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EXPLICIT FORMULAS for the CONSTRUCTION of SOME CLASSES of LOSSLESS TWO-VARIABLE LADDER NETWORKS USING LUMPED and DISTRIBUTED ELEMENTs by SCATTERING APPROACH

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ABSTRACT

In this paper, by using two-variable network functions, the explicit characterizations of lossless ladder networks with lumped and distributed elements are aimed. The explicit two-variable descriptions of some special classes of lossless ladders networks of low-pass, high-pass, band-pass and band-reject types are presented. The explicit formulas define the mixed elements networks under consideration are obtained up to a certain complexity. At the end of this study, the application of the proposed method in the design of two-stage microwave FET amplifier is illustrated with an example.

ÖZ

Bu çalýþmada, kayýpsýz, iki-kapýlý devrelerin iki-deðiþkenli saçýlma fonksiyonlarý yardýmýyla oluþturulmasý problemi ele alýnmýþtýr. Bu amaçla, sýnýrlý bazý karma, toplu ve daðýlmýþ elemanlý iki-kapýlý devre topolojileri incelenerek, ele alýnan devreyi tanýmlayan saçýlma parametrelerine iliþkin açýk formüller türetilmiþtir. Bu makalede, elde edilen açýk formüller, 11.7-12.2 GHz band aralýðýnda, MGF2124 mikrodalga tranzistorlerini içeren çift katlý mikrodalga kuvvetlendirici tasarýmýna uygulanmýþtýr.

Keywords: Two-variable functions, scattering parameters, ladder networks, explicit formulas.

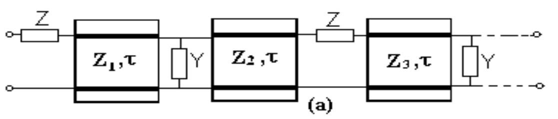
1. Introduction

In the literature, the network design with mixed, lumped and distributed elements has been recognized for a long time [1-4]. In high frequency, high speed communication networks and Monolitic Integrated Circuit (MIC) layout design, the mixed elements network provides more advantages and flexibilities with respect to single element (only lumped or only distributed elements) one. But unfortunately, a general procedure like Darlinghton method in lumped network synthesis still doesn`t exist to design lossless two-port networks with mixed elements. In the most studies, the realizibility conditions for the restricted class of two-variable functions have been examined and the special attention has been devoted to a practical ladder network configurations composed of cascaded lossless simple lumped elements and ideal commensurate transmission lines (UEs) encountered in broadband matching networks and microwave filters, especially.

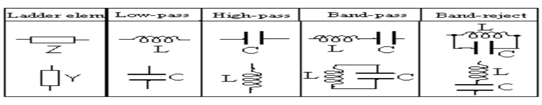
The studies on the scattering parameter description of passive multidimensional (especially two-dimensional) lossless two-ports have given rise to two-variable scattering functions which are called Scattering Hurwitz Polynomials[5-6]. Based on these new functions, some attractive results on the general cascade synthesis problem has been introduced in recent literature[7-11]. The studies showed that apart from the requirements of the positive real or bounded real conditions, additional conditions imposing the topologic restrictions has to be used to ensure realizability of a passive cascade structure for two-variable case.

Starting from this idea, a scattering approach to construct the lossless cascaded two-ports with mixed, lumped and distributed elements is proposed in this paper. Using the topological restrictions of some two-variable structures with simple lumped elements and uniform transmission lines, by means of that approach, it is highlighted that two-variable network functions can be constructed on a scattering basis.

2. Two-Variable Scattering Description of Cascaded Lumped-Distributed Two-Ports



(a)



(b)

Fig. 1a- Generic form of lumped ladder structure with unit elements.
b- Simple lumped elements.

Consider the generic form of cascaded two-port composed of simple lumped, elements and uniform, equidelay ideal transmission lines (Unit Element) shown in Fig.1.

As it is well known, the scattering matrix S describing the mixed element structure can be expressed in terms of two-variable canonic polynomials f(p, λ), h(p, λ) and g(p, λ) as follows:

$$S = \frac{1}{g}\begin{pmatrix} h & s f_* \\ f & -s h_* \end{pmatrix}, \qquad (1)$$

where; the lower asterisk ($_*$) denotes the paraconjugation i.e. $h_* = h(-p,-\lambda)$ and also,

- g,h and f are real polynomials of the complex variables p and λ;
- g(p, λ) is a Scattering Hurwitz polynomial,
- f(p, λ) is monic, σ is a unimodular constant and
- the polynomials are related by the paraunitary condition :

$$g(p,\lambda)g(-p,-\lambda)=h(p,\lambda)h(-p,-\lambda)+f(p,\lambda)f(-p,-\lambda) \qquad (2)$$

For the cascade topology underconsideration, the polynomial f has the form

$$f(p, \lambda)= f_0(p)(1-\lambda^2)^{n_\lambda / 2} \qquad (3)$$

where f_0(p) define the transmission zeros of the lumped cascade without UEs, n_λ denotes the number of UEs in the mixed structure.

In other hand, the two-variable real polynomials g=g(p,λ) and h=h(p,λ) can be expressed in coefficients form as follows:

$$g(p, ë) = p^T Ë_g ë, ;$$

$$h(p, ë) = p^T Ë_h ë,$$

where, $p^T = \left[1 \quad p \quad p^2 \quad \Lambda \quad p^{n_p}\right]$,

$l^T = \left[1 \quad l \quad l^2 \quad \Lambda \quad l^{n_l}\right]$ n_p is the total number of lumped elements in the mixed structure and

$$\Lambda_g = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0n_l} \\ g_{10} & g_{11} & \cdots & g_{1n_l} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n_p0} & g_{n_p1} & \cdots & g_{n_pn_l} \end{bmatrix}$$

$$\Lambda_h = \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n_l} \\ h_{10} & h_{11} & \cdots & h_{1n_l} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_p0} & h_{n_p1} & \cdots & h_{n_pn_l} \end{bmatrix} \quad (4)$$

3. Topologic Properties of Some Lossless Ladder Networks with Unit Elements

By using the losslessness condition (2) of two-ports, the two-variable real canonic polynomials defining the scattering matrix of the ladder networks under consideration are constructed to be characterized the mixed structures shown in Fig.1. Otherwise, it is well known that the scattering matrix and hence the canonic polynomials have to satisfy some additional conditions to ensure the realizability as a passive lossless cascade structure. In this study, the additional conditions which lead to a realizable cascade structure with positive element values are investigated by using the single-variable boundary conditions and the topologic properties of the mixed element structures. For certain classes of two-variable low-pass, high-pass, band-pass and band-reject type ladder structures , the baundary conditions can be established as follows:

Boundary Conditions:

Low-pass type ladder (The transmission zeros of the lumped cascade are all at infinity):

- Putting p=0 in (2), the obtained polynomials $g(0,\lambda)$, $h(0,\lambda)$ and $f(0,\lambda)$ define the cascade of UEs where $g(0,\lambda)$ is strictly Hurwitz and

$$g(0,\lambda)g(0,-\lambda)=h(0,\lambda)h(0,-\lambda)+(1-\lambda^2)^{n_\lambda/2} \quad (5)$$

- Putting λ=0 in (2), the obtained polynomials $g(p,0)$, $h(p,0)$ and $f(p,0)$ define the cascade of lumped section where $g(p,0)$ is strictly Hurwitz and

$$g(p,0)g(p,0)=h(p,0)h(p,0)+1 \quad (6)$$

High-pass type ladder (The transmission zeros of the lumped cascade are all at zero):

- Putting p=∞ in (2) the obtained polynomials $g(\infty,\lambda)$, $h(\infty,\lambda)$ and $f(\infty,\lambda)$ define the cascade of UEs where $g(\infty,\lambda)$ is strictly Hurwitz and

$$g(\infty,\lambda)g(\infty,-\lambda)=h(\infty,\lambda)h(\infty,-\lambda)+)(1-\lambda^2)^{n_\lambda/2} \quad (7)$$

- Putting λ=0 in (2), the obtained polynomials $g(p,0)$, $h(p,0)$ and $f(p,0)$ define the cascade of lumped section where $g(p,0)$ is strictly Hurwitz and

$$g(p,0)\, g(p,0)=h(p,0)\, h(p,0) + p^{n_p} \quad (8)$$

Band-pass type ladder (The trans. zeros are its half at origin and the others at infinity):

- Putting p=0 or p=∞ in (2), the cascade of UEs can not got directly from the band-pass type mixed structure. So, to be established the boundary condition for the distributed subsection (only UEs chain) requires to choose the lossless equation fictitiously as follows:

$$\bar{g}(l)\,\bar{g}(-l)=\bar{h}(l)\,\bar{h}(-l)+(1-\lambda^2)^{n_\lambda/2} \quad (9)$$

where $\bar{g}$, $\bar{h}$ are the fictitious polynomials.

- Putting λ=0in (2), the obtained polynomials $g(p,0)$, $h(p,0)$ and $f(p,0)$ define the cascade of lumped section where $g(p,0)$ is strictly Hurwitz and

$$g(p,0)\, g(p,0)=h(p,0)\, h(p,0) + p^{n_p/2} \quad (10)$$

Band-reject type ladder (The trans. zeros are all at finite jw axis without end points):

- Putting p=0 or p=∞ in (2), the obtained polynomials{g(0,λ), h(0,λ) and f(0,λ)} or {g(∞,λ), h(∞,λ) and f(∞,λ)} define the cascade of UEs where g(0,λ) or g(∞,λ) is strictly Hurwitz (5 or 7).
- Putting λ=0 in (2), the obtained polynomials g(p,0), h(p,0) and f(p,0) define the cascade of lumped section where g(p,0) is strictly Hurwitz and

$$g(p,0)g(p,0)=h(p,0)h(p,0)+(1+k_2p^2+k_4{}^4+..+k_n\,p^{n_p}) \tag{11}$$

where, $k_{np}=1$ is chosen for the sake of transformerless design.

The single variable boundary polynomials define the certain row and column coefficients of the matrices Λ_h and Λ_g defining the two-variable polynomials h(p,λ) and g(p,λ) respectively. These are the first row and the first column for low-pass type, the last row and the first column for high-pass type, only the first column for band-pass type and the first or the last row and the first column for band-reject type restricted mixed structures. At the same time, the two-variable losslessness equation (2) is satisfied with the single-variable boundary conditions given above. Now, the construction problem of the canonic polynomials (h(p,λ) and g(p,λ)) is converted to generate the remaining rows and columns of the coefficient matrices which include the cascade connection information.

4. Construction of Two-Variable Network Functions for Mixed Structures

In this section, the main idea is that the lossless two-port networks composed of lumped and distributed elements are described in terms of the real coefficients of h(p,λ) two-variable polynomial.

As well known, the major difficulty in construction of two-variable network functions is the explicit factorization of two-variable polynomials encountered in the losslessness condition (2) of the two-port.

An alternative solution method be used instead of two-variable factorization is to solve a quadratic equation set called “Fundamental Equation Set” given as follow, obtained by expressing the losslessness relation (2) in the coefficients form given in (4) and equating the same powers of p and λ in it.

Thus, in two-variable case, the problem is reduced to the solving of these quadratic equations(FES) which lead to a realizable cascade structure with positive element values under proper constraints. The solution of FES should be founded such that two-variable g(p,λ) denominator polynomial of the scattering parameters is yielded as Scattering Hurwitz polynomial. In order to solve it, besides the boundary conditions, we need to consider some topological constraints on the coefficients in order to be obtained the practical (realizable) solution. This correspondes to determining of the solution in such a way that the resulting canonic polynomials f(p,λ), g(p,λ) and h(p,λ) yield realizable scattering parameters.

Fundamental Equation Set (FES)

$$g_{0,k}^2 + 2\sum_{l=0}^{k-1}(-1)^{k-l} g_{0,l} g_{0,2k-l} = h_{0,k}^2 + f_{0,k}^2 + 2\sum_{l=0}^{k-1}(-1)^{k-l}(h_{0,l}h_{0,2k-l} + f_{0,l} f_{0,2k-l})$$

$$\vdots \qquad (k = 0,1,\ldots\ldots,n_l)$$

$$\sum_{j=0}^{i}\sum_{l=0}^{k}(-1)^{i-j-l} g_{j,l} g_{i-j,2k-1-l} = \sum_{j=0}^{i}\sum_{l=0}^{k}(-1)^{i-j-l}\left[h_{j,l}h_{i-j,2k-1-l} + f_{j,l} f_{i-j,2k-1-l}\right]$$

$$\vdots \quad (i = 1,3,\ldots,2n_p - 1, \qquad k = 0,1,\ldots\ldots,n_l - 1) \tag{12}$$

$$\sum_{j=0}^{i}(-1)^{i-j}\left(g_{j,k} g_{i-j,k} + 2\sum_{l=0}^{k-1}(-1)^{k-l} g_{j,l} g_{i-j,2k-l}\right) =$$

$$\sum_{j=0}^{i}(-1)^{i-j}\left(h_{j,k}h_{i-j,k} + f_{j,k} f_{i-j,k} + 2\sum_{l=0}^{k-1}(-1)^{k-l}\left[h_{j,l}h_{i-j,2k-l} + f_{j,l} f_{i-j,2k-l}\right]\right)$$

$$\vdots \qquad (i = 2,4,\ldots,2n_p - 2, \qquad k = 0,1,\ldots\ldots,n_l)$$

$$g_{n_p,k}^2 + 2\sum_{l=0}^{k-1}(-1)^{k-l} g_{n_p,l} g_{n_p,2k-l} = h_{n_p,k}^2 + f_{n_p,k}^2 + 2\sum_{l=0}^{k-1}(-1)^{k-l}(h_{n_p,l}h_{n_p,2k-l} + f_{n_p,l} f_{n_p,2k-l})$$

$$\vdots \qquad (k = 0,1,\ldots\ldots,n_l)$$

4.1. Construction of Lossless Two Ports with lumped and Unit Elements (UEs)

The most practical network configurations from the realization point of view are ladder forms constructed of cascaded simple lumped elements and commensurate transmission lines (UEs) as shown in Fig. 2. These types of ladder networks are called to as *low-pass, high-pass, band-pass and band-reject ladder with unit elements* in Fig. 2(a-d) respectively.

By the topologic analysis for each class of cascade mixed structures shown in Fig.2 the invented coefficient constraints are established in Table 1. Utilizing the boundary conditions (5-11) together with these coefficient constraints reflecting the connectivity information, a unique and sufficient explicit solution of FES (12) is obtained up to low-order network complexity using a straightforward algebraic procedure.

The obtained explicit solutions for the mixed structures under consideration are given In Table 2-5, and their correspondence realizations are shown in Fig. 3-6. The two-variable ladder network configurations obtained in this study provide resoanable solutions for the many practical broadband matching problems, even though limited.

For high order ladders, direct derivations of coefficient relations become highly complicated, therefore FES has to be solved numerically. In the reference [10] , a numeric solution method based on Algebraic Decompozition Techniqe is presented.

5. Application

A fundamental application of the results obtained is that of the computer-aided real frequency broadband matching. The real frequency matching is based on the generation of the real scattering parameters defining the matching networks, in terms of a set of independent real parameters and then the numerical determination of these parameters by optimizing the gain performance.of the overall system. So, the scattering representation of the matching
networks is generated from the partially defined numerator polynomial $h(p,\lambda)$ of the unit normalized input reflection function $S_{11}(p,\lambda)= h(p,\lambda) / g(p,\lambda)$. Then, the certain coefficients in the coefficients matrice of $h(p,\lambda)$ polynomial are chosen as the independent unknown parameters and determined to optimize the transducere gain of the system by means of an unconstrained nonlineer search routine. Later, starting from these independent coefficients, we generate the rest of the coefficients matrices of $h(p,\lambda)$ and $g(p,\lambda)$ poly- nomials by using the explicit expressions for the mixed structures introduced in the Sect. 4.

In the following, the use of the obtained explicit solutions for band-pass ladders with unit elements is illustrated by a two-stage microwave FET amplifier design example. As it is well known, the desing of microwave amplifiers can be considered as the design of matching networks called equalizers.

5.1. Two-Stage FET Microwave Amplifier Design

In this example, it is desired to synthesize front-end, interstage and back-end matching networks for a pair of MGF 2124 microwave transistors over the frequency band of 11.7-12.2 GHz (X band). The source and the load rezistances are 50Ω. The scattering parameters of the active device (MGF 2124) is given in [13].

For this amplifier example, it is chosen the BPLU with three section structure for the input and output matching networks, the BPLU with four section for the interstage equalizer. The front-end, interstage and back-end matching networks for the FET are designed successively by using the sequential amplifier design technique described in [6]. The delay lengths of the transmission lines in all matching networks are also chosen as additional free parameters and optimized independently. As a result of optimization, we obtained the following polynomial forms describing the scattering functions and corresponding realizations of the front-end, interstage and back-end equalizers:

The final matched amplifier system has an average gain level of 4.59 dB over the design passband of 11.7-12.2 GHz, which is shown in Fig.8. Within the passband, the maximum and the minumum gain levels are obtained as 5.03 dB and 3.85 dB respectively. Here, the normalized e lements values are given only. Actual element values can be computed by denormalizing with R_0=50 Ohm and f_0=12.2 GHz. In optimization process, the free parameters are initialized by ad-hoc choices as +1 or -1.

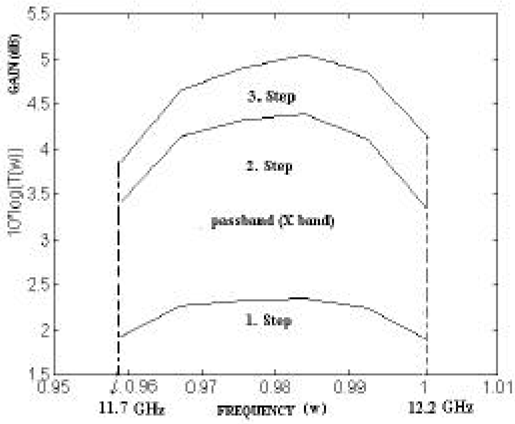
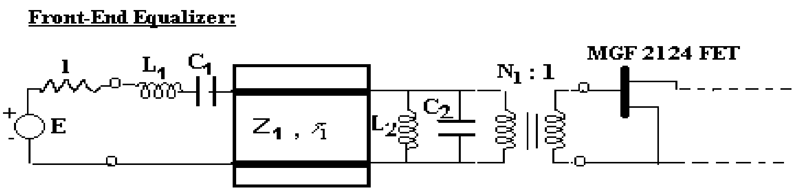
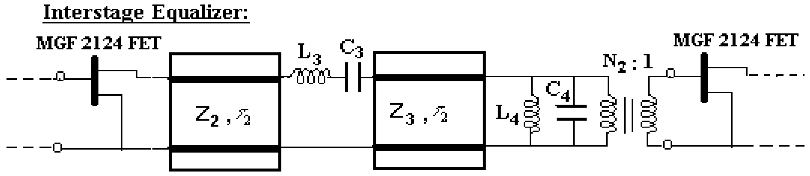


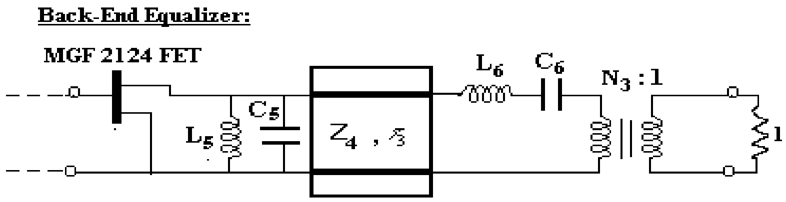
Fig.8.Perform.of double stage FET amp



[L_1=0.6464, C_1= 1.2594, L_2=0.2621, C_2=4.8843,
Z_1=0.2678, t_1=0.3243, N_1=0.538].



[L_3=0.4743, C_3=1.4966, L_4=0.2561, C_4=3.0767,
Z_2=0.9368, Z_3=0.4186, t_2=0.1784, N_2=0.7609]



[L_5=1.6156, C_5= 0.4898, L_6=3.8598, C_6= 0.3318,
Z_4=1.1905, N_3=1.2804, t_3=1.6183].

Fig.7. 2-stage amplifier design (a)Front-End (b)Interstage (c)Back-End Equalizers.

Table 6. After the optimization process, the obtained coefficient matrices

$$\Lambda_h = \begin{pmatrix} 0.8148 & 0 \\ -0.2884 & 1.0725 \\ 1.0464 & -0.7559 \\ -0.7132 & 1.0013 \\ 0.8494 & 0 \end{pmatrix}, \qquad \Lambda_g == \begin{pmatrix} 0.8148 & 0 \\ 1.7641 & 1.0725 \\ 2.9050 & 1.2535 \\ 1.9147 & 1.0013 \\ 0.8494 & 0 \end{pmatrix} \qquad \text{(a)}$$

$$\Lambda_h = \begin{pmatrix} 0.9927 & -1.0597 & 0 \\ -1.0466 & 2.1522 & -1.3122 \\ 1.2102 & -2.0116 & 0.5578 \\ -0.859 & 1.6850 & -0.9832 \\ 0.5552 & -0.5926 & 0 \end{pmatrix}, \qquad \Lambda_g =. \begin{pmatrix} 0.9927 & 1.0597 & 0 \\ 1.9247 & 3.0896 & 1.3122 \\ 2.5243 & 3.7928 & 1.1451 \\ 1.4822 & 2.3503 & 0.9832 \\ 0.5552 & 0.5926 & 0 \end{pmatrix}, \qquad \text{(b)}$$

$$\Lambda_h = \begin{pmatrix} -0.7284 & 0 \\ 0.7806 & -1.2763 \\ -1..2597 & -0.0728 \\ 1.1938 & -1.4938 \\ -0.7383 & 0 \end{pmatrix}, \qquad \Lambda_g = \begin{pmatrix} 0.7284 & 0 \\ 1.5731 & 1.2763 \\ 2.5401 & 1.0026 \\ 1.8209 & 1.4938 \\ 0.7383 & 0 \end{pmatrix} \qquad \text{(c)}$$

For (a) Front-End, (b) Inter-Stage (c) Back-End Equalizers

6. CONCLUSIONS

In this study, for two-variable lossless two-port networks, the explicit characterization is obtained up to a certain complexity using the two-variable scattering functions. Two-variable functional descriptions of mixed element low-pass, high-pass, band-pass and band-reject structures are obtained to be quite useful especially in high speed/high frequency mobile communication subcircuits and the MMIC design packages. It is hoped that the two-variable explicit characterization will provide a flexible tool to designers to construct practical matching networks and construct practical matching networks and microwave amplifiers with lumped elements and transmission lines to be manufactured on the microwave and milimeterwave monolitic integrated circuits

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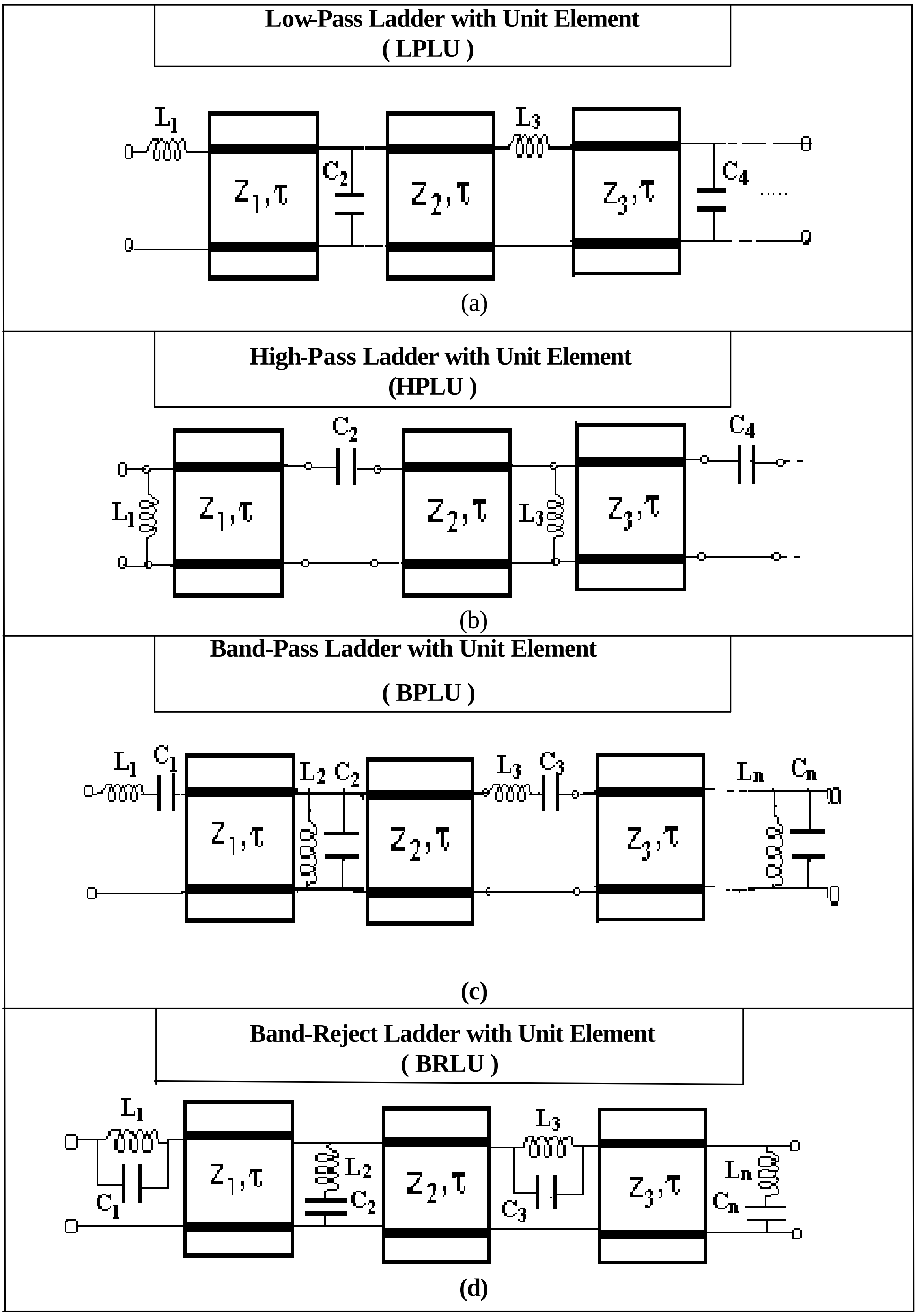


Fig. 2. The ladder network structures with mixed elements

Table 1. Topological properties of lossless two-ports under consideration

LPLU

1. $g_{1,1} = g_{0,1}\, g_{1,0} - h_{0,1}\, h_{1,0}$
2. $g_{k,l} = h_{k,l} = 0$, $(k+l > n_\lambda+1$; $l = 0,1,2,\ldots, n_\lambda$, $k = 0,1,2,\ldots, n_p)$.
3. $h_{k,l} = \mu_2\, g_{k,l}$, $(k+l = n_\lambda+1$; $k, l = 0,1,2,\ldots, n_\lambda)$,
4. $\mu_1 = \mathrm{Sgn}(h_{n_p,0}) = \pm 1$, $\mu_2 = \mathrm{Sgn}(h_{k,l}) = \pm 1$
5. $\mu = \mu_1 = \mu_2 = \pm 1$. for $(n_p = n_\lambda+1)$.

HPLU

1. $g_{n_p-1,1} = g_{n_p,1}\, g_{n_p-1,0} - h_{n_p,1}\, h_{n_p-1,0}$,
2. $g_{n_p-k,l} = h_{n_p-k,l} = 0$, $(k+l > n_\lambda+1$; $l = 0,1,2,\ldots, n_\lambda$, $k = 0,1,2,\ldots, n_p)$.
3. $h_{n_p-k,l} = \mu_2\, g_{n_p-k,l}$ $(k+l = n_\lambda+1$; $k, l = 0,1,2,\ldots, n_\lambda$
4. $\mu_1 = \mathrm{Sgn}(h_{00}) = \pm 1$, $\mu_2 = \mathrm{Sgn}(h_{n_p-k,l}) = \pm 1$
5. $\mu = \mu_1 = \mu_2 = \pm 1$. for $(n_p = n_\lambda+1)$

BPLUs

1. $g_{n_r-k,l} = h_{n_r-k,l} = 0$, $(k+l) > n_\lambda+1$ • $g_{n_r+k,l} = h_{n_r+k,l} = 0$ $(k=0,1,.,n_p)$, $(l=0,1,.,n_\lambda)$.
2. $h_{n_r-k,l} = m_2\, g_{n_r-k,l}$, $(k+l) = n_\lambda+1$ • $h_{n_r+k,l} = m_2\, g_{n_r+k,l}$. $(k=0,1,.,n_p)$, $(l=0,1,.,n_\lambda)$.
3. $\mu_1 = \mathrm{Sgn}(h_{00}) = \mathrm{Sgn}(h_{np,0}) = \pm 1$, $(k+l) = n_\lambda+1$,
   $\mu_2 = \mathrm{Sgn}(h_{n_r-k,l}) = \mathrm{Sgn}(h_{n_r+k,l}) = \pm 1$, $(k=0,.,n_p$, $l=0,.,n_\lambda)$
4. $g_{n_r,n_l} - m_2 h_{n_r,n_l} = (\bar{g}_{0,n_l} - m_2 \bar{h}_{0,n_l}).(g_{n_r,0} - m_1 h_{n_r,0})$ $\quad g_{n_r,n_l} - m_2 h_{n_r,n_l} = (\bar{g}_{0,n_l} - m_2 \bar{h}_{0,n_l})./(g_{n_r,0} - m_1 h_{n_r,0})$
   $g_{n_r,n_l-1} - m_2 h_{n_r,n_l-1} = (\bar{g}_{0,n_l-1} - m_2 \bar{h}_{0,n_l-1})./(g_{n_r,0} - m_1 h_{n_r,0})$ for $(m_1 m_2 = 1)$ $\quad g_{n_r,n_l-1} - m_2 h_{n_r,n_l-1} = (\bar{g}_{0,n_l-1} - m_2 \bar{h}_{0,n_l-1}).(g_{n_r,0} - m_1 h_{n_r,0})$ for $(m_1 m_2 = -1)$
   and (for $n_r \le 2$) . where $(n_r = n_p/2)$.;
5. $\mu = \mu_1 = \mu_2 = \pm 1$ for $(n_r = n_\lambda+1)$ • $g_{n_p,1}\, g_{00} = g_{n_p,0}\, g_{01}$. for $(n_r = n_\lambda)$,

BRLUs

1. $g_{11} = g_{01} g_{10} - h_{01} h_{10}$, $\quad g_{n_p-1,1} = g_{0,1} g_{n_p-1,0} - h_{01} h_{n_p-1,0}$,
2. $h_{i,n_l} = m_2 g_{i,n_l}$, $(i=1,3,5\ldots\ldots,(n_p-1))$,
3. $g_{i,n_l} = k_i\, g_{0,n_l}$, $h_{i,n_l} = k_i\, h_{0,n_l}$, $(i=2,4,.,(n_p-2))$, • $g_{i,n_l-1} - m_2 h_{i,n_l-1} = k_i (g_{0,n_l-1} - m_2 h_{0,n_l-1})$,
4. $\mu_1 = \mathrm{Sgn}(h_{n_r,0}) = \pm 1$, where $(n_r = n_p/2)$ $\mu_2 = \mathrm{Sgn}(h_{i,n_l}) = \pm 1$ $(i=1,3,5\ldots\ldots,(n_p-1))$
   $\mu = \mu_1 = \mu_2 = \pm 1$ for $(n_r = n_\lambda+1$
5. $k_2 = g_{20} - m_1 h_{20}$ for $(n_r \le 2)$
   where k_i $(i=2,4,.,(n_p-2))$ are the coefficients of $f(p,0)$ polynomial that defines the lumped subsection only, $f(p,0) = (1 + k_2 p^2 + k_4 p^4 + \ldots\ldots\ldots + p^n)$.

Entries of Λ_g are nonnegative real numbers

Table 2. Explicit Formulas for Low-Order LPLU

2	$g_{10}=\|h_{10}\|$, $g_{01}=\{h_{01}^2+1\}^{1/2}$, $g_{00}=1$, $h_{00}=0$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=\mu_2 g_{11}$, $\mu_2=Sgn(h_{11})$, $\mu_1=Sgn(h_{10})$
3	$g_{00}=1$, $h_{00}=0$, $g_{10}=\{(h_{10})^2+2g_{20}\}^{1/2}$, $g_{20}=\|h_{20}\|$, $g_{01}=\{(h_{01})^2+1\}^{1/2}$, $\mu=Sgn(h_{20})=\pm 1$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=\mu\, g_{11}$
4	g_{00}, g_{10}, g_{20} (same), $g_{01}=\{(h_{01})^2+2g_{02}+2\}^{1/2}$, $g_{02}=\{(h_{02})^2+1\}^{1/2}$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=(\alpha/\beta)h_{20}+(\beta/\alpha)h_{02}$, $g_{21}=(1/\beta)(g_{20}g_{11}-h_{20}h_{11})$, $h_{21}=\mu_2\, g_{21}$, $g_{12}=(1/\alpha)(g_{11}g_{02}-h_{11}h_{02})$, $h_{12}=\mu_2 g_{12}$, $\alpha=g_{01}-\mu_2 h_{01}$, $\beta=g_{10}-\mu_2 h_{10}$
5	$g_{30}=\|h_{30}\|$, $g_{20}=\{(h_{20})^2+2(g_{30}g_{10}-h_{30}h_{10})\}^{1/2}$, $g_{10}=\{h_{10}^2+2g_{20}\}^{1/2}$, $g_{00}=1$, $h_{00}=0$, $g_{01}=\{(h_{01})^2+2g_{02}+2\}^{1/2}$, $g_{02}=\{(h_{02})^2+1\}^{1/2}$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=(\alpha/\beta)h_{20}+(\beta/\alpha)h_{02}$, $g_{21}=(1/\beta)(g_{20}g_{11}-h_{20}h_{11}-g_{01}g_{30}+h_{01}h_{30})$, $h_{21}=\mu g_{21}$, $\alpha=g_{01}-\mu h_{01}$, $g_{12}=(1/\alpha)(g_{11}g_{02}-h_{11}h_{02})$, $h_{12}=\mu g_{12}$, $\beta=g_{10}-\mu h_{10}$.

Low-order Elementer Mixed Structures

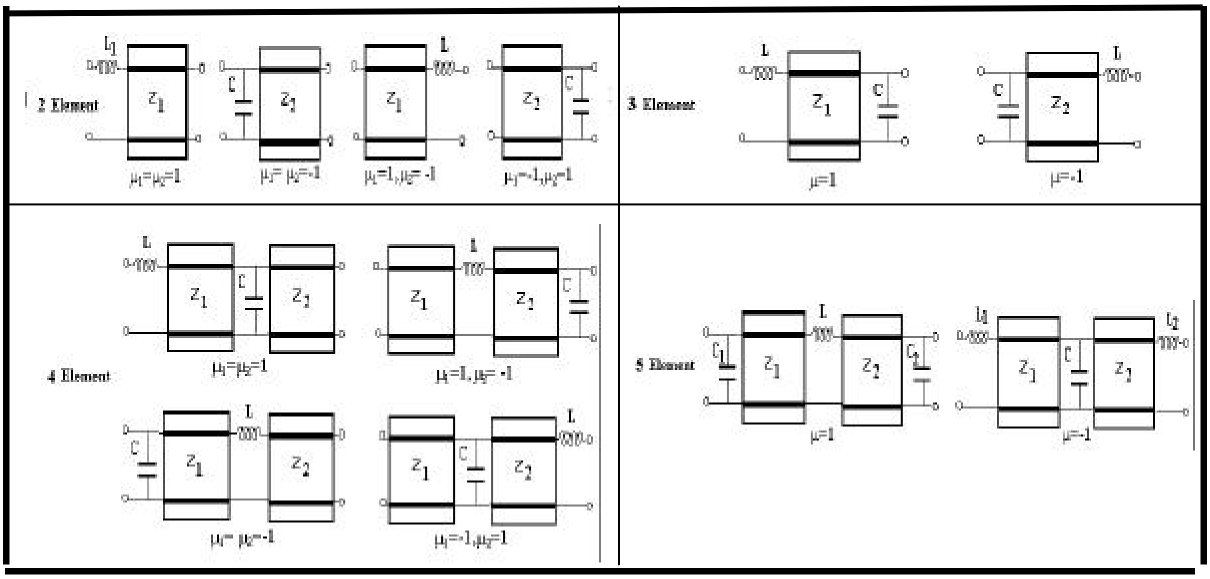


Fig. 3. LPLU network configurations up to 5 Element

Table 3. Explicit Formulas for Low-Order HPLU

2	$g_{00}=\|h_{00}\|$, $g_{11}=\{(h_{11})^2+1\}^{1/2}$, $g_{10}=1$, $h_{10}=0$, $g_{01}=g_{11}g_{00}-h_{11}h_{00}$, $h_{01}=\mu_2 g_{01}$, $\mu_2=\mathrm{Sgn}(h_{01})$, $\mu_1=\mathrm{Sgn}(h_{00})$
3	$g_{00}=\|h_{00}\|$, $g_{10}=\{(h_{10})^2+2g_{00}\}^{1/2}$, $g_{20}=1$, $h_{20}=0$, $g_{11}=g_{21}g_{10}-h_{21}h_{10}$, $h_{11}=\mu g_{11}$; $\mu=\mathrm{Sgn}(h_{00})=\pm 1$,
4	g_{00}, g_{10}, g_{20} (same), $g_{21}=\{(h_{21})^2+2g_{22}+2\}^{1/2}$, $g_{22}=\{(h_{22})^2+1\}^{1/2}$ $g_{11}=g_{21}g_{10}-h_{21}h_{10}$, $h_{11}=(\alpha/\beta)h_{00}+(\beta/\alpha)h_{22}$, $g_{01}=(1/\beta)(g_{00}g_{11}-h_{00}h_{11})$, $h_{01}=\mu_2 g_{01}$, $g_{12}=(1/\alpha)(g_{11}g_{22}-h_{11}h_{22})$, $h_{12}=\mu_2 g_{12}$, $\alpha=g_{21}-\mu_2 h_{21}$, $\beta=g_{10}-\mu_2 h_{10}$
5	$g_{00}=\|h_{00}\|$, $g_{10}=\{(h_{10})^2+2(g_{00}g_{20}-h_{00}h_{20})\}^{1/2}$, $g_{20}=\{(h_{20})^2+2g_{10}\}^{1/2}$, $g_{30}=1$, $h_{30}=0$ $g_{31}=\{(h_{31})^2+2g_{32}+2\}^{1/2}$, $g_{32}=\{(h_{32})^2+1\}^{1/2}$ $g_{21}=g_{31}g_{20}-h_{31}h_{20}$, $h_{21}=(\alpha/\beta)h_{10}+(\beta/\alpha)h_{32}$, $g_{11}=(1/\beta)(g_{10}g_{21}-h_{10}h_{21}-g_{31}g_{00}+h_{31}h_{00})$, $h_{11}=\mu g_{11}$, $\alpha=g_{31}-\mu h_{31}$, $g_{22}=(1/\alpha)(g_{21}g_{32}-h_{21}h_{32})$, $h_{22}=\mu g_{22}$, $\beta=g_{20}-\mu h_{20}$.

Low-order Elementer Mixed Structures

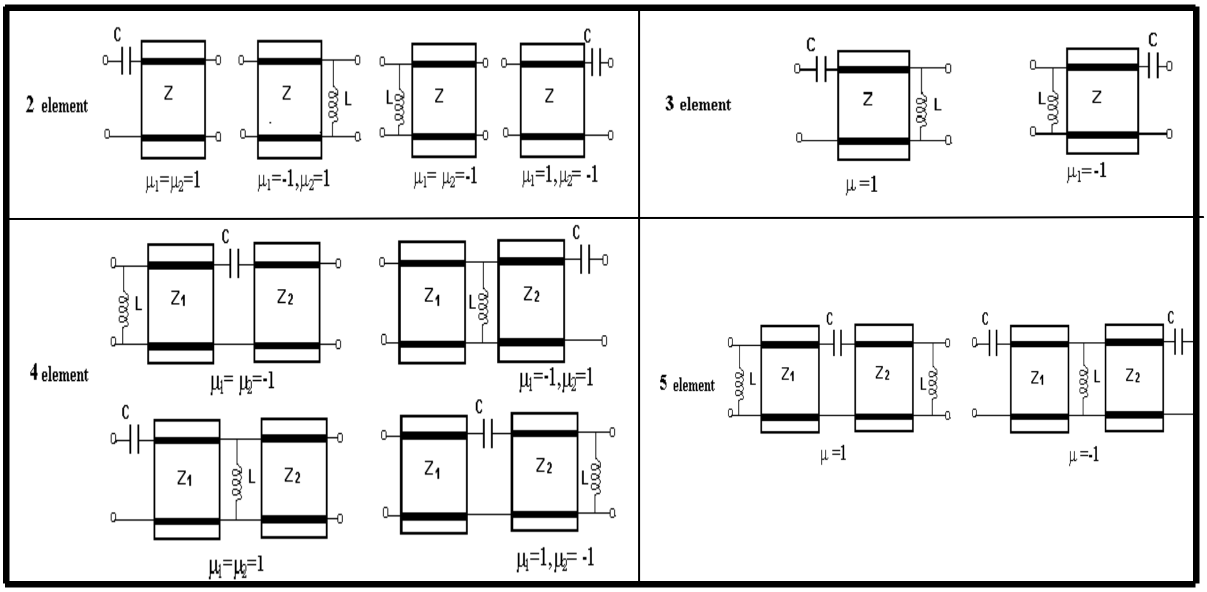
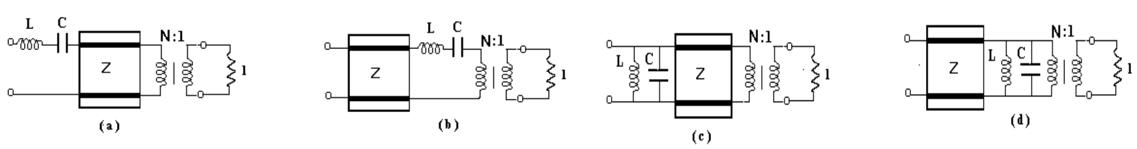


Fig. 4. HPLU network configurations up to 5 Element

Table 4. Explicit Formulas for Low-Order BPLU

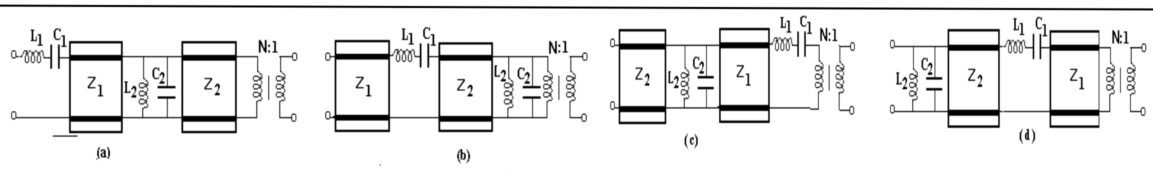
2 SECT.	$g_{00}=\lvert h_{00}\rvert$; $g_{10}=(2(g_{20}g_{00}-h_{20}h_{00})+h_{10}^2+1)^{1/2}$; $g_{20}=\lvert h_{20}\rvert$ $\bar g_{01}=(1+\bar h_{01}^2)^{1/2}$ $h_{11}=g_{11}-\bar\alpha_{01}/N$ (for $\mu_2=1$); ; $h_{11}=\bar\alpha_{01}N-g_{11}$ (for $\mu_2=-1$); $h_{01}=\mu_2g_{01}$, $h_{21}=\mu_2g_{21}$, $g_{01}=(1/\alpha_{10})(g_{00}g_{11}-h_{00}h_{11})$, $g_{21}=(1/\alpha_{10})(g_{20}g_{11}-h_{20}h_{11})$, $N=g_{10}-\mu_1h_{10}$, (for $\mu_1=-1$); $N=1/(g_{10}-\mu_1h_{10})$ (for $\mu_1=1$) $\bar\alpha_{01}=\bar g_{01}-\mu_2\bar h_{01}$, , $\mu_2=\pm1$, $\alpha_{10}=g_{10}-\mu_2h_{10}$; $\mu_1=h_{00}/g_{00}=h_{20}/g_{20}=\pm1$
3 SECT.	$g_{00}=\lvert h_{00}\rvert$; $g_{10}=(2(g_{20}g_{00}-h_{20}h_{00})+h_{10}^2)^{1/2}$; $g_{20}=(2(g_{40}g_{00}-h_{40}h_{00}-g_{30}g_{10}+h_{30}h_{10})+h_{20}^2+1)^{1/2}$ $g_{30}=(2(g_{40}g_{20}-h_{40}h_{20})+h_{30}^2)^{1/2}$; $g_{40}=\lvert h_{40}\rvert$; $\bar g_{01}=(1+\bar h_{01}^2)^{1/2}$ $h_{21}=g_{21}-\bar\alpha_{01}/N$ (for $\mu=1$); $h_{21}=\bar\alpha_{01}N-g_{21}$ (for $\mu=-1$); $g_{21}=(1+h_{21}^2)^{1/2}$ $g_{11}=(1/\alpha_{20})(g_{10}g_{21}-h_{10}h_{21})$, $g_{31}=(1/\alpha_{20})(g_{30}g_{21}-h_{30}h_{21})$, $h_{11}=\mu g_{11}$, $h_{31}=\mu g_{31}$ $\alpha_{20}=g_{20}-\mu h_{20}$., $\mu=\pm1$
4 SECTION	$g_{00}=\lvert h_{00}\rvert$; $g_{10}=(2(g_{20}g_{00}-h_{20}h_{00})+h_{10}^2)^{1/2}$; $g_{20}=(2(g_{40}g_{00}-h_{40}h_{00}-g_{30}g_{10}+h_{30}h_{10})+h_{20}^2+1)^{1/2}$ $g_{30}=(2(g_{40}g_{20}-h_{40}h_{20})+h_{30}^2)^{1/2}$; $g_{40}=\lvert h_{40}\rvert$; $\bar g_{01}=(2(1+\bar g_{02})+\bar h_{01}^2)^{1/2}$ $\bar g_{02}=(1+\bar h_{02}^2)^{1/2}$ $h_{22}=g_{22}-\bar\alpha_{02}/N$, $h_{21}=g_{21}-\bar\alpha_{01}N$ (For $\mu_2=1$); $h_{22}=\bar\alpha_{02}N-g_{22}$ $h_{21}=\bar\alpha_{01}/N-g_{21}$ (for $\mu_2=-1$); $N=1/(g_{20}-\mu_1h_{20})$ $\mu_1=1$, $N=g_{20}-\mu_1h_{20}$ $\mu_1=-1$ $g_{01}=(1/\alpha_{10})(g_{00}g_{11}-h_{00}h_{11})$, $g_{41}=(1/\alpha_{30})(g_{31}g_{40}-h_{31}h_{40})$, $h_{01}=\mu_2g_{01}$, $h_{41}=\mu_2g_{41}$, $g_{12}=(1/\alpha_{21})(g_{11}g_{22}-h_{11}h_{22})$, $g_{32}=(1/\alpha_{21})(g_{31}g_{22}-h_{31}h_{22})$ $h_{12}=\mu_2g_{12}$, $h_{32}=\mu_2g_{32}$, $\mu_2=\pm1$ $g_{11}=(1/g_{20})(g_{10}g_{21}-h_{10}h_{21}+h_{11}h_{20})$, $g_{31}=(1/g_{20})(g_{30}g_{21}-h_{30}h_{21}+h_{31}h_{20})$ $g_{21}=\{2(g_{31}g_{11}-h_{31}h_{11}-\alpha_{10}g_{32}-\alpha_{30}g_{12}+g_{20}g_{22}-h_{20}h_{22}+1)+h_{21}^2\}^{1/2}$, $g_{22}=(h_{22}^2+1)^{1/2}$, $\bar\alpha_{01}=\bar g_{01}-\mu_2\bar h_{01}$, $\bar\alpha_{02}=\bar g_{02}-\mu_2\bar h_{02}$, $\alpha_{10}=g_{10}-\mu_2h_{10}$, $\alpha_{30}=g_{30}-\mu_2h_{30}$, $\alpha_{21}=g_{21}-\mu_2h_{21}$

Low-Order Elementer Mixed Structures



(a) $\mu_1=\mu_2=1$ (b) $\mu_1=1,\mu_2=-1$ **2** (c) $\mu_1=\mu_2=-1$ (d) $\mu_1=-1,\mu_2=1$

$\mu=1$ **3** $\mu=-1$



(a) $\mu_1=\mu_2=1$ (b) $\mu_1=1,\mu_2=-1$ **4** (c) $\mu_1=-1$, $\mu_2=1$ (d) $\mu_1=\mu_2=-1$

Fig. 5. BPLU network configurations up to 4 Section

Table 5. Explicit Formulas for Low-Order BRLU

2	$g_{10}=\lvert h_{10}\rvert$, $g_{01}=(1+h_{01}^2)^{1/2}$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $\mu_1=\mathrm{Sgn}(h_{10})$ $h_{11}=\mu_2 g_{11}$, $h_{21}=h_{01}$ $g_{21}=g_{01}$ $\mu_2=\mathrm{Sgn}(h_{11})$
3	$g_{10}=(2(g_{20}-k_2)+h_{10}^2)^{1/2}$, $g_{30}=(2(g_{20}-k_2)+h_{30}^2)^{1/2}$ $g_{20}=(2(2-g_{30}g_{10}+h_{30}h_{10})+h_{20}^2+k_2^2)^{1/2}$, $g_{41}=g_{01}$, $h_{41}=h_{01}$, $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=\mu\, g_{11}$, $g_{31}=g_{01}g_{30}-h_{01}h_{30}$, $h_{31}=\mu\, g_{31}$, $\mu=\pm1$ $g_{21}=k_2 g_{01}$, $h_{21}=k_2 h_{01}$, $k_2=g_{20}-\mu h_{20}$
4 S E C T.	$g_{01}=(2g_{02}g_{00}+2+h_{01}^2)^{1/2}$, $g_{02}=(1+h_{02}^2)^{1/2}$ $g_{41}=g_{01}$, $g_{42}=g_{02}$, $h_{41}=h_{01}$, $h_{42}=h_{02}$ $g_{11}=g_{01}g_{10}-h_{01}h_{10}$, $h_{11}=\frac{\alpha_{10}}{\alpha_{01}}h_{02}+\frac{\alpha_{01}}{\alpha_{10}}h_{20}$, $g_{21}=h_{21}+k_2(g_{01}-h_{01})$, $h_{21}=\frac{g_{10}(g_{20}g_{31}-h_{20}h_{31})-g_{30}(g_{20}g_{11}-h_{20}h_{11})}{g_{30}h_{10}-h_{30}g_{10}}$ $g_{31}=g_{01}g_{30}-h_{01}h_{30}$, $h_{31}=\frac{\alpha_{30}}{\alpha_{01}}h_{02}+\frac{\alpha_{01}}{\alpha_{30}}h_{20}$, $g_{12}=(1/\alpha_{01})(g_{11}g_{02}-h_{11}h_{02})$, $h_{12}=\mu_2 g_{12}$, $g_{32}=(1/\alpha_{01})(g_{31}g_{02}-h_{31}h_{02})$, $h_{32}=\mu_2 g_{32}$, $\mu_1=\mathrm{Sgn}(h_{20})$. $g_{22}=k_2 g_{02}$, $h_{22}=k_2 h_{02}$, $k_2=g_{20}-\mu_1 h_{20}$. $\alpha_{01}=g_{01}-\mu h_{01}$, $\alpha_{10}=g_{10}-\mu h_{10}$, $\alpha_{30}=g_{30}-\mu h_{30}$, $\mu=\mu_2=\pm1$.

Low-Order Elementer Structures

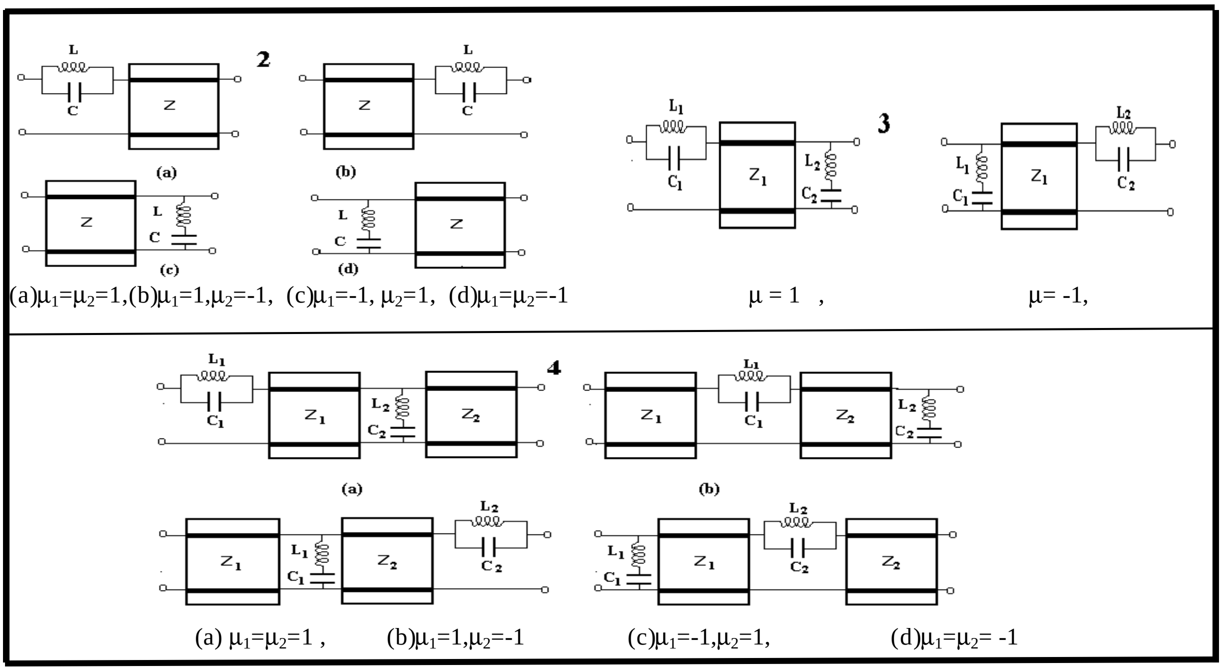


Fig. 6. BRLU network configurations up to 4 Section