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WIENER-HOPF TECHNIQUE AND SOME SPECIAL GEOMETRIES

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ABSTRACT

It is not always possible to obtain rigorous analytical solutions to diffraction problems because of geometrical and physical complexity of most scattering surfaces. Wiener-Hopf (WH) method is applicable to plane discontinuities like two or three –part plane problems, open structures like an infinite set of parallel half-planes and closed structures like parallel plate waveguide. The main purpose of this study is to give an overview of the Wiener-Hopf technique, related factorization and decomposition methods. In order to outline the theory, five types of Wiener-Hopf geometries and solution methods for the diffraction of electromagnetic waves from these geometries were investigated.

Keywords: *Wiener-Hopf,elektromagnetic waves,special geometries*

1. INTRODUCTION

The aim of this study is to investigate Wiener-Hopf (WH) analysis of some diffraction problems and to classify some special WH geometries (a number of simple obstacles). The pioneering study related to the exact solution of diffracted field by a wedge shape obstacles was performed by Sommerfeld in 1896. In his study, he developed a solution method which is known as the Sommerfeld theory of diffraction [1]. In 1931, Wiener-Hopf integral equation, namely Wiener-Hopf technique, was developed by Wiener and Hopf by using the theories of Fourier transforms and functions of a complex variable [2]. Then, a number of pioneering works on WH technique was put forth to develop the progress on the wave scattering and diffraction theory.

Magnus [3], Carlson and Heins [5], Levine and Schwinger [6] applied WH technique via integral representation of the diffraction field for various applications. Clemmow [7] introduced the plane wave spectrum method to solve a pair of dual integral equations.

Jones [8] developed a direct method of formulating WH equations. Hurd [9] formulated the Winer-Hopf-Hilbert (WHH) method to get the exact solution. Lüneberg and Hurd [10], Idemen [11], Uzgören and Büyükaksoy [12], Serbest [13], Rawlings and Williams [14], Kobayashi [15] have studied the Matrix-Wiener-Hopf (MWH) methods using different applications. Rojas [16] made important contributions to this technique regarding of factorization of some kernel functions. A time factor $e^{-i\omega t}$ with ω being the angular frequency is assumed and suppressed throughout the paper.

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2. ANALYSIS

The solution of mixed boundary value problem related to the Helmholtz equation can be given via analytical properties of Fourier integrals as follows

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2)U(x,y)=0 \qquad (1)$$

By using Fourier transforms, Helmholtz equation (1) will be reduced to the functional equation as below which is called the classical Wiener-Hopf (WH) equation. Hence,

$$\phi^+(\nu)+G(\nu)\phi^-(\nu)=H(\nu) \quad , \quad \nu\in B\in(B_+\cap B_-) \qquad (2)$$

where $G(\nu)$ and $H(\nu)$ are known and regular functions in the strip B depicted in Fig.1. The process of the determination of $\phi^{\pm}(\nu)$ functions is so-called WH problem. In diffraction theory, if the formulation of any geometry is reduced to the form of the (2), this geometry is called Wiener-Hopf geometry. The unknown functions $\phi^-(\nu)$ and $\phi^+(\nu)$ appearing in (2) are regular in the half planes $\mathrm{Im}\nu < b$ and $\mathrm{Im}\nu > a$, respectively. Here a and b which determine the boundaries of strip of regularity can be obtained by asymptotic variation of inverse transforms of $\phi^-(\nu)$ and $\phi^+(\nu)$. For sake of simplification, substituting $G(\nu)=-1$ and $H(\nu)=0$ into (2) yields

$$\phi^+(\nu)=\phi^-(\nu) \quad , \quad \nu\in B \qquad (3)$$

As it is well-known from the complex theory of functions, this equality represents the principal of the analytic continuation. Then, one can define an entire function $P(\nu)$ as follows

$$P(\nu)=\begin{cases}\phi^+(\nu) & , \ \nu\in B_+ \\ \phi^-(\nu) & , \ \nu\in B_-\end{cases} \quad , \qquad (4)$$

where $\phi^+(\nu)$ is regular and free of zeros in the upper half plane (B_+) and $\phi^-(\nu)$ is regular and free of zeros in the lower half plane (B_-) of the ν-complex plane. Therefore, $P(\nu)$, for $\nu\to\infty$, is regular all over the ν- complex plane. In other words, $\phi^+(\nu)$ is an analytic continuation of $\phi^-(\nu)$ towards (B_+) over the strip B. Similarly, $\phi^-(\nu)$ is also an analytic continuation of $\phi^+(\nu)$ towards (B_-) over the strip B. The determination of $\phi^{\pm}(\nu)$ depends upon the determination of the entire function $P(\nu)$. By applying the extension of Liouville's theorem together with the asymptotic behavior of the elements for $\nu\to\infty$ to each function on the right hand side of (4) will give the variation of the entire function. Assume that $\phi^{\pm}(\nu)$ functions have algebraic behavior for $\nu\to\infty$. That is, for $\nu\to\infty$, in the region (B_+),

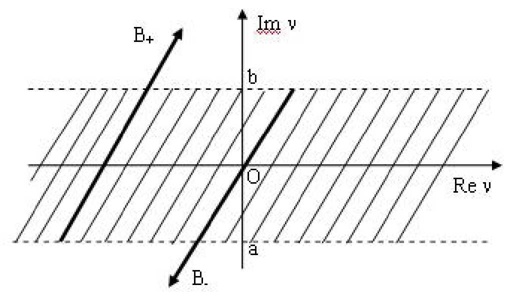

Figure 1. Strip of regularity

$$\phi^+(\nu)=O(\nu^{\alpha}) \quad , \qquad (5)$$

and for $\nu\to\infty$ in the region (B_-),

$$\phi^-(\nu)=O(\nu^{\beta}) \quad , \qquad (6)$$

Then, for $\nu\to\infty$ condition, the degree of entire function $P(\nu)$ will be

$$P(\nu)=O(\nu^{\gamma}) \quad , \quad \gamma=\max(\alpha,\beta) \qquad (7)$$

According to the Liouville's theorem, the degree of $P(\nu)$ will be an algebraic function with, at most, a degree of γ. The constants of the algebraic function can be determined by using physical and/or mathematical conditions depending on actual Wiener-Hopf problem.
If the right hand side of in (2) is zero, the general form of scalar Wiener-Hopf equation is called homogeneous WH equation [17]. Then,

$$\phi^{+}(\nu)+G(\nu)\phi^{-}(\nu)=0 \quad , \quad \nu\in B\in(B_{+}\cap B_{-}) \qquad (8)$$

In order to obtain the solution of WH equation given in the form of (8), on needs to express the kernel function $G(\nu)$ as a product of the functions $G^{-}(\nu)$ and $G^{+}(\nu)$ are regular, free of zeros and algebraic growth at infinity in half planes $\operatorname{Im}\nu < b$ and $\operatorname{Im}\nu < b$, respectively. This procedure explained above is called the WH-factorization. Assume that $G(\nu)$ is regular and free of zeros in the strip defined as $a < \operatorname{Im}\nu < b$. Let us assume that there exist $G^{+}(\nu)$ and $G^{-}(\nu)$ which satisfy equation below,

$$G(\nu)=\frac{G^{-}(\nu)}{G^{+}(\nu)} \qquad (9)$$

By taking logarithm of both sides of (8), one can obtain

$$\log G(\nu)=\log G^{+}(\nu)-\log G^{-}(\nu) \qquad (10)$$

Here, log function is defined such that log1=0. For the condition of $a < \alpha < \operatorname{Im}\nu < \beta < b$, one can write Cauchy's integral formula for factorization (See Fig.2)

$$\log G^{+}(\nu)=\frac{1}{2\pi i}\int_{-\infty+i\alpha}^{\infty+i\alpha}\frac{\log G(\varsigma)}{\varsigma-\nu}d\varsigma \qquad (11)$$

By substituting (9) into (8), one can get homogeneous equation as

$$\phi^{+}(\nu)G^{+}(\nu)+\phi^{-}(\nu)G^{-}(\nu)=0 \qquad (12)$$

According to the analytic continuation, the entire function yields,

$$P(\nu)=\begin{cases}\phi^{+}(\nu)G^{+}(\nu) & , \quad \nu\in B_{+}\\ \phi^{-}(\nu)G^{-}(\nu) & , \quad \nu\in B_{-}\end{cases} \qquad (13)$$

or

$$\phi^{+}(\nu)=\frac{P(\nu)}{G^{+}(\nu)} \qquad (14)$$

and

$$\phi^{-}(\nu)=-\frac{P(\nu)}{G^{-}(\nu)} \qquad (15)$$

It is obvious that the solution of a homogeneous WH equation totally depends on the factorization of $G(\nu)$ function in the sense of WH equation. Taking into account the WH equation with right hand side and assuming that $G(\nu)$ is factorable as in (9), one can obtain the functional equation below,

$$\phi^{+}(\nu)G^{+}(\nu)+\phi^{-}(\nu)G^{-}(\nu)=H(\nu)G^{+}(\nu) \quad , \quad \nu\in B\in(B_{+}\cap B_{-}) \qquad (16)$$

Assume that right hand side of (16) satisfies below equation in the same strip of regularity.

$$H(\nu)G^{+}(\nu)=g^{+}(\nu)-g^{-}(\nu) \qquad (17)$$

Here, the functions $g^{-}(\nu)$ and $g^{+}(\nu)$ are regular, free of zeros and have algebraic growth at infinity in the half planes B_{-} and B_{+}, respectively. The substitution (17) into (16) and rearrangement of the formula yields the homogeneous form of WH equation. That is,

$$\phi^{+}(\nu)G^{+}(\nu)-g^{+}(\nu)+\phi^{-}(\nu)G^{-}(\nu)+g^{-}(\nu)=0 \qquad (18)$$

where $\phi^{+}(\nu)G^{+}(\nu)-g^{+}(\nu)=\psi^{+}(\nu)$ and $\phi^{-}(\nu)G^{-}(\nu)+g^{-}(\nu)=\psi^{-}(\nu)$. By rewriting (18), one can get

$$\psi^{+}(\nu)=-\psi^{-}(\nu) \quad , \quad \nu\in B \qquad (19)$$

and

$$P(\nu)=\begin{cases}\psi^{+}(\nu) & , \quad \nu\in B_{+}\\ \psi^{+}(\nu) & , \quad \nu\in B_{-}\end{cases} \qquad (20)$$

or

$$\phi^{+}(\nu)=\frac{F(\nu)+g^{+}(\nu)}{G^{+}(\nu)} \quad , \quad \phi^{-}(\nu)=\frac{F(\nu)g^{-}(\nu)}{G^{-}(\nu)} \qquad (21)$$

As shown in above equations, one can apply WH technique successfully when the factorization of $G(\nu)$ in terms of $G^{+}(\nu)$ and $G^{-}(\nu)$ and also the decomposition of right hand side of scalar WH equation as in $g^{-}(\nu)$ and $g^{+}(\nu)$ are possible [18]. If possible, the decomposition of the right hand side of scalar WH equation can be done directly, otherwise according Cauchy's theorem, decomposition formula is applied to obtain $g^{\pm}(\nu)$ functions.
Assume that

$$H(\nu)G^{+}(\nu)=g(\nu)=g^{+}(\nu)-g^{-}(\nu) \qquad (22)$$

where $g(\nu)$ is regular and free of zeros in the strip of $a<\mathrm{Im}\,\nu<b$. The strip of regularity and complex integration line is shown in Fig.2.

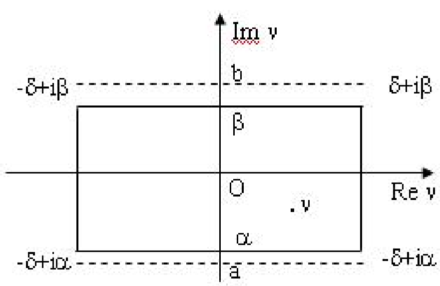

Figure 2. Complex Integration Line for Cauchy's Formula

$$g(\nu)=\frac{1}{2\pi i}\int_{-\infty+i\alpha}^{\infty+i\alpha}\frac{g(\varsigma)}{\varsigma-\nu}d\varsigma-\frac{1}{2\pi i}\int_{-\infty+i\beta}^{\infty+i\beta}\frac{g(\varsigma)}{\varsigma-\nu}d\varsigma \qquad (23)$$

where

$$g(\nu)=\frac{1}{2\pi i}\int_{-\infty+i\alpha}^{\infty+i\alpha}\frac{g(\varsigma)}{\varsigma-\nu}d\varsigma \qquad (24)$$

and

$$g(\nu)=\frac{1}{2\pi i}\int_{-\infty+i\beta}^{\infty+i\beta}\frac{g(\varsigma)}{\varsigma-\nu}d\varsigma \qquad (25)$$

If $\nu.g(\nu)$ goes to zero uniformly for $\nu \to \infty$, then, there will be no contribution to the integrand due to vertical sides of the C for $\delta \to \infty$.

3. SOME SPECIAL WH GEOMETRIES

3.1 First Type WH Geometry (WH1)

Half plane geometry can be considered as an example for this type WH geometry (See Fig.3). Scatterer surface is a semi-infinite surface. The half plane surface may be conductive, resistive, or impedance type.

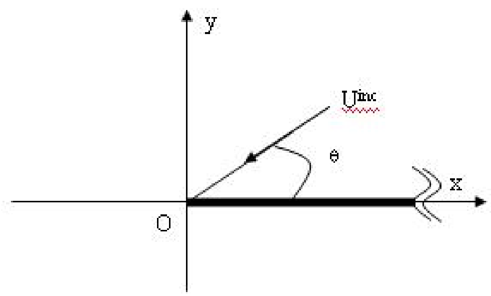

Figure 3. *Half Plane*

3.2 Second Type WH Geometry (WH2)

An example of second type WH geometry (semi-infinite cylinder with hollow) is shown in Fig.4. The surface of the cylinder may represent different material characteristics such as receptivity, conductivity, impedance, or the combination of them. In this type of WH geometry, the axis on which Fourier Transform is applied must not cut the scatterer surface.

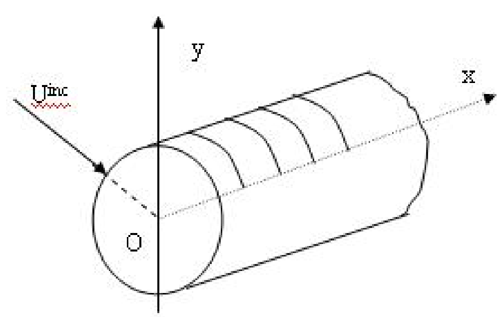

Figure 4. Infinite Cylinder

3.3 Third Type WH Geometry (WH3)

The condition mentioned in the second type WH geometry is satisfied. The condition related the first WH geometry is modified. That is, the scatterer surface is finite. Strip geometry can be given as an example for this type. In the strip problem, the classical WH equation given in (2) takes the form below.

$$G(\nu)P(\nu)+\phi^{-}(\nu)+e^{i\nu l}\phi^{+}(\nu)=H(\nu) \qquad (26)$$

where $P(\nu)$ is an unknown entire function.

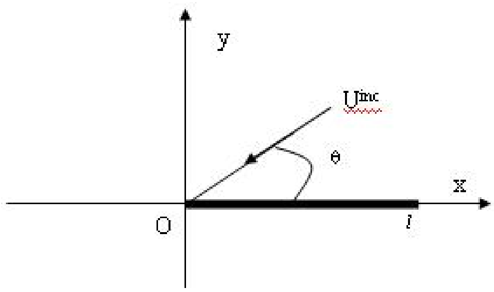

Figure 5. Strip Geometry

3.4 Fourth Type WH Geometry (WH4)

This type of geometry represents the modification of the WH2. The modification is that the axis to which the Fourier Transform is applied passes through the scatterer. Half plane with thickness (Fig.6) or a step discontinuity (Fig.7) can be given as examples of the WH4. The classical WH equation given in (2) takes the form

$$G(\nu)\phi^{+}(\nu)+T(\nu)\phi^{+}(-\nu)+\phi^{-}(\nu)=H(\nu) \qquad (27)$$

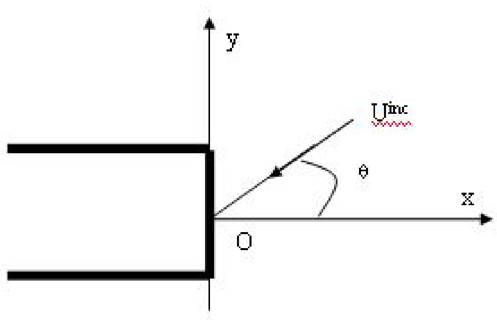

Figure 6. Half Plane with Thickness

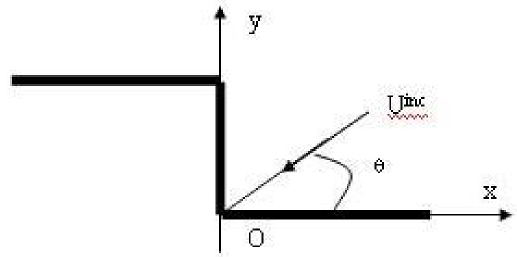

Figure 7. Step Discontinuity

3.5 Fifth Type WH Geometry (WH5)

Both conditions introduced for WH1 and WH2 are modified. Scatterer surface is a semi-infinite surface (WH1). The axis on which Fourier Transform is applied must not cut the scatterer surface(WH2). Double Step Discontinuity (Fig.8) or Strip with Thickness (Fig.9) can be given as examples of the WH5.

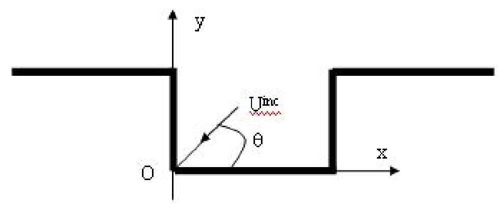

Figure 8. Double Step Discontinuity

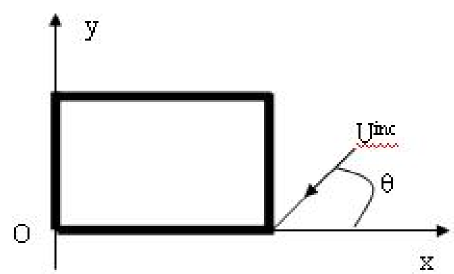

Figure 9. Strip with Thickness

CONCLUSIONS

Wiener-Hopf technique is very important tool in the solution of diffraction problems having special geometrical structures. In this paper we have discussed some important applications of this method for the solution of electromagnetic diffraction problems. Five types of Wiener-Hopf geometries and related solution methods under Wiener-Hopf analysis have been studied. We note that other problems containing such geometries can be solved by taking the analysis given above.

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