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Research Article

Exploring the Potential Role of Reversible Reasoning: Cognitive Research on Inverse Function Problems in Mathematics

Muhammad IKRAM[1, 2], PURWANTO[3], I Nengah PARTA[4], Hery SUSANTO[4]

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Abstract

Researchers have argued that reversible reasoning is involved in all topics in mathematics. The study employed a qualitative research approach, consisted of three sessions (pre-assessment, thinking-aloud, and interview), and involved eight participants enrolled in Algebra class. The aim was to explore the potential role of reversible reasoning on students' inverse functions. The result of study indicated that there three categories of reversible reasoning that refer to the consistency of students in completing inverse function tasks, which are relational-harmonic, relational-visual, and relational-identity. Mental activities performed by the students in constructing and reasoning inverse functions were also explained. In addition, potential aspects of the students' reversible reasoning created during the process of constructing meaning were highlighted. These findings provide perspectives on reversible reasoning, students' understanding of inverse functions, and areas of future research

Keywords:
reversibility, reversible reasoning, inverse function, problem-solving

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[1] Students, Doctoral Study Program of Mathematics Education, Faculty of Mathematics and Science, Universitas Negeri Malang, Jl Semarang No. 5, East Java, Indonesia (muhammad.ikram.1603119@students.um.ac.id). Orcid no: 0000-0002-3763-4299

[2] Department of Mathematics Education, Faculty of Teachers Training and Educations, Universitas Cokroaminoto Palopo, Orcid no: 0000-0002-3763-4299

[3] Professor, Department of Mathematics Education, Faculty of Mathematics and Natural Science, Universitas Negeri Malang, Jl Semarang No. 5, East Java, Indonesia

[4] Doctor, Department of Mathematics Education, Faculty of Mathematics and Natural Science, Universitas Negeri Malang, Jl Semarang No. 5, East Java, Indonesia

Introduction

Mental activities to reverse mathematical concepts are important at all levels of mathematics (e.g., addition and subtraction of the whole number, differentiation and integration in calculus, exponent and logarithm, function and inverse function) (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Haciomeroglu, Aspinwall, & Presmeg, 2010; Ramful, 2014; Simon, Kara, Placa, & Sandir, 2016). Some researchers have explained reversible reasoning in addition and multiplication (Hackenberg, 2010; Steffe & Olive, 2010); however, reversible reasoning in the function domain has not been given much attention. Finding the original function (the initial situation) when the inverse is given is not a trivial task for students (Simon, Kara, et al., 2016). For example, students could find $f(3)$ from a given algebraic formula $f(x) = x^3 - x$, however when asked to determine $f^{-1}(x) = 3$, they showed poor understanding of the relationship between the inverse functions. Reversible reasoning was barely used in this case, nor required, to explain, establish, and provide meaning to the problem. A previous study (Ikram, Purwanto, Parta, & Susanto, 2018) has attempted to explore the ways in which students used reasoning reversible in function compositions. Unlike the previous research, the current study was focused on reversible reasoning in inverse functions.

The terms reversibility and reversible reasoning are frequently found in literature. Therefore, they are often used to depict a cognitive process in mathematical thinking. Despite having different words to describe the process, researchers still adopt Piaget and Krutetskii's definition of reversibility; for example, Wong (1977) using reversible thinking, Paoletti et al (2018) using bidirectional reasoning, and Hackenberg and colleagues (Hackenberg, 2010; Hackenberg & Lee, 2015, 2016) using reciprocal reasoning. The term reversible reasoning is defined by Ramful (Ramful, 2014, 2015) as a deductive reconstruction of a source (an initial situation) from the result (the final situation), while Chun (2017) explains reversible reasoning as an anticipation scheme that is elaborated recursively to return to the initial situation. The definitions of reversible reasoning suggested by Ramful and Chun imply that reversible reasoning involves the discovery, reconstruction, and anticipation schemes of the initial situation.

Reversible reasoning is understood as a mental activity that leads to changes in the direction of thinking. Based on its functional aspect, reversible reasoning is explained by some experts as connections/relations (García-García & Dolores-Flores, 2018; Haciomeroglu, Aspinwall, & Presmeg, 2009; Weber & Thompson, 2014), flexibility (Gray & Tall, 1994; Vilkomir & O'Donoghue, 2009), and schemes (Anderson Norton & Jesse L. M. Wilkins, 2012; Hackenberg, 2010; Simon, Kara, et al., 2016; Steffe & Olive, 2010). However, we notice that the researchers have not discovered an indication of students' reversible reasoning. An example of the concrete action is when the students are asked to verbally identify angle θ that generates $\sin\theta = \frac{1}{2}$, discover functions of which derivatives are graphically or analytically known, and identify which graphical visual representation shows the given function graph. In general, the contribution of reversible reasoning in mathematics need to be investigated further.

Although numerous researchers have explored students' function meaning (Byerley & Thompson, 2017; Carlson & Thompson, 2017; Weber & Thompson, 2014), fewer researchers have focused on students inverse function meaning. Researchers have found that students cannot develop productive and meaningful inverse functions at all stages of development (Paoletti, Stevens, Hobson, Moore, & LaForest, 2018; Wasserman, 2017). For example, Paoletti, et al. (2018) described students' attempt to determine the rule for $f^{-1}(x)$. Instead of connecting the two functions or reversing the function, they merely substituted x and y and solving it for y. Inverse function can be highlighted as an effect of reverse mental action. Therefore, reversible reasoning can play an essential part in students' mental activity to understand a concept.

Topics around inverse function are considered difficult by the majority of secondary school or even university students. One of the barriers to solving these mathematical problems is the confusion to understand "superscript -1" as a reciprocal or inverse function. Zaskis and his colleagues (Kontorovich, 2017; Zazkis & Kontorovich, 2016; Zazkis & Zazkis, 2011) have revealed that when a number is followed by an exponent (superscript -1), it will involve inverse (usually appears in the context of dividing or multiplying fractions); negative superscripts (associated with exponential operations); and inverse functions (associated with function compositions). In addition, Zazkis discovered two interpretations of superscript -1 suggested by students: (1) students considered $\square^{-1}$ as a homonymous symbol in different contexts (fractions or functions), different terminologies (inverse), different procedures and symbols; and (2) students perceived $\square^{-1}$ as a polisemy focused on general words or acts implied from the words.

Carlson et al (Carlson, Madison, & West, 2015) have found that among 601 students participating in their research, 53% of them selected the answer $f^{-1}(t) = \frac{1}{(100)^t}$, when asked to determine the formula of f^{-1} from $f(t) = 100^t$. Similarly, findings from Paoletti et al (2018) suggest that the most common mistake made by students in inverse functions result from their misconceptions of notation "$f^{-1}(x)$" that is usually interpreted as the inverse of $\frac{1}{f(x)}$. Students get confused because of their dichotomy ideas and experience trouble when using different representations. This is indicated by mixed information on superscript -1 as a reciprocal and inverse function. Besides, Paoletti also found that $92\%$of the students were more likely to calculate $f(x)$ by switching and solving, and substituting x in $f(x)$ to generate an unknown result, instead of building an equivalent relationship between $f(x) = y$ and $f^{-1}(y) = x$.

Problems of the Research

The gaps between students' inverse function meanings and reversible reasoning were examined in this study. The aim of the present study was to explore the potential role of reversible reasoning on students' inverse functions. In particular, this study used inverse function tasks and analyzed students' interpretations of these tasks. This study addressed the following research questions: What are the processes that subjects undergo when confronted with the inverse function task of a novel nature? What characterizes their solution approaches? Is it reversible or irreversible?

A think-aloud protocol was developed to identify the participants' interpretation of inverse function problems (Simon et al (2016) and Ramful, 2014). This conceptual framework has been used in the previous study (Ikram et al., 2018). The present study extended upon the research literature through the use of an empirically derived protocol. Therefore, this study does not only contribute valuable concepts into the development of students' inverse function meaning but also establish an awareness of the importance of reversible reasoning in mathematics education.

Method

Research Design

In studying individual's apprehension for a various mathematical idea, researchers are at a disadvantage and we cannot see the way of their thinking. Rather, we only have identified what an individual says with think-aloud, writes, and gestures when involving in mathematical tasks. Therefore, in our research, we focus on individuals expressed of ideas, or the spontaneous expression that an individual state about an idea. From these utterances, we can make inferences about how individuals have managed her experiences with the notion. As a result, we conducted qualitative research refers to studies in which various cases are examined in order to highlight a particular issue in depth. At last, the study is defined as a collective study, where the analyses of cases aim to serve as a tool for extending a more general understanding with respect to some phenomena or theory (Yin, 2014).

In light of the study's aim and the questions introduced above, task-based interviews were used to investigate the variety of their answers, conceptions, and reflections of a small sample of students. The collection of these interviews and their interpretation constitute the core of this qualitative research.

Participants

This study involved 8 students aged 18-22 years old from Department of Mathematics Education who were enrolled in Algebra class at one of the state universities in East Java. Since the participants had studied inverse functions in basic calculus, algebra, and algebraic structures, it was assumed that they already possessed rich and deep conceptual knowledge on this topic. We chose the students from a convenience sample of students accessible to the studies team in terms of program, location, and scheduling. Furthermore, we have considered situations to select the participants include students who had completed the first to an advanced calculus course, and experience in working with calculus in analytics and graphics context. The pseudonyms given to the participants are to help to analyze what they worked in task given. The students are known in this paper as S1, S2, S3, S4, S5, S6, S7, and S8 were asked to answer the task. Table 1 summarizes most of the background information for each participant

Table 1.
Participants Descriptions

	M/F	Undergraduate major	Number of undergrand math courses	Undergrad math grade point average	Score Pre-assement test
S1	M	Math	18	3.72	95
S2	M	Math	16	3.47	85
S3	F	Math-Edu	17	3.55	80
S4	F	Math	21	3.75	90
S5	M	Math-Edu	18	3.81	95
S6	F	Math-Edu	19	3.63	90
S7	F	Math	18	3.58	85
S8	F	Math-Edu	19	3.78	90

Data Collection

A think-aloud protocol was developed to identify patterns of students' mental activity in reversible reasoning while solving inverse function tasks, and interviews were conducted to clarify the students' answers. This study consisted of three sessions: pre-assessment, thinking-aloud, and interview. At the initial stage, pre-assessment was conducted (e.g., is $f(x) = x^2$ a function or not?). The students were asked to (1) identify whether the given function had an inverse or not; (2) identify whether the given statements were true or false; and (3) find unknown values in a function and an inverse. Instruments used in the pre-assessment stage had undergone a validation process and therefore could be used to measure students' conceptual knowledge of inverse function problems. Participants who represented high, medium, and low ability categories were selected based on the pre-assessment scores and the students' Grade Point Average (GPA). The participants consisted of three students from the Department of Mathematics and five students from the Department of Mathematics Education.

Three inverse function tasks were developed from the pre-assessment and validated by two professors who are expert in mathematics education. A field test was conducted to examine the reliability of the tasks. Based on the results of the validity and field tests, the form of the task questions was changed from closed ended into open-ended. The development of the tasks referred to Paoletti (2018), Dubinsky (2002), and Moore (2014). Task 1 aimed to reveal the students' mental activity in finding x in $h^{-1}(2) = x$ and their understanding of function composition $h = f \circ g$. Similarly, task 2 aimed to reveal how the students determined the value of $f^{-1}(4)$ if h and g were known. Finally, Task 3 aimed to reveal how students interpreted relationships between analytical or graphical functions. The three tasks were used in the thinking-aloud session (Figure 1).

To reveal the students' reversible reasoning process, a think-aloud protocol was developed. The think-aloud protocol was conducted to observe how the students activated, anticipated, coordinated, and verified problems. The students' activating, anticipating, coordinating, and verifying activities were identified based on indicators presented in Table 1. When an activity could not be encoded using the existing indicators, a new indicator was added to the protocol. In short, the think-aloud protocol was created as a typology to identify the students' reversible reasoning.
In order to ground the study, constant comparative methods was employed.

Task 1	Is there $x \in R$, so that $h^{-1}(2) = x$ for $f(x) = \frac{1}{x}$ and $g(x) = \frac{x^2+x-5}{x^2+2x+6}$ and $h = f \circ g$?
Task 2	If $h(x) = \frac{x-5}{x+1}$ and $g(x) = 2x - 1$, and h is the composite of $h = f \circ g$, what is $f^{-1}(4)$?
Task 3	If known $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ and $g(x) = -\frac{\pi}{2} + arc\sin x$. What can be concluded from $f(x)$ and $g(x)$?

Figure 1.
Tasks for Think Aloud

Activities performed by the students when solving the inverse function problems were observed and identified based on indicators presented in Table 2. *Activating* captures the process of forming an initial idea with recognition of a certain situation; *Anticipating* points out the process of reflecting ideas when recursively using the result of the scheme to produce a cause; *Coordinating* illustrates the process of matching ideas, comparing information between two ideas, bridging prior knowledge to make inferences that yield a coherent mental action, or identifying a relationship between two sources; and *Verifying* indicates the processes of reflecting on the entire solution process, identifying critical features, and developing confidence in handling the process. An elaborated instance of how the rubric was applied in data analysis is illustrated in the next section.

During the interview, the interviewer's role was restricted to providing the tasks to the participants and clarifying ideas when needed. To elicit ideas, the interviewer used prompts such as "why do you think so?"; "can you tell me what you are thinking about?"; "can you explain the markings on your answer?"; "can you give other reasons?"; "can you show that on the other representations?" and "do you think your answer is correct?".

Table 2.
Components of Reversible Reasoning

Activity	Indicators
Activating	• Read the task • Analyze the information • Clarify what needs to be accomplished
Anticipating	• Compare problems faced with problems that have been resolved before • Clarify concepts needed to produce a solution • Make assumptions to "simplify" the problem • List assumptions • Make sketches that correspond to stated or implied conditions/relationships/ assumptions • Change mathematical ideas with reversing situations, operations, relations, or representations • Consider whether the chosen idea could answers the question posed • Realize limitations of problem situations • Embody outside knowledge to help with any of above
Coordinating	• Bridge prior knowledge to make inferences • Organize the relation of two sources in problems situation • Consider Coherence of information from one representation to another • Make visualization from coordination results to solve the problem • Interpret findings
Verifying	• Reflect on the appropriateness of actions • Compare an answer to a known result • Ensure that the goal of the problem has been reached • Ascertain that there are other alternative processes to finding the result

Data Analysis

Throughout the analysis, we followed a constant comparative method (Creswell & Guetterman, 2018). The interview was videotaped, and all written work was captured. The interview video, capturing words, and gestures were transcribed. To produce descriptive categories, we used open and axial coding to construct models of the students' thinking. As the process evolved, continuous comparisons were made between each of the categories and the emerging new categories.

Each researcher analyzed the written work of the participants, describing each sign of their thinking that indicated reversible reasoning and discussing our reviewing, looking for common techniques on specific tasks or types of tasks. Transcripts of think-talk aloud data, interview transcripts, and student answers were provided as the source of data. To analyze the data, think-talk aloud data transcripts and interviews were reduced to fragments containing student explanations. The explanation is coded based on the think-aloud protocol, which has been developed as an analytical tool. Data is encoded, sorted, and repeatedly read to answer research questions

Trustworthiness was increased through (a) ensuring that the collected data are rigorous and comprehensive, by way of managing the task in a written form and producing a verbal transcription of each interview shortly after its recording; and (b) validating the process of coding and recoding of the different categories via discussions with several mathematics education specialists. One professor and two doctoral lecturers in the field of mathematics education. In the results section, we discuss findings based on emerging themes by looking for similarities and differences with previous research findings.

Coding was done according to the following instructions:

- If a participant immediately recognizes the inverse functions as an action of switching x and y and solving it for y
- If a participant sees the inverse functions as reverse mapping of a function or if $f(x) = y$, then $f^{-1}(y) = x$
- If a participant sees the inverse functions as analogous geometrical interpretations or reflection over the line, $y = x$
- If a participant sees the inverse functions as descriptor of function and reverse the sequence

We sighted how their thoughts changed from one perspective to others and how consistent their thoughts were while addressing various tasks. For instance, if a student determined an inverse function by establishing equivalent relationships between $f(x) = y$ with $f^{-1}(y) = x$ through diagram venn representation and describe the graphs, we then classified that students as having reversible with relational-harmonic. If a student determined an inverse function by visualizing problem situation, describe the graphs of function and its inverse, and reverse coordinates (x, y) to be (y, x), we then classified that students as having reversible with relational-visual. Furthermore, if a students determined an inverse function by switching x and y and solving it for y, coordinate two function to obtain identity, and reverse order of $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, we then classified that students as having reversible with relational-identity. This analysis supported our recognizing the extent to which a student's reversible reasoning in various contexts to connect with a problem situation

Results

At first, the coding of students' mental actions in completing the inverse function task is described, then the mental action code of students based on the think-aloud protocol and compare the mental actions of each student is presented to figure out the indications of students doing reversible reasoning for inverse function problems and the process of reversible reasoning. The analysis process generates data categorization with the main theme; they are reversible relational-harmonic, reversible relational-visual, reversible relational-identity.

Overall, 3 of 8 students (Table 5) shows a consistent technique for determining inverse functions by analyzing problems analytically and involving visual aspects to reverse the problem situation by constructing an equivalent relationship between functions and inverses. One student consistently acts exchanging x and y variables and completing for y and justifying that the act of exchanging variables x and y is based on previous experience. Two students consistently involve coordinating two functions to obtain identity and reverse the order to obtain the inverse function. Meanwhile, the other two students consistently show different techniques, namely building a relationship between functions and inverses using venn graphs and diagram representations.

One of the students, Adjie, for assignment 1 acted exchanging the x and y variables, but experienced obstacles and was unable to continue the completion process because it was unable to describe the form of the quadratic equation obtained. So that the analysis is done by involving the definition of an inverse function, analogous to it visually by using a venn diagram representation to construct the idea of the equivalent relationship between the function and its inverse. We summarize the mental actions of other students in.

In the second part, think-talk aloud of some students who represent three categorizations as long as they complete the task of inverse functions adjusted to the aloud think the protocol is described. Also, the interview transcript was used as supporting data to explain why they did these mental actions and things that did not appear in the process of completing the task.

The first attention goes to Budi's case, which represents a reversational-harmonic category, Erik which represents a relational-geometric reversible category, and Adjie which represents a reversible-identity category (overall, can be seen in Table 3). Overall, the think-aloud protocol for verifying is not visible; interviews are conducted as a crosschecking tool. In completing tasks 1 and 2, Budi recalled all the problems regarding inverse functions that had been resolved before. He was initially hesitant after composing $f \circ g$ for task 1 because it produced a fraction function. He identifies the existence of the inverse function of the composition of the value whether it is single or not, besides that it also involves the representation of venn diagrams to understand the basic concepts of functions and inverses. In this case, Budi outlines the nature of the results of the composition and makes an initial guess to limit the domain to the function so that it matches the definition of the inverse function. Furthermore, he realizes the relationship between $h^{-1}(2) = x$ equivalent to $h(x) = 2$, where $h(x)$ is the result of the composition of $f \circ g$ which is limited to positive or negative integers.

Similar to assignment 2, after analyzing the whole function known analytically, it involves the nature of wisdom and compositions of its two functions ($h(x) = f\big(g(x)\big)$, $h(x) = \frac{x-5}{x+1}$ and $g(x) = 2x - 1$). Next, he recognizes the relationship implied in $f^{-1}(4)$ which is analogous to x. He states that $f(2x - 11) = h(x)$ is equivalent to $f^{-1}(h(x)) = 2x - 1$, where the value of x at (x-5) / (x + 1) must be found the result is 4. In this case, Budi reverses the problem situation by involving the equivalent relationship between the function and the inverse. As for task 3, he identifies one of the functions known by describing the graph (i.e. $f(x)$). However, for other functions, it is difficult to interpret " arc". After making an analogy through another example outside the context of the problem, he questions whether $\sin^{-1}\left(\frac{1}{2}\right)$ is equivalent to the $arc\sin\left(\frac{1}{2}\right)$. In this case, he realized that the meaning of "superscript -1" in the trigonometric inverse function ($\sin^{-1} x$) is equivalent to the $arc \sin x$. Next, he views $f(x)$ and $g(x)$ as functions which are mutually inverse as evidenced by using the equivalent relationship between $f(x) = y$ and $f^{-1}(y) = x$.

Erik represents reversible relational-visual because he consistently involves the representation of pictures, graphs, and venn diagrams to analogize the relationship between functions and inverses. Just like Budi, it also involves mental actions by recalling ideas that have been learned about inverse functions, for example inverses in fraction functions (tasks 1 and 2) and trigonometry (task 3), understanding interpreting the power of -1 representing an inverse function and the sign "$\circ$" which represents the composition of a function. He made visualizations using graphics, after finding the composition results in the form of rational functions (tasks 1 and 2). He uses other mental actions by outlining his ideas about how to describe graphs of rational functions, involving the concept of limits as horizontal asymptotes (e.g. $|x| \to \infty$), vertical asymptotes, and points of inflection. He made a guess based on the results of the graph sketch for task 1 that there are two x values that yield 4, so he realizes the need to limit the domain to x. Likewise, with task 3, he draws one graph ($f(x) = \sin\left(x + \frac{\pi}{2}\right)$), understands that for $f(x) = \sin x$ is defined as a function with a domain limited to intervals $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and conclude that the domain at function $f(x)$ is limited to $[0, \pi]$. It reverses the domain and range f (x) to sketch the inverse graph of $f(x)$, namely the domain $[-1,1]$ with a range of $[0, \pi]$, comparing the two functions ($f(x)$ and $g(x)$) structural uses the equivalent relationship between function and inverse and concludes that one of the functions ($g(x)$) is the inverse of another function ($f(x)$).

Finally, Adjie represents reversible relational-identity, where he consistently takes action to exchange x and y variables to find inverse functions, coordinate two functions to obtain identity, and reverse the sequence $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Suppose for task 1, because what is asked is h^{-1}, then it changes $h^{-1}(x) = (f \circ g)^{-1}$. Likewise with task 2, because what is asked is f^{-1}, so it uses the identity element g^{-1} (e.g., $h \circ g^{-1} = f \circ g \circ g^{-1}$) and reverse the order to determine the inverse function ($f^{-1}(x) = (h \circ g^{-1})^{-1}$). Also, he interpreted the structural relationship formed from $h(x) = f\big(g(x)\big)$, Where he thought of how x at $f(2x - 1)$ to produce $f(x)$. For task 3, he assumes that $g(x)$is the inverse of $f(x)$ and proves it using the statement $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$. Suppose that $\sin\left(-\frac{\pi}{2} + \sin^{-1}(x) + \frac{\pi}{2}\right)$ is equal with $-\frac{\pi}{2} + \sin^{-1}\left(\sin\left(x + \frac{\pi}{2}\right)\right)$. Overall, students who represent reversible relational-identity do not involve the definition of inverse functions in the completion process, especially in tasks 1 and 3.

Table 3.
Category of students mental actions for inverse function tasks

Category	Descriptions
Relational-Harmonic	• Remember all the ideas that have been learned about inverses based on specific functions known • interpret superscript -1 which represents the inverse function • determine whether a function that is known to have an inverse function • compose two known functions • understand the sign "∘" as a composition of a function • declares two inverse functions • establish equivalent relationships between $f(x) = y$ with $f^{-1}(y) = x$ through diagram venn representation • describe the graph
Relational-Visual	• remember all the ideas that have been learned about inverses based on specific functions known • interpret superscript -1 which represents the inverse function • understand the sign "∘" as a composition of a function • declares two inverse functions • describe the graph of function and this inverse • reverse coordinates (x, y) to be (y, x)
Relational-identity	• remember all the ideas that have been learned about inverses based on specific functions known • interpret superscript -1 which represents the inverse function • understand the sign "∘" as a composition of a function • compose two known functions • Exchange x and y and settlement for y • Coordinate two functions to obtain identity • Exchange order of $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Discussion

Currently, fewer literature has been found related to reversible reasoning investigations in conceptual relationships for inverse function problems. The focus of previous researchers are more on the operational aspects, for example by identifying error reversals that students make for the problem of "students and professors" (González-Calero, Arnau, & Laserna-Belenguer, 2015; Soneira, González-Calero, & Arnau, 2018; Tunç-Pekkan, 2015), reversible multiplication relationships (Hackenberg, 2010), cognitive conflict and insufficient mental processes to reverse problem situations (Ramful, 2014), the type of task that causes reversible reasoning (B. Dougherty, Bryant, Bryant, & Shin, 2017; B. J. Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel, 2015; Sangwin & Jones, 2017; Simon, Kara, et al., 2016; Vilkomir & O'Donoghue, 2009). Likewise, with investigations about inverse functions, most focus more on errors made by students in solving problems (Carlson et al., 2015; Kontorovich, 2017; Paoletti et al., 2018; Zazkis & Kontorovich, 2016; Zazkis & Zazkis, 2011). In this section, the researchers conclude reversible reasoning during the process of solving inverse function problems. The writers divide the discussion into three parts: reversible reasoning indications for inverse function problems; reversible reasoning category; Reasoning implications are reversible in the learning process.

Reversible Reasoning Indication for Inverse Function Problems

Reversible reasoning is a mental action by changing the direction of thinking, by looking back at the problem analytically, involving anticipation, and reversing operations, relations, situations, or representations. These mental actions are considered as the main prerequisites for many problems at each level of mathematics (Hackenberg, 2010; Ramful, 2014; Ramful & Olive, 2008; Steffe & Olive, 2010). Steffe & Olive (2010) through the von Glasersfeld scheme theory reveals that reversible reasoning is a mental action using the output to obtain the initial situation. Especially for inverse function problems, we found that out of 8 students participating, they showed mental actions by reversing operations, relationships, situations, and representations. This is different from the findings of Ramful (2014) that reversible reasoning is sensitive to the numerical properties of the problem parameters. Also, their mental actions for each task showed consistency, so we concluded that reversible reasoning was not caused by the type of questions such as typology developed by Simon et al. (2016) or Dougherthy et al. (2015). However, it is caused by different characteristics of each student, such as involving defining, visual representation, or identity elements ($f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$).

Four components of reversible reasoning that are used to analyze mental actions were applied by students when they were solving inverse function problems, namely activating, anticipating, coordinating, and verifying. This is based on Hackenberg's opinion (2010) that anticipation is a requirement for reversible reasoning. However, in this article, we identify three indications that students do reversible reasoning, namely: analytic problem solving involving prior knowledge; there is a mental attitude towards concepts related to problems outside the context of the problem at hand; and the existence of mental actions by reversing operations, relations, situations or representations. In particular, mental actions by reversing operations and relations are in line with Piaget's reversibility definition (1958) which is called negation and compensation, while mental actions by reversing the situation are in line with Ramful's findings (2014).

It is also found that students indicated a reversal of operations when involving elements of identity to solve problems; students are indicated to do a reversal of relations when they use an equivalent relationship between functions ($f(x) = y$) and it's inverse ($f^{-1}(y) = x$); students indicated the reversal of the situation when they use verbal expressions in interpreting pieces of information in inverse functions in different ways (e.g., "$f(2x - 1)$" is interpreted as how x becomes $f(x)$); and students who indicated reversing representations were marked by constructs involving venn diagrams, graphics, images, to obtain inverse function ideas. In the anticipating stage, students reflect the structure of the problem based on the scheme they have by developing the initial idea. The anticipating stage causes abstraction by involving efforts to achieve the results of previous experiences by producing the initial situation (Ron Tzur, 2011; Simon, Placa, & Avitzur, 2016; Tzur, 2007; Von Glasersfeld, 1995).

When students reflect or recall a concept that has been formed, the scheme is active (activating stage), which is characterized by the ability to access prior knowledge through the introduction of problem situations. The activating process occurs internally but can be marked when students remember all the ideas learned about inverses. Pino-Fan et al. (2017) emphasize that the activation of the scheme involves verbal and symbolic meanings of the structure of the problem. However, schemes that have been formed are sometimes unable to be activated by students and depend on processing information that occurs in long-term memory (LTM) (Sun, 2006). In this case, activating activity is limited to identifying the problem situation, the quality of attention given to the situation, and the strength of the relationship of each activated scheme.

The formation of a new conception at the coordinating stage causes the reversal of operations, relations, situations, and representations. In this case, there is a restructuring of the old scheme in which students are not familiar with the problem situation at hand, so they need to adjust the information on the structure of the problem with the scheme they have. In this case, the influence of the old scheme was needed to build a new scheme, as revealed by Bagley et al. (2015). The findings at the coordinating stage are also in line with the opinions of Hackenberg (2010) and Ramful (2014) who state that mental actions by changing the direction of thinking trigger the accommodation process. Although, there has been no investigation linking the accommodation-assimilation process with reversible reasoning.

Stage applying functions as execution or work mathematically, where they involve different representations to find solutions and draw conclusions. The different representations of each student are described in the next section. Finally, the verifying stage was not revealed through aloud think-talk; the interview is done by asking whether they were re-verification or not and whether they were sure about the solution and others. However, the eight students revealed that verification was carried out in conjunction with the selection of ideas at the activating stage of applying.

Category of Reversible Reasoning

For inverse function problems, we categorize three reversible reasoning models of students, namely relational-harmonic, relational-visual, relational identity. This is based on a pattern that is consistently shown by students in solving problems. Students are categorized as relational-harmonics (Budi, Dian, Gina) when they do analytic analysis by involving definers, for example, whether the function fulfils the nature of the property, how the definition of the composition functions. They also involve visual representation, for example when analogizing the relationship between functions and inverses through a venn diagram expressed by notation $D(f) = R(f^{-1})$ dan $R(f) = D(f^{-1})$ dan $f(x) = y$ dan $f^{-1}(y) = x$. Also, they sketch the function and inverse graphs to reinforce the findings. In this case, students categorized as relational-harmonics utilize their conceptual knowledge of inverse functions not only isolated from the analytic aspects, but the visual aspects also play a role. Thus, we support the assumptions implied by some previous findings (Haciomeroglu et al., 2010; Natsheh & Karsenty, 2014) that strong mathematical knowledge is not only dominated by symbolic or analytical manipulation but can also utilize its visual aspects and can minimize students' difficulties in solving various types of problems.

In contrast to relational-harmonics, students (Erik and Cindy) are categorized as relational-visual, showing dominance in the visual aspect before reversing operations, situations, relations, or representations. This is indicated by the sketch of the fraction function graph (in tasks 1 and 2) which is accompanied by its parameters, for example, a flat asymptote and a sloping asymptote, and a cut off point on the x -axis and y -axis. Likewise, with graphs for trigonometric functions (in task 3). Although it does not explicitly describe the definition of an inverse function, they understand the nature of the property, build relationships between functions and inverses based on the results of sketches, and domains and ranges (e.g., $\sin x$ has a domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1,1]$). Especially for inverse trigonometric functions, these research findings are in line with the opinion of Martinez-planell & Delgado (2016) that the notion of trigonometric inverse functions related to construction varies and raises its difficulties for students.

The relational-visual category fulfils three components of the type of visual reasoning, namely the visual display through sketches of graphs or diagrams, the presence of visual actions by looking at problem situations can be represented graphically or diagrams, and the visual purpose that visualization results can reach a solution (Natsheh & Karsenty, 2014). Also, graphical representations and diagrams play an important role in developing mental visualization which is the basis for building formal understanding in depth (Delice & Kertil, 2013; Hoffkamp, 2011; Törner, Potari, & Zachariades, 2014).

Finally, students (Adjie, Fira, Hasri) categorized as relational-identity view identity elements as an important part of finding inverses (e.g., $f \circ f^{-1}(x) = x = f^{-1} \circ f(x)$), understands the meaning of composition, reverses the order (e.g., $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)), and the act of exchanging x and y and resolving it for y. Although different from the previous two categories, students in this category regard definition and visualization as complex. One student (Fira) uses the metaphor "$(g \circ f)^{-1}$" which is illustrated by wearing socks (f) then wearing shoes (g), in reverse (inverse) is removing shoes (g^{-1}), then the socks (f^{-1}). In this case, students categorized as relational-identity, in accordance with a developed genetic decomposition model (Marmur & Zazkis, 2018; Paoletti et al., 2018; Wasserman, 2017)., Namely the scheme "inverse function as coordination of two functions to obtain identity "and" inverse function as the act of exchanging x and y and completing it for y ". Wasserman (2017) asserts that the existence of an element of identity can help students in abstract algebra, where inverses are considered descriptors in Group concepts and recognize the function identity of $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x = i(x)$.

Implication of Reversible Reasoning for Education Learning Process

Although the existing literature has provided little theoretical guidance for promoting reversible reasoning, such as Simon et al. (2016), Ramful (2014), and Hackenberg (2010), developing this type of reasoning in the learning process has not been elaborated. Simon et al. (2016) noted that one way to develop it was by compiling a task typology using basic concepts that aimed to build the opposite through re-interiorization. In this case, the findings are not similar, where students independently have to develop relationships that are reversible based on the concepts learned.

Based on the three categories of reversible reasoning found, all three may be able to help teachers to minimize their habits in teaching the concept of inverse functions which only focus on the act of exchanging x and y and solving them for y. This causes isolated students to interpret inverse functions and does not involve the importance of defining visual aspects and the role of identity. It is in accordance to the findings of the previous study (Paoletti et al., 2018; Wasserman, 2017), which suggested that researchers need to survey to assess teachers' mathematical knowledge in interpreting inverse functions. So the factors that cause students to experience confusion in the rank -1 (Kontorovich, 2017; Zazkis & Kontorovich, 2016; Zazkis & Zazkis, 2011) and students interpret functions and inverses analytically and graphically (Carlson et al., 2015) can be described in detail.

From a cognitive point of view, genetic decomposition of inverse functions needs to be integrated with the learning process and allow other reversible reasoning categories to emerge. This is consistent with the mental actions in this study and the investigations carried out by previous researchers (As'ari, Kurniati, Abdullah, Muksar, & Sudirman, 2019; Paoletti et al., 2018; Thahir, Komarudin, Hasanah, & Rahmahwaty, 2019; Wasserman, 2017; Yasin et al., 2019). Where is the complexity faced by students when understanding inverse functions, namely: (1) definition of inverse function; (2) means the power of -1 which represents a function or inverse; (3) developing the definition of inverse function through the composition function; (4) interpreting inverse functions through algebraic, geometric, structural, identity, or reversing interpretations.

The implications of this research for educators are as follows: The need for awareness of the benefits of reversible reasoning so that it can automatically increase the mental flexibility of educators; Minimize the tendency that students are only able to solve problems without meaning rather than analytically and visually; Use of technology (e.g., Geogebra) may be able to help students to construct the concept of functions and inverses; and Lastly and most importantly, teachers need to be trained to use assignments that require students to do reversible reasoning.

Conclusion

The findings have been developed in various ways. First, it is proven that students are said to do reversible reasoning that is better than previous studies. Second, theoretically reversible components of reasoning that are used as the think-aloud protocol are developed. Third, it is found that there are three categories of reversible reasoning that refer to the consistency of students in completing inverse function tasks, which are relational-harmonic, relational-visual, and relational identity. By linking the findings to the previous research, important implications have been drawn for the need to involve reversible reasoning in the learning process. In addition, it is noted that some limitations might be used for further research, such as (1) the need for an explanation of the reversible reasoning process for each category through the comognitive framework, assimilation-accommodation, Action-process-object-scheme (APOS), or reflective abstraction; (2) investigate whether the components and categories found are also relevant for other problems, such as exponents and logarithms, or derivatives and antiretrovirals; and (3) there has been a question whether the eight participating students are students who have a high initial test score, so it is still possible to conduct investigations for students who are of moderate or low ability. It is expected that the results of this study function as a long-term study, considering that only a small number of researchers focus on the relational aspect to express reversible reasoning.

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Biodata of the Authors

Muhammad ikram is a lecturer and researcher at universitas cokroaminoto palopo, jalan latammacelling no. 19, kota palopo, South Sulawesi, Indonesia, which is a college that specifically produces prospective teacher. He is currently studying mathematics education doctoral program student at universitas negeri malang. His research focus is on mathamatics education especially in mental construction of students in solving problem and reversible reasoning. **Affiliation:** Universitas Cokroaminoto Palopo,
Email: muhammad.ikram.1603119@students.um.ac.id **Phone:** +6285242002088
ORCID Number: 0000-0002-3763-4299
SCOPUS ID: - **WoS Researcher ID :** -

Prof. Purwanto, Ph.D is a professor in mathematics education. He is lecturer and researcher at the faculty of mathematics and natural sciennce, Universitas Negeri Malang, East Java, Indonesia. His research focus is on mathematics Education and in discrete mathematics.
Affiliation: Universitas Negeri Malang
Email: purwanto.fmipa@um.ac.id
SCOPUS ID: - **WoS Researcher ID :** -

Dr. I Nengah Parta, M.Si is a lecturer and researcher at the faculty of mathematics and natural sciennce, universitas negeri malang, east java, Indonesia. His research focus is on mathematics education especially in research development of learning models. **Affiliation:** Universitas Negeri Malang
Email: nengah.parta.fmipa@um.ac.id
SCOPUS ID: - **WoS Researcher ID :** -

Dr. Hery Susanto, M.Si is a lecturer and researcher at the faculty of mathematics and sciennce, universitas negeri malang, east java, Indonesia. His research focus is on mathematics education and in algebra
Affiliation: Universitas Negeri Malang
Email: hery.susanto.fmipa@um.ac.id
SCOPUS ID: - **WoS Researcher ID :** -

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