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A COMPARISON OF DIFFERENT ESTIMATION METHODS FOR THE PARAMETERS OF THE WEIBULL LINDLEY DISTRIBUTION

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ABSTRACT

In this study, we consider different estimation methods for the parameters of Weibull Lindley distribution introduced by Ashgarzadeh et al. [1]. In this context, maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramer Von Mises (CVM) and Anderson Darling (AD) estimation methods are utilized. The main focus of this study is to examine performances of these estimation methods. For this purpose, we carry out a Monte-Carlo simulation study based on different parameter settings and various values of the sample size. Results show that the AD estimators are almost preferable. Two real life data sets taken from the literature are also considered at end of the study.

Keywords: Weibull Lindley distribution, Parameter estimation, Bias, Efficiency

1. INTRODUCTION

There has been a great interest to Lindley distribution, proposed by Lindley [2], in statistics and many related areas because of its tractable properties. For example, its probability density function (pdf), cumulative density function (cdf) and hazard rate function (hrf) have simple and closed forms. Furthermore, it provides flexibility in terms of modelling, see for example [3]. These features of Lindley distribution lead to researchers to study extensions and generalizations of it. In the related literature, there are many papers considering its different generalizations [4-8]. Weibull is another widely-used and well-known distribution in statistics and other areas such as engineering, reliability and so on [9]. Although it is used in lifetime analysis frequently, it is not suitable to model all data sets [10, 11]. This is because of the fact that its hazard rate function is monotone decreasing or increasing according to value of the shape parameter. In other words, data sets having bathtub hazard rates cannot be modelled using Weibull distributions. Therefore, different extensions/generalizations of Weibull distribution is considered in the related literature, see for example [12-17].

It is clear that extended or generalized versions of a distribution are more flexible than its simplest version in terms of accommodating the skewness and/or kurtosis. However, most of the generalized distributions include a large number of parameters which may bring some difficulties in studying mathematical and/or statistical features especially for small sample sizes. Asgharzadeh et al. [1] propose Weibull Lindley (WL) distribution as a new generalization of Lindley distribution. As its name refers, the WL distribution is obtained by compounding the well-known Weibull and Lindley distributions. The advantage of the WL distribution is that it includes three parameters.

The derivation of the WL distribution is explained as follows [1]: Assume that independent random variables Y and Z have Lindley(λ) and Weibull(α,β) distributions, respectively. Then, the random variable X defined as the minimum of Y and Z, i.e. $X=\min(Y,Z)$, has the WL distribution. The pdf of X is given by

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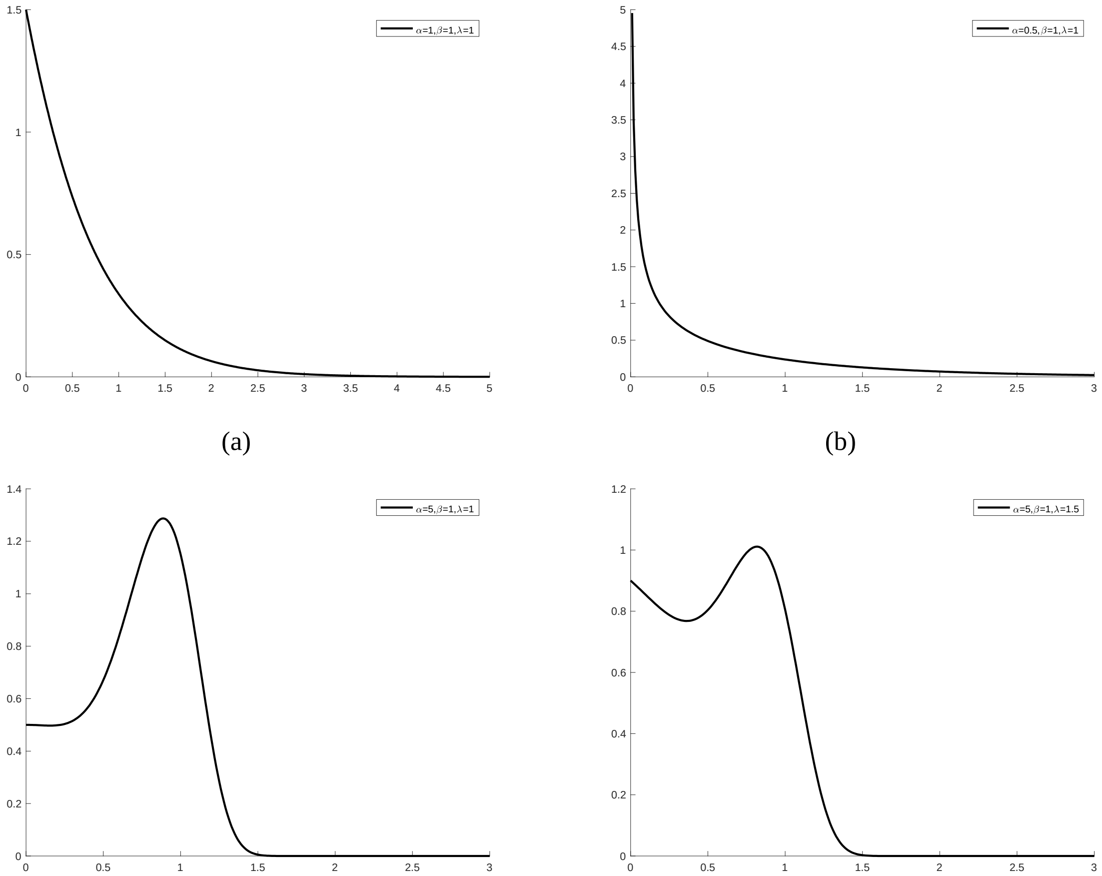
$$f(x) = \frac{1}{1+\lambda}\left[\alpha\lambda(\beta x)^{\alpha} + \alpha\beta(1+\lambda)(\beta x)^{\alpha-1} + \lambda^2(1+x)\right]e^{-\lambda x-(\beta x)^{\alpha}} \tag{1}$$

where $x > 0, \alpha > 0, \beta \geq 0$ and $\lambda \geq 0$. The random variable X having the WL distribution with parameters α, β and λ is shortly denoted by $X \sim WL(\alpha, \beta, \lambda)$. The WL distribution reduces to the Weibull and Lindley distributions when $\lambda = 0$ and $\beta = 0$, respectively. It is clear that the Rayleigh and exponential distributions are also special cases of the WL distribution, i.e. Rayleigh distribution is obtained for $\lambda = 0$ and $\beta = 2$ while the exponential distribution occurs when $\lambda = 0$ and $\alpha = 1$. The WL distribution can be bimodal for different parameter settings. The plots of the WL distribution are given in Figure 1 for some selected values of the parameters.



(a) (b) (c) (d)

Figure 1. The pdf plots of the WL distribution for some selected values of the parameters (α, β, λ):
(a) (1,1,1) (b) (0.5,1,1) (c) (5,1,1) (d) (5,1,1.5)

The cdf and the hrf of X are formulated as

$$F(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x-(\beta x)^{\alpha}} \tag{2}$$

and

$$h(x) = \frac{\lambda^2(1+x)}{1+\lambda+\lambda x} + \alpha\beta(\beta x)^{\alpha-1}, \tag{3}$$

respectively. The superiority of the WL distribution over many other generalizations of the Lindley distributions is that it has a bathtub shaped hazard rate besides it can be decreasing or increasing. On the other hand, none of the known generalizations of the Lindley distribution is able to model bathtub shaped hazard rate [1]. We refer to Asgharzadeh et al. [1] for further details about the WL distribution.

In spite of the fact that the WL distribution has attractive properties, it has not been considered in the context of evaluating the performances of the different estimation methods as far as we know. Indeed, the maximum likelihood (ML) estimation is applied by Asgharzadeh et al. [1]. In this study, we also consider least squares (LS), weighted LS (WLS), Cramer Von-Mises (CVM) and Anderson Darling (AD) methods which can be seen in the context of minimum distance estimation methods. In these methods, the distance between the estimated and empirical cdf is minimized with respect to the parameters of interest. Norms and the goodness of fit statistics are mostly used as distance criteria, see e.g. [18-20] for further details.

The aim of this study is to compare the efficiencies of the ML, LS, WLS, CVM and AD methods which are used to estimate the unknown parameters of the WL distribution. We carry out an extensive Monte-Carlo simulation study to compare the performances of the considered estimators. It should also be noted that there are many papers in which the performances of the different estimation methods for the parameters of a distribution are compared. One can see for example Mazucheli et al. [21] and Akgul and Senoglu [22] in this context.

The rest of the paper is organized as follows. Descriptions of the ML, LS, WLS, CVM and AD methods are briefly given in Section 2. Section 3 is reserved to Monte-Carlo simulation study and its results. In Section 4, two real data sets are considered to show the implementation of the different estimation methods in estimating the parameters of the WL distribution. The paper is finalized with a conclusion section.

2. METHODS FOR PARAMETER ESTIMATION

This section includes the parameter estimation methods, i.e. we briefly review the ML, LS, WLS, CVM and AD methods. It should be noted that the ML estimation of the parameters of the WL distribution has already been considered by Asgharzadeh et al. [1]. To the best of our knowledge, other estimation methods are firstly considered in this study to estimate the unknown parameters of the WL distribution.

2.1. ML Method

Let $X_1, X_2, \dots, X_n$ be a random sample from WL(α, β, λ) distribution. Then, the loglikelihood ($\log L$) function is given by

$$\log L = \sum_{i=1}^{n} \log[\alpha\lambda(\beta x_i)^{\alpha} + \alpha\beta(1+\lambda)(\beta x_i)^{\alpha-1} + \lambda^2(1+x_i)] - \lambda \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} (\beta x_i)^{\alpha} - n\log(1+\lambda). \tag{4}$$

It is well-known that the ML estimates of the unknown parameters are points that $\log L$ function attains its maximum. Therefore, the partial derivates of $\log L$ function with respect to the parameters of interest should be taken to obtain the following likelihood equations. They are given as follows:

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{M(x_i)} \{[\beta(1+\lambda) + \alpha\beta(1+\lambda)\log(\beta x_i)](\beta x_i)^{\alpha} + [\lambda + \alpha\lambda \log(\beta x_i)^{\alpha}]\} - \lambda \sum_{i=1}^{n} (\beta x_i)^{\alpha} \log(\beta x_i) = 0,$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\beta M(x_i)} [\alpha\beta(1+x_i)(\beta x_i)^{\alpha-1} + \alpha^2\lambda(\beta x_i)^{\alpha}] - \frac{\alpha}{\beta}\sum_{i=1}^{n} (\beta x_i)^{\alpha} = 0,$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{1}{M(x_i)} [\alpha(\beta x_i)^{\alpha} + \alpha\beta(\beta x_i)^{\alpha-1} + 2\lambda(1+x_i)] - \sum_{i=1}^{n} x_i - \frac{n}{1+\lambda} = 0$$

where $M(x) = \lambda^2(1+x) + \alpha\lambda(\beta x)^{\alpha} + \alpha\beta(1+\lambda)(\beta x)^{\alpha-1}$. It is clear that likelihood equations cannot be solved explicitly. Therefore, numerical methods should be performed. See Ashgarzadeh et al. [1] for more details about the ML estimators of the parameters of the WL distribution.

2.2. LS Methodology

The LS estimators of the parameters of the WL distribution are obtained by minimizing the following function:

$$S = \frac{1}{n}\sum_{i=1}^{n} \left(F(x_{(i)}) - \frac{i}{n+1}\right)^2 \tag{5}$$

with respect to the parameters α, λ and β. Here, $F(\cdot)$ is the cdf of the WL distribution given in Equation (2) and $x_{(i)}$ stands for i −th ordered observation. The LS estimates of the parameters α, λ and β are obtained as solutions of the following nonlinear equations:

$$\frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n} \left(F(x_{(i)}; \alpha, \beta, \lambda) - \frac{i}{n+1}\right) \Lambda_1(x_{(i)}; \alpha, \beta, \lambda) = 0,$$

$$\frac{\partial S}{\partial \beta} = \sum_{i=1}^{n} \left(F(x_{(i)}; \alpha, \beta, \lambda) - \frac{i}{n+1}\right) \Lambda_2(x_{(i)}; \alpha, \beta, \lambda) = 0,$$

$$\frac{\partial S}{\partial \lambda} = \sum_{i=1}^{n} \left(F(x_{(i)}; \alpha, \beta, \lambda) - \frac{i}{n+1}\right) \Lambda_3(x_{(i)}; \alpha, \beta, \lambda) = 0$$

where

$$\Lambda_1(x_{(i)}; \alpha, \beta, \lambda) = \frac{(1+\lambda+\lambda x_{(i)})(\beta x_{(i)})^{\alpha} \ln(\beta x_{(i)})\, e^{-\lambda x_{(i)} - (\beta x_{(i)})^{\alpha}}}{(1+\lambda)}, \tag{6}$$

$$\Lambda_2(x_{(i)}; \alpha, \beta, \lambda) = \frac{\alpha(1+\lambda+\lambda x_{(i)})(\beta x_{(i)})^{\alpha} e^{-\lambda x_{(i)} - (\beta x_{(i)})^{\alpha}}}{\beta(1+\lambda)}, \tag{7}$$

$$\Lambda_3(x_{(i)}; \alpha, \beta, \lambda) = \frac{(2+\lambda+\lambda x_{(i)} + x_{(i)})\lambda x_{(i)} e^{-\lambda x_{(i)} - (\beta x_{(i)})^{\alpha}}}{(1+\lambda)^2}. \tag{8}$$

It is clear that LS estimators should also be obtained using numerical methods. We refer to Swain et al. [23] for details about LS estimation method.

2.3. WLS Methodology

The WLS estimators of the parameters of the WL distribution are obtained by minimizing the following function:

$$S_w = \frac{1}{n}\sum_{i=1}^{n} w_i \left(F(x_{(i)}) - \frac{i}{n+1}\right)^2 \tag{9}$$

where $F(\cdot)$ is the cdf of the WL distribution given in Equation (2) and $x_{(i)}$ denotes i −th ordered observation. The weights denoted by w_i are formulated as follows:

$$w_i = \frac{1}{Var\left[F(x_{(i)})\right]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}, i = 1,2,\dots,n.$$

After taking partial derivatives of the S_W with respect to the parameters of interest, i.e. α, λ and β, following nonlinear equations are obtained:

$$\frac{\partial S_w}{\partial \alpha} = \sum_{i=1}^{n} w_i \left(F(x_{(i)};\alpha,\beta,\lambda) - \frac{i}{n+1}\right)\Lambda_1(x_{(i)};\alpha,\beta,\lambda) = 0,$$

$$\frac{\partial S_w}{\partial \beta} = \sum_{i=1}^{n} w_i \left(F(x_{(i)};\alpha,\beta,\lambda) - \frac{i}{n+1}\right)\Lambda_2(x_{(i)};\alpha,\beta,\lambda) = 0,$$

$$\frac{\partial S_w}{\partial \lambda} = \sum_{i=1}^{n} w_i \left(F(x_{(i)};\alpha,\beta,\lambda) - \frac{i}{n+1}\right)\Lambda_3(x_{(i)};\alpha,\beta,\lambda) = 0$$

where Λ_1, Λ_2 and Λ_3 are given in Equations (6), (7) and (8), respectively. The solutions of these equations give the WLS estimates of the parameters α, λ and β. Similar to LS estimators, WLS estimators should also be obtained using numerical methods. See Swain et al. [23] for more details about WLS estimation method.

2.4. CVM Method

The CVM estimators of the parameters α, β and λ are obtained by minimizing the following function [24]:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)}) - \frac{2i-1}{2n}\right)^2 \tag{7}$$

where $F(\cdot)$ is the cdf of the WL distribution given in Equation (2) and $x_{(i)}$ is i −th ordered observation. It should be noted that CVM estimators cannot be obtained explicitly as in other methods since the following estimating equations include nonlinear functions of the parameters:

$$\frac{\partial W^2}{\partial \alpha} = \sum_{i=1}^{n} \left(F(x_{(i)};\alpha,\beta,\lambda) - \frac{2i-1}{2n}\right)\Lambda_1(x_{(i)};\alpha,\beta,\lambda) = 0,$$

$$\frac{\partial W^2}{\partial \beta} = \sum_{i=1}^{n} \left(F(x_{(i)};\alpha,\beta,\lambda) - \frac{2i-1}{2n}\right)\Lambda_2(x_{(i)};\alpha,\beta,\lambda) = 0,$$

$$\frac{\partial W^2}{\partial \lambda} = \sum_{i=1}^{n} \left(F\left(x_{(i)}; \alpha, \beta, \lambda\right) - \frac{2i-1}{2n} \right) \Lambda_3\left(x_{(i)}; \alpha, \beta, \lambda\right) = 0$$

where Λ_1, Λ_2 and Λ_3 are the same as in the LS and WLS methods. Therefore, numerical methods should be utilized to obtain the CVM estimates.

2.5. AD Method

The AD estimators of the parameters of the WL distribution are obtained by minimizing the following function [25]:

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log F\left(x_{(i)}\right) + \log\left(1 - F\left(x_{(n+i-1)}\right)\right) \right\} \tag{8}$$

with respect to the parameters α, λ and β. Here, $F(\cdot)$ is the cdf of the WL distribution given in Equation (2) and $x_{(i)}$ stands for i −th ordered observation. The AD estimates of the parameters are solutions of the following nonlinear equations:

$$\frac{\partial A^2}{\partial \alpha} = -\frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \frac{\left(1+\lambda+\lambda x_{(i)}\right) \ln\left(\beta x_{(i)}\right) \left(\beta x_{(i)}\right)^{\alpha}}{(1+\lambda) e^{\lambda x_{(i)} + \left(\beta x_{(i)}\right)^{\alpha}} - \lambda\left(1 + x_{(i)}\right) - 1} - \left(\beta x_{(n+i-1)}\right)^{\alpha} \ln\left(\beta x_{(n+i-1)}\right) \right\} = 0,$$

$$\frac{\partial A^2}{\partial \beta} = -\frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \frac{\alpha\left(1+\lambda+\lambda x_{(i)}\right)\left(\beta x_{(i)}\right)^{\alpha}}{\beta(1+\lambda) e^{\lambda x_{(i)} + \left(\beta x_{(i)}\right)^{\alpha}} - \beta\left(1+\lambda+\lambda x_{(i)}\right)} - \alpha {x_{(n+i-1)}}^{\alpha} \beta^{\alpha-1} \right\} = 0,$$

$$\frac{\partial A^2}{\partial \lambda} = -\frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ -\frac{\lambda x_{(i)}\left(2+\lambda+x_{(i)}+\lambda x_{(i)}\right)}{(1+\lambda)\left(1+\lambda+\lambda x - (1+\lambda) e^{\lambda x_{(i)} + \left(\beta x_{(i)}\right)^{\alpha}}\right)} - \frac{\lambda x(2+\lambda+x+\lambda x)}{(1+\lambda)(1+\lambda+\lambda x)} \right\} = 0.$$

Since the AD estimators cannot be obtained explicitly, numerical methods should be used.

3. SIMULATION STUDY

In this section, we provide a Monte-Carlo simulation study which is carried out to compare the efficiencies of the estimation methods described in the previous section. We generate random samples from the WL distribution using the algorithm given by Asgharzadeh [1] for several values of the sample size (n) and different parameter settings. The sample size is taken to be $n = 20, 50, 100, 200$ and 500. Without loss of generality, we use the following scenarios in which different parameter values are considered:

	Scenario			
Parameter	1	2	3	4
α	1	0.5	5	5
β	1	1	1	1
λ	1	1	1	1.5

The computation of the estimates is done using "`fminsearch`" function which is available in the optimization toolbox of MATLABR2017b software. The performances of the estimators are compared using mean and mean squared error (MSE) criteria based on 1000 Monte Carlo runs. The deficiency (DEF) criterion is also considered to compute the joint efficiencies of the estimators, see e.g. Kantar and Senoglu [9], Akgul et al. [10]. The DEF is formulated as follows:

$$DEF = MSE(\hat{\alpha}) + MSE(\hat{\beta}) + MSE(\hat{\lambda}). \tag{9}$$

The results of the Monte-Carlo simulation study are tabulated in Tables 1 – 4.

Results according to Table 1

- The ML, LS, WLS, CVM and AD estimators of α have larger bias values except $n = 500$. Furthermore, the LS, WLS and AD estimators are relatively unbiased when $n = 200$. In terms of MSE, the LS estimator of α is the best for small sample size for $n = 20$ and 50. However, the AD estimator is more preferable for the remaining values of the sample size since it has the minimum MSE value.

- The ML method overestimates β for $n = 20$ and has high bias values for the remaining sample sizes. The LS, WLS, CVM and AD estimators are almost unbiased for $n = 100$ and $n = 200$. While the AD estimator of β is more efficient when the sample size is 20, the CVM estimator gains efficiency for the other sample sizes according to the MSE criterion. The LS estimator also gives promising results while estimating β.

- λ is generally underestimated by all methods and they are biased in most of the cases. The ML and CVM estimators have negligible biases for $n = 200$ and $n = 500$, respectively. The MSEs of the LS and CVM estimators of λ are close to each other except $n = 20, 50$ and 200. They perform better than the others in terms of MSE criterion in most of the cases. The AD estimator also provides a promising performance.

- The LS estimator is the best in terms of DEF criteria for scenario 1 when the sample size is 20, 200 and 500. It is followed by the AD estimator which is more preferable for $n = 50$ and 100.

Results according to Table 2

- The biases of the estimators of α are negligible for all values of the sample size. The ML estimator has minimum MSE value for parameter α. The AD estimator is the second best.

- The LS, WLS, CVM and AD estimators of β have higher bias values for $n = 20$ and 50. However, they become unbiased as the sample size increases. The ML estimator of β is also biased for $n = 20$ and 50 but its bias decreases when $n \geq 100$. The AD estimator is almost the best one according to the MSE criterion. The LS and the WLS, ML estimators also provide satisfactory results when $n = 20$ and 500, respectively.

- The estimators of λ have larger biases values except the ML for small n. Indeed, the ML estimator is unbiased for all values of the sample size. While the AD has the smallest MSE values in estimating λ for $n = 20$ and 50, the ML method is more preferable for other values of the sample size.

- The DEF values for scenario 2 show that the LS and the ML estimators are more preferable in terms of MSE criterion for $n = 20$ and 500, respectively. In the remaining values of the sample size, the AD estimator outperforms its rivals.

Results according to Table 3

- All estimators of α have larger biases when $n \leq 100$. However, they become unbiased as the sample size increases. The AD estimator of α is more preferable for the values of sample size 20, 50 and 100. Otherwise, the ML method has the smallest MSE value but it is followed by the AD.

- The LS,WLS, CVM and AD estimators of β are unbiased for all values of the sample size. The MSEs of these estimators are also close to each other except $n = 20$. Therefore, all estimators can be preferred while estimating β.

- The biases of the all estimators of λ are almost unbiased. The MSEs of the estimators are more or less the same as for larger values of the sample sizes. The WLS has the smallest MSE in majority of the cases.

- According to the DEF criterion the AD estimator has the best performance for $n \leq 100$ for scenario 3. Otherwise, the ML estimator can be preferred.

Results according to Table 4

- α is overestimated and its estimators have larger bias values for $n \leq 200$. The AD estimator of α provides a better performance than the others for $n = 20$, 50 and 100 in terms of MSE criterion. However, the ML method gains efficiency for larger values of the sample size.

- The biases of all estimators of β are negligible for all considered cases. The WLS estimator has the minimum MSE value for $n = 20$ and 50. However, all estimators are efficient when the sample size increases.

- All estimators of λ have small amount of bias value except $n = 20$ and 50. The WLS estimator has satisfactory results in terms of MSE values but all estimators gain efficiency when n gets larger.

- The DEF values imply that the AD and ML estimators are more preferable for scenario 4 when $n \leq 100$ and $n \geq 200$, respectively.

Overall, we suggest to use the AD method for estimating the parameters of the WL distribution since it has the minimum DEF values when the sample size is less and equal than 100. It should be noted that the ML method is more preferable for larger values of n. However, it is not efficient for small sample sizes. We therefore suggest to use the AD for larger values of sample sizes since it is the second best after the ML method.

Table 1. Simulated means and MSEs of the estimators based on Scenario 1.

	$\alpha = 1$		$\beta = 1$		$\lambda = 1$		
	MEAN	MSE	MEAN	MSE	MEAN	MSE	DEF
Method				$n = 20$			
ML	1.7144	12.5855	1.0346	0.9664	0.7997	1.1884	14.7403
LS	1.3937	1.7736	1.1518	0.5576	0.7365	0.8040	3.1352
WLS	1.3564	2.0566	1.1208	0.6067	0.7418	0.8753	3.5387
CVM	1.6018	2.4777	1.1889	0.5613	0.7751	0.8608	3.8998
AD	1.3203	1.9618	1.1329	0.5318	0.7322	0.8608	3.3545
				$n = 50$			
ML	1.4573	6.1572	0.9719	0.9200	0.8807	1.0800	8.1572
LS	1.2152	0.6291	1.1614	0.4671	0.7054	0.8310	1.9272
WLS	1.2255	0.7477	1.1054	0.4954	0.7595	0.8340	2.0771
CVM	1.3056	0.8177	1.1785	0.4358	0.7233	0.7952	2.0487
AD	1.1868	0.4712	1.1261	0.5188	0.7328	0.8856	1.8756
				$n = 100$			
ML	1.2378	1.3111	0.8766	1.2996	0.9543	1.4770	4.0877
LS	1.1734	0.3933	1.0898	0.4238	0.7746	0.7998	1.6170
WLS	1.1609	0.3803	1.0578	0.4597	0.7955	0.8426	1.6826
CVM	1.2390	0.4738	1.0938	0.4045	0.8071	0.7892	1.6676
AD	1.1537	0.2897	1.1011	0.4293	0.7405	0.8257	1.5447
				$n = 200$			
ML	1.1913	1.8614	0.8207	1.3046	1.0334	1.5864	4.7523
LS	1.0721	0.1665	1.0553	0.4640	0.7910	0.8289	1.4594
WLS	1.0850	0.2388	1.0023	0.5220	0.8427	0.8790	1.6398
CVM	1.1110	0.1912	1.0533	0.5057	0.8024	0.8958	1.5927
AD	1.0579	0.1309	1.0357	0.5091	0.7997	0.8706	1.5106
				$n = 500$			
ML	1.0608	0.2721	0.7336	1.6002	1.1371	1.9167	3.7890
LS	1.0084	0.0586	0.9596	0.4761	0.9100	0.7670	1.3017
WLS	1.0132	0.0719	0.9788	0.4851	0.8717	0.7913	1.3483
CVM	1.0430	0.1132	0.9131	0.4607	0.9902	0.7451	1.3190
AD	1.0206	0.0583	0.9890	0.4686	0.8662	0.7832	1.3101

Table 2. Simulated means and MSEs of the estimators based on Scenario 2.

	$\alpha = 0.5$		$\beta = 1$		$\lambda = 1$		
	MEAN	MSE	MEAN	MSE	MEAN	MSE	DEF
Method				$n = 20$			
ML	0.5398	0.0304	0.9640	1.8145	1.0952	0.6365	2.4814
LS	0.4883	0.0668	1.3437	1.4233	0.7379	0.6037	2.0939
WLS	0.4925	0.1159	1.3114	1.4551	0.7837	0.5614	2.1325
CVM	0.5365	0.0978	1.3762	1.7262	0.8311	0.6716	2.4956
AD	0.4903	0.0363	1.2835	1.5615	0.8784	0.5592	2.1570
				$n = 50$			
ML	0.5241	0.0157	0.9865	1.0581	1.0331	0.3954	1.4692
LS	0.4844	0.0208	1.2031	0.8569	0.8451	0.4011	1.2788
WLS	0.4901	0.0192	1.1807	0.8239	0.8745	0.3674	1.2105
CVM	0.4993	0.0221	1.2177	0.8955	0.8718	0.4169	1.3345
AD	0.4964	0.0180	1.1693	0.8027	0.9084	0.3479	1.1685
				$n = 100$			
ML	0.5106	0.0082	1.0294	0.5338	0.9866	0.1893	0.7313
LS	0.4923	0.0133	1.1272	0.5802	0.8792	0.2909	0.8845
WLS	0.4952	0.0110	1.0990	0.5034	0.9175	0.2352	0.7496
CVM	0.5015	0.0135	1.1405	0.5910	0.8894	0.2940	0.8985
AD	0.4988	0.0106	1.0966	0.4870	0.9330	0.2188	0.7165
				$n = 200$			
ML	0.5050	0.0044	1.0356	0.2868	0.9865	0.0943	0.3856
LS	0.4908	0.0075	1.0470	0.3564	0.9591	0.1655	0.5294
WLS	0.4974	0.0056	1.0529	0.2970	0.9662	0.1198	0.4224
CVM	0.4957	0.0074	1.0541	0.3586	0.9646	0.1665	0.5325
AD	0.4995	0.0054	1.0562	0.2901	0.9690	0.1121	0.4075
				$n = 500$			
ML	0.5025	0.0019	1.0303	0.1049	0.9897	0.0330	0.1398
LS	0.5013	0.0036	1.0523	0.1759	0.9644	0.0686	0.2481
WLS	0.5016	0.0024	1.0385	0.1274	0.9790	0.0430	0.1729
CVM	0.5029	0.0036	1.0536	0.1762	0.9681	0.0680	0.2477
AD	0.5024	0.0024	1.0411	0.1265	0.9786	0.0422	0.1710

Table 3. Simulated means and MSEs of the estimators based on Scenario 3.

	$\alpha = 5$		$\beta = 1$		$\lambda = 1$		
	MEAN	MSE	MEAN	MSE	MEAN	MSE	DEF
Method				$n = 20$			
ML	6.9767	37.7485	1.0295	0.0144	0.9351	0.2061	37.9690
LS	6.4599	16.5876	1.0032	0.0168	0.9900	0.1503	16.7547
WLS	6.1292	14.0433	1.0060	0.0105	0.9856	0.1437	14.1976
CVM	6.4664	17.0314	1.0374	0.0131	0.9137	0.1675	17.2119
AD	5.9396	10.5258	1.0231	0.0108	0.9442	0.1502	10.6868
				$n = 50$			
ML	5.5136	3.3891	1.0091	0.0040	0.9914	0.0684	3.4615
LS	5.4676	3.5508	1.0013	0.0035	1.0060	0.0577	3.6120
WLS	5.3344	2.6685	1.0020	0.0033	1.0029	0.0555	2.7274
CVM	5.4617	3.5553	1.0134	0.0038	0.9777	0.0616	3.6207
AD	5.2485	2.0609	1.0083	0.0034	0.9861	0.0582	2.1225
				$n = 100$			
ML	5.2054	0.8425	1.0038	0.0017	0.9988	0.0286	0.8728
LS	5.1692	1.1703	1.0012	0.0017	1.0004	0.0291	1.2011
WLS	5.1280	0.9411	1.0010	0.0016	1.0012	0.0278	0.9706
CVM	5.1670	1.1759	1.0071	0.0018	0.9869	0.0300	1.2078
AD	5.0925	0.8398	1.0038	0.0017	0.9938	0.0282	0.8697
				$n = 200$			
ML	5.0719	0.3394	1.0027	0.0009	0.9916	0.0141	0.3543
LS	5.0590	0.4948	1.0007	0.0009	0.9943	0.0139	0.5096
WLS	5.0456	0.3871	1.0013	0.0009	0.9931	0.0138	0.4017
CVM	5.0572	0.4962	1.0036	0.0009	0.9876	0.0142	0.5113
AD	5.0237	0.3625	1.0026	0.0009	0.9893	0.0139	0.3772
				$n = 500$			
ML	5.0337	0.1243	1.0004	0.0003	0.9993	0.0053	0.1299
LS	5.0332	0.1795	0.9996	0.0003	1.0005	0.0054	0.1852
WLS	5.0299	0.1425	0.9998	0.0003	1.0003	0.0054	0.1482
CVM	5.0324	0.1796	1.0007	0.0003	0.9979	0.0054	0.1853
AD	5.0191	0.1385	1.0003	0.0003	0.9986	0.0054	0.1442

Table 4. Simulated means and MSEs of the estimators based on Scenario 4.

	$\alpha = 5$		$\beta = 1$		$\lambda = 1.5$		
	MEAN	MSE	MEAN	MSE	MEAN	MSE	DEF
Method				$n = 20$			
ML	10.7829	640.6584	1.0596	0.0395	1.3838	0.3260	641.0239
LS	7.2579	40.8133	1.0193	0.0381	1.3974	0.2649	41.1163
WLS	6.9039	43.6619	1.0246	0.0315	1.3935	0.2650	43.9584
CVM	7.2576	45.5161	1.0756	0.0476	1.3143	0.3270	45.8907
AD	6.5222	29.4139	1.0508	0.0438	1.3617	0.2932	29.7510
				$n = 50$			
ML	6.2880	25.7493	1.0203	0.0094	1.4748	0.1081	25.8668
LS	5.9292	7.7802	1.0127	0.0080	1.4712	0.0964	7.8846
WLS	5.7026	6.3834	1.0121	0.0080	1.4724	0.0939	6.4854
CVM	5.8972	8.3087	1.0313	0.0095	1.4427	0.1048	8.4229
AD	5.4778	4.3840	1.0217	0.0091	1.4556	0.1013	4.4945
				$n = 100$			
ML	5.3802	1.9756	1.0077	0.0032	1.5008	0.0430	2.0219
LS	5.3585	3.2012	1.0051	0.0033	1.4925	0.0453	3.2498
WLS	5.2169	2.0631	1.0053	0.0033	1.4922	0.0449	2.1113
CVM	5.3314	3.2104	1.0142	0.0037	1.4789	0.0473	3.2613
AD	5.1334	1.6198	1.0094	0.0035	1.4859	0.0461	1.6694
				$n = 200$			
ML	5.1467	0.6357	1.0035	0.0015	1.4930	0.0232	0.6604
LS	5.1749	1.1602	1.0020	0.0016	1.4914	0.0247	1.1865
WLS	5.1036	0.8405	1.0022	0.0016	1.4906	0.0247	0.8668
CVM	5.1611	1.1572	1.0064	0.0016	1.4849	0.0252	1.1840
AD	5.0708	0.7700	1.0040	0.0016	1.4876	0.0250	0.7966
				$n = 500$			
ML	5.0846	0.2547	1.0020	0.0006	1.4988	0.0088	0.2641
LS	5.0692	0.4460	1.0014	0.0006	1.4966	0.0091	0.4557
WLS	5.0633	0.3185	1.0015	0.0006	1.4976	0.0090	0.3281
CVM	5.0633	0.4454	1.0030	0.0006	1.4941	0.0092	0.4552
AD	5.0450	0.3053	1.0021	0.0006	1.4963	0.0090	0.3149

4. APPLICATION

In this section, two real data sets considered by Asgharzadeh et al. [1] are reanalyzed to show the implementation of the different estimation methods.

4.1. Data set 1

This data set contains times to reinfection of sexually transmitted diseases (STD) for 877 patients, see Klein and Moeschberger [26]. Asgharzadeh et al. [1] model this data set using the WL distribution and compare its modeling performance with seven different statistical distributions. In modeling the data, they use the ML methodology to obtain the unknown parameters of the corresponding distributions. At the end of their analysis, it is obtained that the WL distribution models the data better than its rivals in terms of the some criteria such as Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS) test statistic and etc.

Different from Asgharzadeh et al. [1], in this study the LS, WLS, CVM and AD estimation methods are considered in addition to ML method to model the data using WL distribution. The estimates of the unknown parameters are given in Table 5. It is clear from Table 5 that the estimates do not differ so much. We suggest to use AD estimators since it has the smallest KS test statistic value.

Table 5. The parameter estimates for the data set 1.

	ML	LS	WLS	CVM	AD
α	0.6434	0.6074	0.6305	0.6087	0.6136
β	0.0017	0.0017	0.0017	0.0017	0.0017
λ	0.0023	0.0022	0.0023	0.0022	0.0023
KS	0.0324	0.0265	0.0291	0.0263	0.0238

4.2. Data set 2

This data set is also taken from Klein and Moeschberger [26] and contains times to death of twenty-six psychiatric patients. Asgharzadeh et al. [1] model this data using eight different distributions including WL distribution. They show that the WL distribution provides a better fitting performance than other considered distributions. In this study, we also fit WL distribution to this data set using different estimation methods. The resulting parameter estimates are given in Table 6. It is clear from Table 6 that the ML, LS, WLS, CVM and AD estimates are more or less the same. As in the first data set, we suggest to use the AD method in terms of KS test statistic value. The ML method is also promising.

Table 6. The parameter estimates for data set 2.

	ML	LS	WLS	CVM	AD
α	9.8995	10.2507	9.4119	10.5498	10.7058
β	0.0283	0.0282	0.0281	0.0284	0.0280
λ	0.0436	0.0432	0.0429	0.0423	0.0457
KS	0.1085	0.1086	0.1087	0.1089	0.1078

5. CONCLUSION

In recent years, many new distributions have been proposed. Most of them have intractable pdfs and/or hazard rates besides having many parameters. Although the modelling performance of the distribution will increase when the number of parameters increases, there may occur some computational problems while studying mathematical and/or statistical properties. Therefore, distributions including less number of parameters and exhibiting good fitting performance have a great interest in terms of practical aspects.

In this context, the WL distribution, proposed by Asgharzadeh et al. [1], can be seen as an attractive alternative since it has three parameters and tractable statistical properties. For example, its hazard rate function is expressed analytically and take different forms (bath-tubed, increasing and decreasing) according to different parameter settings. The WL distribution has good statistical properties but it has not been considered in the context of different estimation methods as far as we know. We therefore consider the ML, LS, WLS, CVM and AD estimation methods to estimate the unknown parameters of the WL distribution. We conduct a Monte-Carlo simulation study to compare the efficiencies of the estimators.

Results of the simulation study show that the AD method outperforms its rivals in most of the considered cases. The CVM is the almost the worst estimator. The performances of the LS and WLS estimators are promising since they are the best or follow AD estimators in a limited number of cases. Although the ML method exhibits a satisfactory performance for large n, it is not efficient so longer for small values of sample size. In the applications, it is also obtained that the AD method is preferable to other estimation methods according to the KS test. We therefore suggest to use AD in estimating the parameters of the WL distribution.

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