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Catalan Transform of The k −Lucas Numbers

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Öz

Bu çalışmada, k–Lucas dizisinin $L_{k,n}$ Catalan dönüşümünün $CL_{k,n}$ tanımı verildi. k–Lucas dizisinin $L_{k,n}$ Catalan dönüşümünün $CL_{k,n}$ geren fonksiyonu elde edildi. Ayrıca, $CL_{k,n}$ dönüşümü, alt üçgen matris olan Catalan matrisi C ile $n \times 1$ tipindeki L_k matrisinin çarpımı olarak yazıldı. Hankel fonksiyonu kullanılarak $CL_{k,n}$ ler ile oluşturulan matrislerin determinantları hesaplandı.

Anahtar Kelimeler: k −Lucas dizisi, k −Fibonacci dizisi, Catalan dönüşümü, Hankel dönüşümü.

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Abstract

In this study, the $CL_{k,n}$ description of Catalan transformation of k −Lucas $L_{k,n}$ sequences was given. The $CL_{k,n}$ generating function of Catalan transformation of k −Lucas $L_{k,n}$ sequences was obtained. And also, $CL_{k,n}$ transformation was written as the multiplying of Catalan matris C which is the lower triangular matris, and the L_k matris of $n\,x\,1$ type. Determinants of matrices which were formed with $CL_{k,n}$ by using Hankel transform were calculated.

Keywords: k −Lucas sequence, k −Fibonacci sequence, Catalan Transform, Hankel Transform.

1. Introduction

For any integer n, it is called a generalized Fibonacci-type sequence any recurrence sequence of the following form $G(n+1) = aG(n) + bG(n-1)$, $G(0) = m, G(1) = t$,where m, t, a and b are any complex numbers. There is an extensive work in the literature concerning Fibonacci-type sequences and their applications in modern science (see e.g.[Horadam, 1961; Bruhn, et al., 2015; Anuradha, 2008; Özkan, et al., 2018; Falcon and Plaza, 2007; Özkan and Taştan, 2019] and the references therein) The known Lucas numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc [Koshy, 2001; Özkan and Altun, 2019;

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Özkan, et al., 2017; Özkan, 2014; Taştan and Özkan, 2020; Özkan, et al., 2019; Özkan and Taştan, 2019].

There exist generalizations of the Lucas numbers. This paper is an extension of the work of Falcon [Falcon, 2013]. Falcon [Falcon, 2013] gave an application of the Catalan transform to the k −Fibonacci sequences. In this paper, we put in for Catalan transform to the k −Lucas sequence and present application of the Hankel transform to the Catalan transform of the k −Lucas sequence.

The other section of the paper is prepared as follows. The following, we introduce some fundamental definitions of k–Lucas numbers. In section 3, Catalan transform of k −Lucas sequence is given. In section 4, generating function of Catalan transformation of k −Lucas sequence is obtained. Finally, we give Henkel transform of the new sequence and its determinant.

2. k −Lucas numbers

Let k be any positive real number. Then the k-Lucas sequence is defined recurrently by

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \text{ for } n \geq 1$$

where $L_{k,0} = 2$ and $L_{k,1} = k$. We will show the sequence with such that $\{L_{k.n}\}_{n\in N}$ from now on.

When $k = 1$, the known Lucas sequence is optained.

Characteristic equation of the sequence is

$$r^2 - k.r - 1 = 0.$$

Its characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2}$$

and

$$r_2 = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Characteristic roots verify the properties

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\Delta} = \delta$$

$$r_1 + r_2 = k$$

$$r_1.r_2 = -1$$

Binet's formula for $L_{k,n}$ is

$$L_{k,n} = r_1^n + r_2^n.$$

k −Lucas sequence as numbered;

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}$$

$L_{k,0} = 2,$

$L_{k,1} = k,$

$L_{k,2} = kL_{k,1} + L_{k,0} = k^2 + 2,$

$L_{k,3} = kL_{k,2} + L_{k,1} = k(k^2 + 2) + 1 = k^3 + 3k,$

$L_{k,4} = kL_{k,3} + L_{k,2} = k(k^3 + 3k) + k^2 + 2,$

$L_{k,4} = k^4 + 4k^2 + 2,$

$L_{k,5} = kL_{k,4} + L_{k,3} = k(k^4 + 4k^2 + 2) + k^3 + 3k,$

$L_{k,5} = k^5 + 5k^3 + 5k,$

$L_{k,6} = kL_{k,5} + L_{k,4} = k(k^5 + 5k^3 + 5k) + k^4 + 4k^2 + 2,$

$L_{k,6} = k^6 + 6k^4 + 9k^2 + 2,$

$$L_{k,7} = kL_{k,6} + L_{k,5}$$

$$= k(k^6 + 6k^4 + 9k^2 + 2) + k^5 + 5k^3 + 5k$$

$$= k^7 + 7k^5 + 14k^3 + 7k.$$

2.1 Catalan Numbers

For $n \geq 0$, the n^{th} Catalan number is showed by [Barry, 2005]

$$C_n = \frac{1}{n+1}\binom{2n}{n} \text{ or } C_n = \frac{(2n)!}{(n+1)!n!}$$

and its generating function is given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

For some n, the first Catalan numbers are

$\{1, 1, 2, 5, 14, 132, 429, 1430, 4862, 16796, \ldots\}$

,from now on OEIS, as A000108 in http://en.wikipedia.org/wiki/Catalan number.

3. Catalan transform of the k −Lucas sequence

Following [Barry, 2005], we define the Catalan transform of the k −Lucas sequence $\{L_{k,n}\}$ as

$$CL_{k,n} = \sum_{i=1}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i}L_{k,i} \text{ for } n \geq 1$$

with $CL_{k,0} = 0$.

We can give some members of Catalan transform of the first k −Lucas numbers. These are the polynomials in k:

$$CL_{k,1} = \sum_{i=1}^{1} \frac{i}{2-i}\binom{2-i}{1-i} L_{k,i} = k,$$

$$CL_{k,2} = \sum_{i=1}^{2} \frac{i}{4-i}\binom{4-i}{2-i} L_{k,i} = k^2 + k + 2,$$

$$CL_{k,3} = \sum_{i=1}^{3} \frac{i}{6-i}\binom{6-i}{3-i} L_{k,i} = k^3 + 2k^2 + 5k + 4,$$

$$CL_{k,4} = \sum_{i=1}^{4} \frac{i}{8-i}\binom{8-i}{4-i} L_{k,i} = k^4 + 3k^3 + 9k^2 + 14k + 12,$$

$$CL_{k,5} = \sum_{i=1}^{5} \frac{i}{10-i}\binom{10-i}{5-i} L_{k,i} = k^5 + 4k^4 + 14k^3 + 30k^2 + 46k + 36,$$

$$CL_{k,6} = \sum_{i=1}^{6} \frac{i}{12-i}\binom{12-i}{6-i} L_{k,i} = k^6 + 5k^5 + 20k^4 + 53k^3 + 107k^2 + 151k + 114,$$

$$CL_{k,7} = \sum_{i=1}^{7} \frac{i}{14-i}\binom{14-i}{7-i} L_{k,i} = k^7 + 6k^6 + 27k^5 + 84k^4 + 204k^3 + 378k^2 + 509k + 372.$$

We can show $\{L_{k,n}\}$ as the $n \times 1$ matrix L_k and the product of the lower triangular matrix C as

$$\begin{bmatrix} CL_{k,1} \\ CL_{k,2} \\ CL_{k,3} \\ CL_{k,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 5 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} L_{k,1} \\ L_{k,2} \\ L_{k,3} \\ L_{k,4} \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} k \\ k^2 + k + 2 \\ k^3 + 2k^2 + 5k + 4 \\ k^4 + 3k^3 + 9k^2 + 14k + 12 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 5 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} k \\ k^2 + 2 \\ k^3 + 3k \\ k^4 + 4k^2 + 2 \\ \vdots \end{bmatrix}$$

where

$$C_{i,j} = \sum_{r=j-1}^{i-1} C_{i-1,r}.$$

The lower triangular matrix $C_{n,n-i}$ is known as the Catalan triangle and its elements verify the formula

$$C_{n,n-i} = \frac{(2n-i)!\,(i+1)}{(n-i)!\,(n+1)!}$$

with $0 \leq i \leq n$.

4. Generating function

We know that the generating function of the k −Lucas polynomials and the generating function of the Catalan numbers, respectively, are $L_k(x) = \frac{2-kx}{1-kx-x^2}$ and $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ in [1, 14].

Let $C(x)$ and $A(x)$, respectively, be the generating function of the sequence of the Catalan numbers $\{C_n\}$ and and the generating function of the sequence $\{a_n\}$. It is proved that $A(x * c(x))$ is the generating function of the Catalan transform of this last sequence [Barry, 2005]. Consequently, we obtain that the generating function of the Catalan transform of the k −Lucas sequence $\{L_{k,n}\}$ is

$$CL_k(x) = L_k(x * C(x)) = \frac{4 - k + k.\sqrt{1-4x}}{1 + 2x - k + (k+1)\sqrt{1-4x}}.$$

5. Hankel Transform

$A = \{a_0, a_1, a_2, \dots,\}$ is a sequence of real numbers [Cvetkovi´c, et al., 2002; Layman, 2001]. The Hankel transform of the sequence A is the sequence of determinants $H_n = Det[a_{i+j-2}]$, i.e.,

$$H_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{vmatrix}.$$

The upper-left $n \times n$ subdeterminant of H_n is called the Hankel determinant of order n of the sequence A.

The Hankel transform of the Catalan sequence is the sequence $\{1, 1, 1, \dots\}$ [Sloane, 2007] and the Hankel transform of the sum of consecutive generalized Catalan numbers is the bisection of classical Fibonacci sequence [Rajkovi´c, et al.,2007]. It is very interesting the study of the Catalan transform of this k-Lucas sequence, as we will see in the sequel.

Considering the Catalan transform of the k −Lucas sequence of the preceding subsection, we find out:

$$HCL_1 = Det[k] = k$$

$$HCL_2 = \begin{vmatrix} k & k^2+k+2 \\ k^2+k+2 & k^3+2k^2+5k+4 \end{vmatrix} = -4$$

$$HCL_3 = \begin{vmatrix} k & k^2+k+2 & k^3+2k^2+5k+4 \\ k^2+k+2 & k^3+2k^2+5k+4 & k^4+3k^3+9k^2+14k+12 \\ k^3+2k^2+5k+4 & k^4+3k^3+9k^2+14k+12 & k^5+4k^4+14k^3+30k^2+46k+36 \end{vmatrix}$$

$$= k^3 - 4k^2 - 8k - 16.$$

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