold. While doing this, firstly the Spin−structure is constructed on the manifold and a manifold endowed with this structure is called Spin−manifold. Also with the help of this structure, a spinor bundle can be constructed (Friedrich, 2000; Naber, 1997). On the spinor bundle A. Lichnerowicz (Lichnerowicz, 1963) defined the Schrödinger−Lichnerowicz formula as follows:

$$D^*D = \nabla^*\nabla + \frac{R}{4}, \qquad (1)$$

where R is the scalar curvature of M, ∇^* is the adjoint of the spinorial Levi−Civita connection ∇ and D^* is the adjoint of the Dirac operator D. Considering this formula, one can obtain subtle informations about the scalar curvature of the manifold and geometry of it.

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As well as optimal lower bounds to the eigenvalues of the Dirac operator can be obtained as in (Hijazi, 1991; Hijazi et al., 2001; Hijazi et al., 2001, Nakad et al., 2013; Zhang, 1998). Moreover, by considering the limiting case of the eigenvalue estimation one can be find whether the manifold is Einstein or not. As mentioned above, on the compact Spin−manifold, the lower bound estimates for the eigenvalues of the Dirac operator was firstly introduced by A. Lichnerowicz as follows (Lichnerowicz, 1963):

$$\lambda^2 > \frac{1}{4} \inf_M R, \qquad (2)$$

where R is the scalar curvature of M.

Later on, T. Friedrich in (Friedrich, 2001) modified the spinorial Levi−Civita connection based on the eigenvalue of the Dirac operator and improved the lower bound given in (2). At this point, eigenvalue estimation has gained a different dimension by O. Hijazi. Because O. Hijazi developed these eigenvalue estimates based on the first eigenvalue of the Yamabe operator using some identities coming with respect to the conformal change of the metric (Hijazi, 1986). Moreover, O. Hijazi improved his estimates in terms of the square of the Energy−momentum tensor (Hijazi, 1995) and extended his estimates with respect to the conformal change of the metric. As mentioned above, eigenvalue estimates are not only studied on the Spin−manifolds. Also it has been studied on the manifold endowed with Spin−structure whose codimension is gretar or equal to 1 as in (Hijazi et al., 2001; Hijazi, et al., 2001; Zhang, 1998; Zhang, 1999).

In this paper, we improve the following estimates for the eigenvalue λ_H of the submanifold Dirac operator D_H defined by O. Hijazi and X. Zhang in section 5 and page 173 of (Hijazi et al., 2001) for the compact

Riemannian Spin−submanifolds $M^n \subset N^{n+r}$ of dimension $n \geq 2$ whose normal bundle is also Riemannian Spin:

$$\lambda_H^2$$

$$\geq \begin{cases} \frac{1}{4} \sup_\beta \inf_{M_\Phi} \left(\frac{R + R_{\perp,\Phi} + 4|Q_\Phi|^2}{1 + n\beta^2 - 2^*\beta} - C_{n,\beta,\Phi} \frac{(n-1)}{(1-n\beta)^2} \parallel H \parallel^2 \right), \\ \frac{1}{4} \sup_{\beta,u} \inf_{M_\Phi} \left(\frac{(\overline{R} + \overline{R}_{\perp,\Psi})e^{2u} + 4|Q_\Phi|^2}{1 + n\beta^2 - 2^*\beta} - C_{n,\beta,\Phi} \frac{(n-1)}{(1-n\beta)^2} \parallel H \parallel^2 \right), \end{cases}$$

$$(3)$$

where $\parallel H \parallel = \sqrt{\sum_A H_A^2}$ is the norm of the mean curvature denoted by H_A and for any spinor field $\Phi \in \Gamma(\mathbb{S})$,

$$R_{\perp,\Phi} = -\frac{1}{2} (\sum_{i,j,A,B} R_{ijAB} e^i \cdot e^j \cdot e^A \cdot e^B \cdot \Phi, \frac{\Phi}{|\Phi|^2})$$

is defined on the complement set of zeros M_Φ,

$$C_{n,\beta,\Phi} = \frac{n\beta^2 - 2\beta + 1}{1 + n\beta^2 - 2^*\beta}, \quad 2^* = 2\left(1 - \left(\frac{Re(\epsilon\omega \cdot \Phi,\Phi)}{|\Phi|^2}\right)\right),$$

where $\epsilon = (-1)^{r-1}$.

This paper is organized as follows. At first, we introduce some basic facts concerning compact Riemannian Spin−submanifold Dirac operator (Hijazi et al., 2001). Then, we obtain a new lower bound for the eigenvalue of the submanifold Dirac operator in terms of the mean curvature, Energy−momentum tensor and it trace. Finally, we improve these estimates in terms of the Yamabe number, Energy−Momentum tensor and its trace.

In the following section, some fundemantal definiton and properties are given. For details see (Hijazi et al., 2001).

2. Submanifold Dirac Operator

Assume that N is an $(n + r)$ −dimensional compact Riemannian Spin−manifold and M its n −dimensional compact Riemannian Spin−submanifold. As it is known, the Spin−structure on the tangent bundle of the $(n + r)$ −dimensional N whose codimension greater than 1 and the Spin−structure on the tangent bundle of M determine the Spin−structure constructed on the normal bundle of M. Moreover, the Spin−structure constructed on the normal bundle of M is unique. Here, the Spin−structure defined on

the tangent bundle of N is the total sum of the Spin−structure defined on the tangent bundle of M and the Spin−structure constructed on its normal bundle. Accordingly, the spinor bundle $\mathbb{S}$ defined on N is also globally defined on M. Therefore, the spinor bundle defined on N and M is denoted by the same symbol. Thus, two different spinorial Levi−Civita connections can be defined on $\mathbb{S}$. The spinorial Levi−Civita connections constructed on N and M are represented by $\widetilde{\nabla}$ and ∇, respectively. Similarly, the Dirac operators connected with the spinorial Levi− Civita connections $\widetilde{\nabla}$ and ∇ are indicated by $\widetilde{D}$ and D, respectively. Also, on the spinor bundle $\mathbb{S}$, it can be defined the following well−known positive definite Hermitian metric which satisfies, $w \in \Gamma(T^*N)$, and any spinor field $\Phi, \Psi \in \Gamma(\mathbb{S})$, the relation (Lawson et al., 1989)

$$(w \cdot \Phi, w \cdot \Psi) = |w|^2 (\Phi, \Psi), \tag{4}$$

where $' \cdot '$ denotes the Clifford multiplicapion. Recall that, with respect to the metric (,) $\widetilde{\nabla}$ is globally defined along M. Also, D is formally self−adjoint with respect to this metric (Hijazi, 2001).

The identities used in this paper are given as follows without the need for proof since they are the identities mentioned in (Hijazi, 2001). As in (Hijazi, 2001)., fix a point $s \in M$ and an orthonormal basis $\{e_\alpha\}$ of $T_s N$ with $\{e_A\}$ normal and $\{e_i\}$ tangent to M such that $(\nabla_i e_j) = 0$. Throughout the whole paper indices ranges are given as follows:

$$1 \leq \alpha, \beta, \gamma, \eta \leq n + r; 1 \leq i, j, k, l \leq n,$$

$$n + 1 \leq A, B \leq n + r. \tag{5}$$

Let e^α be the coframe at point s. Then the relation between $\widetilde{\nabla}$ and ∇ is given by

$$\widetilde{\nabla}_i = \nabla_i + \frac{1}{2} h_{Aij} e^A \cdot e^j \cdot, \tag{6}$$

where $h_{Aij} = h_{Aji} = (\widetilde{\nabla}_i e_A, e_j)$ is the second fundemantal form. According to this relation, ∇ is metric compatible with the metric (,). Moreover, $\widetilde{R}_{\alpha\beta\gamma\eta}, R_{ijkl}$ be the curvature tensor of N and M, respectively. Also, R_{ijAB} be the curvature of the normal bundle of M. Recall that the normal bundle of M is flat if and only if $R_{ijAB} = 0$. With respect to the (5), $\widetilde{D}$ is given as

$$\widetilde{D} = D + \frac{1}{2} H_A e^A \cdot, \tag{7}$$

where $H_A = \sum_{i=1}^n h_{Aii}$ is the component of the mean curvature of M. On the submanifold M, for any spinor field $\Phi \in \Gamma(\mathbb{S})|_M$, the Schrödinger−Lichnerowicz−type formula is described as follows

$$\int_M |D\Phi|^2 v_g = \int_M (|\nabla\Phi|^2 + \frac{1}{4}(R + R_{\perp,\phi})|\Phi|^2) v_g. \tag{8}$$

Considering the operator $\omega_\perp$ defined on $\mathbb{S}$ by,

$$\omega_\perp := (-1)^{(r(r-1)4)} e^{A_1} \cdot e^{A_2} \dots e^{A_r} \cdot, \tag{9}$$

where $\{e^{A_i}\}$ is an orthonormal coframe, the submanifold Dirac operator D_H satisfies the following relation

$$D_H = \omega_\perp \cdot \widetilde{D}. \tag{10}$$

Here D_H is formally self−adjoint with respect to the metric (,) and satisfies $\widetilde{D}^* \widetilde{D} = D_H^2$, where $\widetilde{D}^* \Phi = D\Phi - \frac{1}{2} H_A e^A \cdot \Phi$ (Hijazi, 2001).

3. Eigenvalue Estimates For the Spin Submanifold Dirac Operator

In this section, we consider the compact Spin−submanifold Dirac operator defined by O. Hijazi and X. Zhang in (Hijazi, 2001) as follows:

$$D\Phi = -\lambda_H \epsilon \omega_\perp \cdot \Phi - \frac{H_A}{2} e^A \cdot \Phi, \tag{11}$$

where $\epsilon = (-1)^{r-1}$ and λ_H is the real constant or real function eigenvalue of D_H. After a simple calculation we get

$|D\Phi|^2 = \lambda_H^2|\Phi|^2 + \frac{\|H\|^2}{4}|\Phi|^2 + \lambda_H H_A Re(\epsilon\omega_\perp \cdot \Phi, e^A \cdot \Phi).$ (12)

To simplify calculations in the way used to obtain an estimate for the eigenvalue λ_H of the Dirac operator D_H, we integrate (12) over M and combine with (8) to get

$\int_M |\nabla\Phi|^2 v_g = \int_M \left(\lambda_H^2|\Phi|^2 + \frac{\|H\|^2}{4}|\Phi|^2 + \lambda_H H_A Re(\epsilon\omega_\perp \cdot \Phi, e^A \cdot \Phi) - (\frac{R+R_{\perp,\Phi}}{4})|\Phi|^2\right) v_g.$ (13)

In the following theorem, we obtain an optimal lower bound to the eigenvalue λ_H of the Dirac operator D_H by using the modified spinorial Levi−Civita connection constructed with the Energy−Momentum tensor Q_Φ and its trace trQ_Φ.

Theorem 3.1 *On a compact Riemannian Spin−submanifold $M^n \subset N^{n+r}$ of dimension $n \geq 2$ whose normal bundle is also Spin, any square of the eigenvalue λ_H of the Dirac operator to which attached an eigenspinor Φ satisfies*

$\lambda_H^2 \geq \frac{1}{4} \sup_{\beta,\kappa_{n,\beta}} \inf_M \left(\frac{R+R_{\perp,\Psi}+4\kappa_{n,\beta}|trQ_\Phi|^2}{n\beta^2-2\beta+1} - \frac{(n-1)\|H\|^2}{(1-n\beta)^2}\right)$ (14)

where β real function, $\beta = \frac{1}{n}$ if $H_A \neq 0$ for some A and $\kappa_{n,\beta} := \frac{(1-n\beta)^2}{n}$.

Proof 3.2 *For any real fuctions α, β, γ, we define the modified spinorial Levi−Civita connection in terms of the trace of the Energy−Momentum tensor Q_Φ as follows:*

$\nabla_i^{\alpha,\beta,\gamma}\Phi = \nabla_i\Phi + \frac{\alpha}{2}H_A e^i \cdot e^A \cdot \Phi - \beta\lambda_H \epsilon e^i \cdot \omega_\perp \cdot \Phi + \gamma trQ_\Phi e^i \cdot \Phi.$ (15)

For any $1 \leq i \leq n$, the square norm of $\nabla_i^{\alpha,\beta,\gamma}\Phi$ is

$|\nabla_i^{\alpha,\beta,\gamma}\Phi|^2 = |\nabla_i\Phi|^2 + \alpha H_A Re(\nabla_i\Phi, e^i \cdot e^A \cdot \Phi) - 2\beta\lambda_H Re(\nabla_i\Phi, \epsilon e^i \cdot \omega_\perp \cdot \Phi) + 2\gamma trQ_\Phi Re(\nabla_i\Phi, e^i \cdot \Phi)$
$+ \frac{\alpha^2}{4}\|H\|^2|\Phi|^2 + \gamma^2|trQ_\Phi|^2|\Phi|^2 + \beta^2\lambda_H^2|\Phi|^2$
$- \alpha\beta\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi) + \alpha\gamma H_A trQ_\Phi Re(e^A \cdot \Phi, \Phi)$
$- 2\beta\gamma\lambda_H trQ_\Phi Re(\epsilon\omega_\perp \cdot \Phi, \Phi).$ (16)

Summing over i, one gets

$|\nabla^{\alpha,\beta,\gamma}\Phi|^2 = |\nabla\Phi|^2 + (\frac{n\alpha^2+2\alpha}{4})\|H\|^2|\Phi|^2$
$+ (\alpha - \beta - n\alpha\beta)\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi)$
$+ (n\beta^2 - 2\beta)\lambda_H^2|\Phi|^2 + (n\gamma^2 - 2\gamma)|trQ_\Phi|^2|\Phi|^2$
$- 2n\beta\gamma\lambda_H trQ_\Phi Re(\epsilon\omega_\perp \cdot \Phi, \Phi).$ (17)

Integrating over M and using (13), we get

$\int_M |\nabla^{\alpha,\beta,\gamma}\Phi|^2 v_g = \int_M ((n\beta^2 - 2\beta + 1)\lambda_H^2|\Phi|^2$
$+ (\frac{n\alpha^2+2\alpha+1}{4})\|H\|^2|\Phi|^2 + (\alpha - \beta - n\alpha\beta + 1)$
$\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi) + (n\gamma^2 - 2\gamma - 2n\beta\gamma)|trQ_\Phi|^2$
$- (\frac{R+R_{\perp,\Phi}}{4})|\Phi|^2) v_g.$ (18)

Taking $\alpha = \frac{1-\beta}{n\beta-1}$ and $\gamma = \frac{1-n\beta}{n}$, we have

$\int_M \lambda_H^2|\Phi|^2 v_g \geq \int_M \left(\frac{1}{(n\beta^2-2\beta+1)}\left(\frac{R+R_{\perp,\Phi}}{4} + \kappa_{n,\beta}|trQ_\Phi|^2\right) - \frac{n-1}{(1-n\beta)^2}\|H\|^2\right)|\Phi|^2) v_g.$ (19)

This give us (14).

4. Extending Eigenvalue Estimation Based on Conformal Metric

Let $\bar{g}_N = e^{2u}g_N$ be a conformal change of the metric for any real function u on N. Depending on the g_N and $\bar{g}_N$, one can construct two SO_n principal bundles as $SO_{g_N}N$ and $SO_{\bar{g}_N}N$, respectively. As well as the isometry G_u can be defined between them. Also, G_u induced an isometry between $Spin_{g_N}$ and $Spin_{\bar{g}_N}$ principal bundles which are defined regard to the g_N and $\bar{g}_N$, respectively. As in (Hijazi, 1986; Hijazi, 2001), with the help of the $Spin_{g_N}$ and $Spin_{\bar{g}_N}$ principal bundles one can construct $\mathbb{S}$ and $\overline{\mathbb{S}}$ spinor bundles, respectively. On the spinor bundles $\mathbb{S}$ and $\overline{\mathbb{S}}$ one can define Hermitian metrics $(\ ,\ )_{g_N}$ and $(\ ,\ )_{\bar{g}_N}$ respectively, satisfies

$(\Psi, \Phi)_{g_N} = (\overline{\Psi}, \overline{\Phi})_{\bar{g}_N},$ (20)

where $\Psi, \Phi \in \Gamma(\mathbb{S})$ and $\overline{\Psi} = G_u\Psi, \overline{\Phi} = G_u\Phi \in \Gamma(\overline{\mathbb{S}})$.

The Clifford multiplication on $\Gamma(\overline{\mathbb{S}})$ is defined as follows

$$\overline{e^i \cdot \overline{\Psi}} := \overline{e^i \cdot \Psi}. \tag{21}$$

Also, the regular class of N denoted by $\mathcal{U}$ is given as

$$\mathcal{U} = \{u \in C^\infty(N, \mathbb{R}), \; du(e_A)|_M = 0, \text{for all A}\}. \tag{22}$$

Considering the regular conformal metric $\overline{g} = e^{2u}|_M g$ one gets the following identities (Hijazi, 2001):

$$\overline{D}_H(e^{-((n-1)2)u}\overline{\Phi}) = e^{-((n+1)2)u}\overline{D_H\Phi} \tag{23}$$

$$\overline{H}_{ijA} = e^{-u}(H_{ijA} + du(e_A)). \tag{24}$$

Also, with respect to the conformal metric $\overline{g}_N = e^{2u}g_N$, one has $\overline{R}_{ijAB} = e^{-2u}R_{ijAB}$ and $\overline{R}_{\perp,\Psi} = e^{-2u}R_{\perp,\Phi}$ for $\Psi = e^{-((n-1)2)u}\Phi$.

Recall that, $D_H\Phi = \lambda_H\Phi \Rightarrow \overline{D}_H \; \overline{\Psi} = \lambda_H e^{-u}\overline{\Psi}$ where $\overline{\Psi} = e^{-((n-1)2)u}\overline{\Phi}$ and $\overline{H}_A = e^{-u}H_A$.

Considering $\overline{g}_N = e^{2u}g_N$ and applying $\overline{\Phi}$ to the equation (19), we get

$$\int_M (1 + n\beta^2 - 2\beta)e^{-2u}\lambda_H^2|\Psi|^2_{\overline{g}}v_{\overline{g}} = \int_M (|\overline{\nabla}^{\alpha,\beta,\gamma} \; \overline{\Psi}|^2_{\overline{g}} + \frac{1}{4}(\overline{R} + \overline{R}_{\perp,\Psi})|\overline{\Psi}|^2_{\overline{g}} - (\frac{(1+n\beta^2-2\beta)(1-n)}{4(1-n\beta)^2})e^{-2u}\|\overline{H}\|^2 |\Psi|^2_{\overline{g}} + \frac{(1-n\beta)^2}{n}e^{-2u}|trQ_\Phi|^2_{\overline{g}}|\overline{\Psi}|^2_{\overline{g}})v_{\overline{g}}. \tag{25}$$

According to this, the following corollaries are obtained.

Corollary 4.1 *Under the same conditions as in Theorem* 3.1, *if $n \geq 2$, then any eigenvalue λ_H of the submanifold Dirac operator D_H to which is attached an eigenspinor Φ satisfies*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\beta,\kappa_{n,\beta}}\inf_M \left(\frac{(\overline{R}+\overline{R}_{\perp,\Psi})e^{2u}+4\kappa_{n,\beta}|trQ_\Phi|^2}{n\beta^2-2\beta+1} - \frac{(n-1)\|H\|^2}{(1-n\beta)^2}\right), \tag{26}$$

where $\overline{R}$ is the scalar curvature of M associated to a regular conformal metric $\overline{g} = e^{2u}|_M g$, for some real functions β, u on N.

In the following, the inequality obtained in (21) is extended in terms of the first eigenvalue μ_1 of the Yamabe operator $L := 4\frac{n-1}{n-2}\Delta_g + R$.

Considering the relation between μ_1 and the scalar curvature $\overline{R}$ given by $\mu_1 = \sup_u \inf_M(\overline{R}e^{2u})$, one gets the following inequality.

Corollary 4.2 *Under the same conditions as in Theorem* 3.1, *if $n \geq 3$, then*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\beta,\kappa_{n,\beta}}\inf_{M_\Phi} \left(\frac{(\mu_1 + R_{\perp,\Phi} + 4\kappa_{n,\beta}|trQ_\Phi|^2)}{n\beta^2-2\beta+1} - \frac{(n-1)}{(1-n\beta)^2}\|H\|^2\right), \tag{27}$$

where μ_1 is the first eigenvalue of the Yamabe operator.

As in the above corallary, by considering the relation between $\overline{R}$ and the area of M given as $\sup_u \inf_M(\overline{R}e^{2u}) = \frac{8\pi}{Area(M)}$, one gets the following inequality.

Corollary 4.3 *Under the same conditions as in Theorem* 3.1, *if M is compact surface of genus zero and*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\beta,\kappa_{2,\beta}}\inf_{M_\Phi} \left(\frac{1}{2\beta^2-2\beta+1}(\frac{8\pi}{Area(M)} + R_{\perp,\Phi} + 4\kappa_{2,\beta}|trQ_\Phi|^2) - \frac{1}{(1-2\beta)^2}\|H\|^2\right), \tag{28}$$

where $\kappa_{2,\beta} = \frac{(1-2\beta)^2}{2}$ and $Area(M)$ is denoted the area of M.

In the following theorem we get a new an eigenvalue estimate to the eigenvalue of the submanifold Dirac operator D_H by constructing modified spinorial Levi−Civita connection in terms of the Energy−Momentum tensor and its trace.

Theorem 4.4 *Under the same conditions as in Theorem* 3.1, *if* $n \geq 2$, *then any eigenvalue of* D_H *satisfies*

$$\lambda_H^2 \geq \frac{1}{4} \sup_{\beta} \inf_{M_\Phi} \left(\frac{(R + R_{\perp, \Psi}) + 4((n\beta^2 - 2\beta)|trQ_\Phi|^2 + |Q_\Psi|^2)}{n\beta^2 - 2\beta + 1} - \frac{(n-1)\|H\|^2}{(1 - n\beta)^2} \right). \quad (29)$$

Proof 4.5 *For real fuctions* α, β, γ, *we define*

$$\nabla_i^{\alpha, \beta, \gamma} \Phi = \nabla_i \Phi + \frac{\alpha}{2} H_A e^i \cdot e^A \cdot \Phi - \beta \lambda_H \epsilon e^i \cdot \omega_\perp \cdot \Phi + \gamma tr Q_\Phi e^i \cdot \Phi + Q_{ij,\Phi} e^j \cdot \Phi. \quad (30)$$

For any $1 \leq i \leq n$, the square norm of $\nabla_i^{\alpha, \beta, \gamma}$ is

$$|\nabla_i^{\alpha, \beta, \gamma} \Phi|^2 = |\nabla_i \Phi|^2 + \alpha H_A Re(\nabla_i \Phi, e^i \cdot e^A \cdot \Phi) - 2\beta \lambda_H Re(\nabla_i \Phi, \epsilon e^i \cdot \omega_\perp \cdot \Phi) + 2\gamma tr Q_\Phi Re(\nabla_i \Phi, e^i \cdot \Phi) + 2 Re(\nabla_i \Phi, Q_{ij,\Phi} e^j \cdot \Phi) + \frac{\alpha^2}{4} \| H \|^2 |\Phi|^2 + \gamma^2 |tr Q_\Phi|^2 |\Phi|^2 - \alpha\beta\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi) + \alpha\gamma H_A tr Q_\Phi Re(e^A \cdot \Phi, \Phi) + \alpha H_A Re(e^i \cdot e^A \cdot \Phi, Q_{ij,\Phi} e^j \cdot \Phi) + \beta^2 \lambda_H^2 |\Phi|^2 - 2\beta\gamma\lambda_H tr Q_\Phi Re(\epsilon\omega_\perp \cdot \Phi, \Phi) - 2\beta Re(\epsilon e^i \cdot \omega_\perp \cdot \Phi, Q_{ij} e^j \cdot \Phi) + 2\gamma tr Q_\Phi Re(e^i \cdot \Phi, Q_{ij,\Phi} e^j \cdot \Phi) + |Q_{ij,\Phi}|^2 |\Phi|^2. \quad (31)$$

Summing over i, one gets

$$|\nabla^{\alpha, \beta, \gamma} \Phi|^2 = |\nabla \Phi|^2 + (\frac{n\alpha^2 + 2\alpha}{4} \| H \|^2 |\Phi|^2 + (\alpha - \beta - n\alpha\beta)\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi) + (n\beta^2 - 2\beta)\lambda_H^2 |\Phi|^2 + (n\gamma^2 - 2\gamma + 2\gamma)|tr Q_\Phi|^2 |\Phi|^2 - (2\beta + 2n\beta\gamma)\lambda_H tr Q_\Phi Re(\epsilon\omega_\perp \cdot \Phi, \Phi) - |Q_\Phi|^2 |\Phi|^2. \quad (32)$$

Integrating over M and using (11), we get

$$\int_M |\nabla^{\alpha, \beta, \gamma} \Phi|^2 v_g = \int_M ((n\beta^2 - 2\beta + 1)\lambda_H^2 |\Phi|^2 + (\frac{n\alpha^2 + 2\alpha + 1}{4}) \| H \|^2 |\Phi|^2 - |Q_\Phi|^2 |\Phi|^2 + (\alpha - \beta - n\alpha\beta + 1)\lambda_H H_A Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi) + (n\gamma^2 - 2\gamma + 2\gamma)|tr Q_\Phi|^2 |\Phi|^2 - (2\beta + 2n\beta\gamma)\lambda_H tr Q_\Phi Re(\epsilon\omega_\perp \cdot \Phi, \Phi) - (\frac{R + R_{\perp, \Phi}}{4})|\Phi|^2) v_g. \quad (33)$$

Taking $\alpha = \frac{1 - \beta}{n\beta - 1}$, $\gamma = -\beta$, we get

$$\int_M (n\beta^2 - 2\beta + 1)\lambda_H^2 |\Phi|^2 v_g \geq \int_M ((\frac{R + R_{\perp, \Phi}}{4})|\Phi|^2 - \frac{n - 1}{4(1 - n\beta)^2} \| H \|^2 |\Phi|^2 + (n\beta^2 - 2\beta)|tr Q_\Phi|^2 |\Phi|^2 + |Q_\Psi|^2 |\Phi|^2) v_g. \quad (34)$$

This give us the desired result given in (29).

Under the conformal change of the metric $\overline{g} = e^{2u}|_M g$ the following corollaries are obtained.

Corollary 4.6 *Under the same conditions as in Theorem* 4.4, *if* $n \geq 2$, *then any eigenvalue* λ_H *of the submanifold Dirac operator* D_H *to which is attached an eigenspinor* Φ *satisfies*

$$\lambda_H^2 \geq \frac{1}{4} \sup_{\beta} \inf_{M_\Phi} \left(\frac{(\overline{R} + \overline{R}_{\perp, \Psi})e^{2u} + 4((n\beta^2 - 2\beta)|trQ_\Phi|^2 + |Q_\Phi|^2)}{n\beta^2 - 2\beta + 1} - \frac{(n-1)\|H\|^2}{(1 - n\beta)^2} \right), \quad (27)$$

where $\overline{R}$ is the scalar curvature of M associated to a regular conformal metric $\overline{g} = e^{2u}|_M g$, for some real functions β, u on N.

As in the corollaries (4.2) and (4.3), the inequality obtained in (35) can be improved in terms of the fist eigenvalue μ_1 of the Yamabe number and area of M as in the following corollaries.

Corollary 4.7 *Under the same conditions as in Theorem* 4.4, *if* $n \geq 3$, *then*

$$\lambda_H^2 \geq \frac{1}{4} \sup_{\beta} \inf_{M_\Phi} \left(\frac{(\mu_1 + R_{\perp, \Phi} + 4((n\beta^2 - 2\beta)|trQ_\Phi|^2 + |Q_\Phi|^2)}{n\beta^2 - 2\beta + 1} - \frac{(n-1)}{(1 - n\beta)^2} \| H \|^2 \right) \quad (36)$$

where μ_1 is the first eigenvalue of the Yamabe operator.

Corollary 4.8 *Under the same conditions as in Theorem* 4.4, *if* M *is compact surface of genus zero and*

$$\lambda_H^2 \geq \frac{1}{4} \sup_{\beta} \inf_{M_\Phi} \left(\frac{1}{2\beta^2 - 2\beta + 1}(\frac{8\Pi}{Area(M)} + R_{\perp, \Phi} + 4((2\beta^2 - 2\beta)|trQ_\Phi|^2 + |Q_\Phi|^2) - \frac{1}{(1 - 2\beta)^2} \| H \|^2 \right) \quad (37)$$

where $Area(M)$ is denoted the area of M.

If m is odd, all results are the same as in (Hijazi, 2001).

5. References

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