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A Classification of Strict Walker 3-Manifold

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Abstract

In this paper we construct two special families of ruled surfaces in a three dimensional strict Walker manifold. The local degeneracy (resp. non-degeneracy) to one of this family has a strong consequence on the geometry of the ambiant Walker manifold.

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1. Introduction

The study of submanifolds of a given ambiant space is a naturel interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here the ambiant space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Among these, the significant Walker manifolds are the examples of the non-symmetric and non-homogeneous Osserman manifolds ([4]).
Three dimensional geometry plays a central role in the investigation of many problem in Riemannian and Lorentzian geometry. The fact that Ricci operator completly determines the curvature tensor is crucial to these investigations ([1]). The strict Walker manifolds are described in terms of a suitable coordinates (x,y,z) of the manifolds $\mathbb{R}^3$ and their metric depends on an arbitrary function of two variables $f = f(y,z)$ and their metric tensor is given by

$$g_f^{\varepsilon} = \varepsilon dy^2 + 2dxdz + fdz^2 \tag{1.1}$$

where $\varepsilon = \pm 1$. These manifolds are denoted by (M, g_f^{ε}).
Curvature properties and a complete characterization of locally symmetric or locally conformally flat three dimensional Walker manifolds have been studies in [3]. Also, in [6] the authors obtained a complete classification of parallel surfaces in a Lorenztian three strict Walker manifold (i.e. admetting a parallel null vector field) as the ambiant space. Some results on minimal graphs on three dimensional Walker manifolds can be found in [5].
In this paper we present two special classes of ruled surfaces in (M, g_f^{ε}) which look like to ruled surfaces in the Euclidean space $\mathbb{E}^3$ and semi-Euclidean $\mathbb{E}_1^3$. These ruled surfaces are made by a one-parameter family of affine straight lines which are the geodesics of $\mathbb{E}^3$ (resp. $\mathbb{E}_1^3$). The study of ruled surfaces of a given ambiant space is a naturel and interesting problem. A surface Σ in M is said to be ruled if every point of Σ is on (a open geodesic segment) in M that lies in Σ (see [8]). Locally a ruled surface is made by a one parameter family of geodesic segments [2]. Several authors are studied problems on ruled surfaces (see [7], [11]).
The paper is organised as follow: in section 2, we recall some preliminaries results for Walker manifold (M, g_f^{ε}). In the section 3, we give some basic formula for immersed surface in (M, g_f^{ε}) and we construct the two families of ruled surfaces in (M, g_f^{ε}) which are used in the main result. In the last section we give the proof of the main theorem. We show that the local degeneracy (respectively non-degeneracy) of one surface of this family has a strong consequence on the geometry of the ambiant space. Moreover, the surfaces of one of these families are flats. A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker ([10]). Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^{ε}) with coordinates (x,y,z) is expressed as

$$g_f^{\varepsilon} = dx \circ dz + \varepsilon dy^2 + f(x,y,z)dz^2 \tag{1.2}$$

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and its matrix form as

$$g_f^\varepsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ with inverse } (g_f^\varepsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x,y,z)$, where $\varepsilon = \pm 1$ and thus $D = \mathrm{Span}\partial_x$ as the parallel degenerate line field. Notice that when $\varepsilon = 1$ and $\varepsilon = -1$ the Walker manifold has signature $(2,1)$ and $(1,2)$ respectively, and therefore is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (1.2) is given by:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x + f_z)\partial_x + \frac{1}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z \end{aligned} \tag{1.3}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. Hence, if (M, g_f^ε) is a strict Walker manifolds i.e., $f(x,y,z) = f(y,z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \ \nabla_{\partial_z}\partial z = \frac{1}{2}f_z\partial_x - \frac{\varepsilon}{2}f_y\partial_y. \tag{1.4}$$

The non-zero components of the curvature tensor of (M, g_f^ε) are given by

$$\begin{aligned} R(\partial_x,\partial_z)\partial_x &= \frac{1}{2}f_{xx}\partial_x, \ R(\partial_x,\partial_z)\partial_y = \frac{1}{2}f_{xy}\partial_x, \ R(\partial_y,\partial_z)\partial_y = -\frac{1}{2}f_{yy}\partial_x, \\ R(\partial_x,\partial_z)\partial_z &= \frac{1}{2}ff_{xx}\partial_x - \frac{\varepsilon}{2}ff_{xy}\partial_y - \frac{1}{2}ff_{xx}\partial_z, \ R(\partial_y,\partial_z)\partial_x = \frac{1}{2}f_{xy}\partial_x, \\ R(\partial_y,\partial_z)\partial_z &= \frac{1}{2}ff_{xy}\partial_x - \frac{\varepsilon}{2}f_{yy}\partial_y - \frac{1}{2}f_{xy}\partial_z. \end{aligned} \tag{1.5}$$

Note that the existence of a null parallel vector field (i.e $f = f(y,z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ε as follows:

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2}f_y, \ \Gamma_{33}^1 = \frac{1}{2}f_z, \ \Gamma_{33}^2 = -\frac{\varepsilon}{2}f_y \tag{1.6}$$

and

$$R(\partial_y,\partial_z)\partial_y = -\frac{1}{2}f_{yy}\partial_x, \quad R(\partial_y,\partial_z)\partial_z = -\frac{\varepsilon}{2}f_{yy}\partial_y \tag{1.7}$$

Let now u and v be two vectors in M. Denoted by (e_1,e_2,e_3) the canonical frame in $\mathbb{R}^3$. The vector product of u and v in (M, g_f^ε) with respect to the metric g_f^ε is the vector denoted by $u \times v$ in M defined by

$$g_f^\varepsilon(u \times v, w) = \det(u,v,w) \tag{1.8}$$

for all vector w in M, where $\det(u,v,w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ then by using (1.8), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) e_1 - \varepsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_3$$

2. Fundamental equation for surfaces

Let (M,g) be a pseudo-Riemannian manifold with its canonical Levi-Civita connection denoted by ∇. We will also denoted g by $\langle \cdot,\cdot \rangle$. Let $\Sigma \subset M$ be a semi-Riemannian surface of (M,g). We denoted by ξ a unit normal vector field on Σ with sign $\varepsilon_1 = \langle \xi,\xi \rangle = \pm 1$. Let D be the Levi-Civita connection of the induced metric on Σ by the inclusion $i : \Sigma \hookrightarrow M$. If X, Y and Z are tangent vector field to Σ then, the Gauss and the Weingarten equations are given by

$$\nabla_X Y = D_X Y + h(X,Y)\xi \tag{2.1}$$
$$-\nabla_X \xi = SX \tag{2.2}$$

where h and S are respectively the second fundamental form and the shape operator of Σ, which are related by:

$$g_f^\varepsilon(SX,Y) = h(X,Y)\varepsilon_1 = g_f^\varepsilon(\nabla_X Y, \xi). \tag{2.3}$$

If we denoted by R^M the curvature of (M,g) and by R the curvature of (Σ, i^*g), then R^M and R satisfied

$$\begin{aligned} \langle R^M(X,Y)Z,W \rangle &= \langle R(X,Y)Z,W \rangle + \varepsilon_1 \Big(h(Y,Z)h(X,W) \\ &\quad - h(X,Z)h(Y,W) \Big) \end{aligned} \tag{2.4}$$
$$\langle R^M(X,Y)Z,\xi \rangle = \varepsilon_1 \Big((\nabla h)(Y,X,Z) - (\nabla h)(X,Y,Z) \Big) \tag{2.5}$$

where

$$(\nabla h)(X,Y,Z) = X\Big(h(Y,Z)\Big) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.6}$$

Next, we will also use the classical version of the fundamental equation (2.4) for parametrized surfaces in (M,g_f^{ε}) which are isometric immersions. Let $\mathscr{D}$ be an open subset of the plane $\mathbb{R}^2$ satisfying this interval condition: horizontal or vertical lines intersect $\mathscr{D}$ in intervals (if at all). A *two-parameter map* is a smooth map $\varphi : \mathscr{D} \longrightarrow M$. Thus φ is composed of two interwoven families of *parameter curves*:

1. the u-parameter curves $v = v_0$ of φ is $u \longrightarrow \varphi(u,v_0)$.
2. the v-parameter curves $u = u_0$ of φ is $v \longrightarrow \varphi(u_0,v)$.

The partial velocities $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$ are vector fields on φ. Evidently $\varphi_u(u_0,v_0)$ is the velocity vector at u_0 of the u-parameter curve $v = v_0$, and symmetrically for $\varphi_v(u_0,v_0)$. If φ lies in the domain of a coordinate system $x^1,\ldots,x^n$, then its coordinate functions $x_i \circ \varphi$ $(1 \leq i \leq n)$ are real-valued functions on $\mathscr{D}$ and

$$\varphi_u = \sum \frac{\partial x^i}{\partial u}\partial_i, \quad \varphi_v = \sum \frac{\partial x^i}{\partial v}\partial_i.$$

So far M could be a smooth manifold: now suppose it is semi-Riemannian. If Z is a smooth vector field on φ, its partial covariant derivatives are: $Z_u = \frac{DZ}{\partial u}$, the covariant derivative of Z along u-parameter curves, and $Z_v = \frac{DZ}{\partial v}$, the covariant derivative of Z along v-parameter curves. Explicitly, $Z_u(u_0,v_0)$ is the covariant derivative at u_0 of the vector field $u \longrightarrow Z(u,v_0)$ on the curve $u \longrightarrow \varphi(u,v_0)$.
In terms of coordinates, $Z = \sum Z^i\partial_i$, where each $Z^i = Z(x^i)$ is a real valued function on $\mathscr{D}$. Then

$$Z_u = \sum_k \Big\{\frac{\partial Z^k}{\partial u} + \sum_{i,j}\Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u}\Big\}\partial_k \tag{2.7}$$

In the special case $Z = \varphi_u$ the derivative $Z_u = \varphi_{uu}$ gives the accelerations of u-parameter curves, while φ_{vv} gives v-parameter accelerations. With coordinate notation as above, we have

$$\varphi_{uv} = \sum_k \Big\{\frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i,j}\Gamma_{ij}^k \frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\Big\}\partial_k. \tag{2.8}$$

Now we will assume that φ is an isometric immersion. The first fondamental form of the immersion φ is given by

$$\begin{cases} E = g_f\left(\varphi_*(\partial_u),\varphi_*(\partial_u)\right) \\ F = g_f\left(\varphi_*(\partial_u),\varphi_*(\partial_v)\right) \\ G = g_f\left(\varphi_*(\partial_v),\varphi_*(\partial_v)\right). \end{cases} \tag{2.9}$$

The coefficients of the second fundamental form of φ are

$$\begin{cases} L &= \varepsilon_1 g_f\left(\varphi_{uu},\xi\right) \\ M &= \varepsilon_1 g_f\left(\varphi_{uv},\xi\right) \\ N &= \varepsilon_1 g_f\left(\varphi_{vv},\xi\right) \end{cases} \tag{2.10}$$

where $\varepsilon_1 = g_f^{\varepsilon}(\xi,\xi)$ the sign of the unit normal ξ. We recall the two most important curvature functions for submanifolds: mean and Gauss curvatures. The mean curvature is given by

$$H = \varepsilon_1 \frac{1}{2}\Big(\frac{LG - 2MF + NE}{EG - F^2}\Big). \tag{2.11}$$

We put $X = \varphi_u$ and $Y = \varphi_v$, then the sectional curvature $K(X,Y)$ of $\mathscr{D}$ and the sectional curvature $K^M(X,Y)$ of M are related by

$$K(X,Y) = K^M(X,Y) + \varepsilon_1 \frac{LN - M^2}{EG - F^2}. \tag{2.12}$$

The sectional curvature of $K^M(X,Y)$ is defined by

$$K^M(X,Y) = \frac{g_f^{\varepsilon}(R^M(X,Y)X,Y)}{g_f^{\varepsilon}(X,X)g_f^{\varepsilon}(Y,Y) - (g_f^{\varepsilon})^2(X,Y)}. \tag{2.13}$$

The equations (2.4) and (2.5) (Gauss-Codazzi) take the following form

$$\begin{aligned} \varphi_{uuv} - \varphi_{uvu} &= R^M(\varphi_u,\varphi_v)\varphi_u \\ \varphi_{vuv} - \varphi_{vvu} &= R^M(\varphi_v,\varphi_u)\varphi_v \end{aligned} \tag{2.14}$$

see [9] for details. To end this section, we construct the two families of ruled surfaces in (M,g_f^{ε}) which are used in the main result. From (1.6), a curve $\gamma(t) = (\gamma_1(t),\gamma_2(t),\gamma_3(t))$ is a geodesic of (M,g_f^{ε}) if the following relation are satisfied:

$$\begin{cases} \frac{d^2\gamma_1(t)}{dt^2} &= f_y \frac{d\gamma_2}{dt}\frac{d\gamma_3}{dt} + \frac{1}{2}f_z\left(\frac{d\gamma_3}{dt}\right)^2 \\ \frac{d^2\gamma_2(t)}{dt^2} &= -\frac{\varepsilon}{2}f_y\left(\frac{d\gamma_3}{dt}\right)^2 \\ \frac{d^2\gamma_3(t)}{dt^2} &= 0. \end{cases} \tag{2.15}$$

These equations have the following trivial solutions: $\gamma_1(t) = a_1t + b_1, \gamma_2(t) = a_2t + b_2$ et $\gamma_3(t) = b_3$ où $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. From these solutions one gets the following ruled surfaces made by affine straight line.
Let $r \in \mathbb{R}$ and $b : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function. We denote by $\Sigma_1(r,b)$ the surface in M defined by the equation

$x + \varepsilon ry - \varepsilon r^2 z - b(z) = 0.$

The surface $\Sigma_1(r,b)$ can be parametrised by the map

$$\begin{array}{rcl} \varphi : \mathbb{R}\times\mathbb{R} & \to & M \\ (u,v) & \mapsto & u(-\varepsilon r, 1, 0) + (b(v), rv, v). \end{array} \tag{2.16}$$

Let now $c : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function. We denote by $\Sigma_2(c)$ the surface in M defined by $y = c(z)$ where a parametrisation is given by

$$\begin{array}{rcl} \psi : \mathbb{R}\times\mathbb{R} & \to & M \\ (x,z) & \mapsto & x(1,0,0) + (0, c(z), z). \end{array} \tag{2.17}$$

3. Main results

The main results of this work is the following theorems:

Theorem 3.1. *Let M be a three-dimensional strict Walker manifold.*

1. *The following assertions are equivalents:*
   - *(a) There exist a surface $\Sigma_1(r,b)$ which is degenerate (anywhere) in some open neighborhood W of a point p, $(W \subset \Sigma_1(r,b)$.*
   - *(b) There exist a neighborhood U of point $p \in M$ $(U \subset M)$ on which the function f depends only on z.*
   - *(c) There exist an open set $\Omega \subset M$ on which we have $f_y = 0$.*
2. *If any one of the three assertions (a), (b) and (c) doesn't hold then $\Sigma_1(r,b)$ is not extrinsecally flat; and if one of the surfaces $\Sigma_1(r,b)$ is flat then the function f is solution of the differential equation $2|f + 2b'(z) + \varepsilon r^2|\varepsilon f_{yy} + \varepsilon_1\varepsilon f_y^2 = 0$ on $\Sigma_1(r,b)$.*
3. *A surface $\Sigma_2(c)$ is flat and minimal.*

Proof. 1. we prove that (a)$\Rightarrow$ (b)$\Rightarrow$ (c)$\Rightarrow$ a).
Let $\Sigma_1(r,b)$ be a surface of M given by its equation: $x + \varepsilon ry - \varepsilon r^2 z - b(z) = 0$. The map given in (2.16) is a parametrisation of $\Sigma_1(r,b)$. We put $X = \varphi_u$ and $Y = \varphi_v$. An easy computation gives

$$\begin{cases} X = -\varepsilon r\partial_x + \partial_y \\ Y = b'\partial_x + r\partial_y + \partial_z \end{cases} \tag{3.1}$$

where $b'(v) = \frac{db}{dv}$.
By using the metric given by (1.2), one gets

$$\begin{cases} g_f^\varepsilon(X,X) = \varepsilon \\ g_f^\varepsilon(Y,Y) = f + 2b' + \varepsilon r^2 \\ g_f^\varepsilon(X,Y) = 0 \end{cases} \tag{3.2}$$

Let us show that (a)$\Rightarrow$ (b).
Assume that a surface $\Sigma_1(r,b)$ is degenerate anywhere in a neighborhood W of a point $p \in \Sigma_1(r,b)$.
Since $\Sigma_1(r,b)$ is embedded in $M = \mathbb{R}^3$ then, W is the trace of an open set Ω' of M, that is $W = \Omega' \cap \Sigma_1(r,b)$.
From (3.2), the degeneracy (everywhere) in W is equivalent to say that

$$f(u + rv, v) + 2b'(v) + \varepsilon r^2 = 0 \tag{3.3}$$

in W. Since u, v are coordinate in W then $f(u + rv, v) + 2b'(v) + \varepsilon r^2 = 0$ for (u,v) belong to an open set in $\mathbb{R}^2$. The fact that f does not depends on the variable x, the relation above show that f must be depends only on z. This shows (b).
(b) $\Rightarrow$ (c) is trivial.
Let us show that (c) $\Rightarrow$ (a).
Assume that there exist an open set $\Omega \subset M$ such that $f_y = 0$. Then there is an open set Ω' in Ω on which f depends only on z, i.e $f = f(z)$.
We choose $\Omega' \subset \Omega$ as follow:

$$\Omega' =]a_1, a_2[\times]b_1, b_2[\times]c_1, c_2[.$$

Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a bump function of a point $z_0 \in]c_1, c_2[$, that is

$$\begin{cases} \theta = 1 \text{ on }]z_0 - \delta, z_0 + \delta[\subset]c_1, c_2[\\ Supp(\theta) \in]c_1, c_2[, \end{cases}$$

where δ is a positive real number. We define $b : \mathbb{R} \longrightarrow \mathbb{R}$ by $b(z) = -\frac{1}{2}\theta\left(\varepsilon r^2 z + \int_{z_0}^{z} f(t)dt\right)$. With this construction one can see that the surface $\Sigma_1(b,r)$ is degenerate everywhere in $\Omega' \cap \Sigma_1(b,r)$ by (3.3). Thus (a)$\Rightarrow$ (b)$\Rightarrow$ (c)$\Rightarrow$ (a).

2. Now assume that one of the assertion (a), (b) and (c) doesn't hold. Then a $\Sigma_1(r,b)$ is non-degenerate (locally) and it induced a metric given by
$$\begin{pmatrix} \varepsilon & 0 \\ 0 & G \end{pmatrix}, \tag{3.4}$$
where $G = f(u+rv,v)+2b'(z)+\varepsilon r^2$. Let us computate now the shape operator and the curvature of the surface $\Sigma_1(r,b)$. We denote by
$$\begin{aligned} X = \varphi_u &= -\varepsilon r\partial_x + \partial_y \\ Y = \varphi_v &= b'\partial_x + r\partial_y + \partial_z. \end{aligned} \tag{3.5}$$
Using the relations (2.8), (2.16) and (1.6) one can get
$$\begin{aligned} \varphi_{uu} &= 0 \\ \varphi_{uv} &= \frac{1}{2}f_y\partial_x \\ \varphi_{vv} &= (b''+rf_y+\frac{1}{2}f_z)\partial_x - \frac{\varepsilon}{2}f_y\partial_y. \end{aligned} \tag{3.6}$$
And the relation (1.8) and (3.5) give
$$\varphi_u \times \varphi_v = (-(\varepsilon r^2+b')-f, r, 1). \tag{3.7}$$
Using (2.10) we get the following relation
$$\begin{cases} \varepsilon_1 DL = g_f^{\varepsilon}(\varphi_u\times\varphi_v, \varphi_{uu}) = 0 \\ \varepsilon_1 DM = g_f^{\varepsilon}(\varphi_u\times\varphi_v, \varphi_{uv}) = \frac{1}{2}f_y \\ \varepsilon_1 DN = g_f^{\varepsilon}(\varphi_u\times\varphi_v, \varphi_{vv}) = b''+\frac{1}{2}rf_y+\frac{1}{2}f_z \end{cases} \tag{3.8}$$
where $D=\sqrt{|f+2b'(z)+\varepsilon r^2|}$. The curvature K^M of the manifold M is obtained by the relation (2.13) and we have
$$K^M(X,Y) = -\frac{\varepsilon f_{yy}}{2(f+2b'(z)+\varepsilon r^2)}. \tag{3.9}$$
We have by (2.13), (3.9) and (2.12)
$$K(X,Y) = \frac{-2D^2\varepsilon f_{yy} - \varepsilon_1\varepsilon f_y^2}{D^2(f+2b'(z)+\varepsilon r^2)} \tag{3.10}$$
Then if we suppose that a surface $\Sigma_1(r,b)$ is flat then f is solution of the differential equation $2D^2\varepsilon f_{yy}+\varepsilon_1\varepsilon f_y^2=0$.
3. Now we show that $\Sigma_2(c)$ is flat

We have $\psi_x = \partial_x$ and $\psi_z = c'\partial_y+\partial_z$. With simple calculus we have
$$\begin{aligned} g_f^{\varepsilon}(\psi_x,\psi_x) &= 0 \\ g_f^{\varepsilon}(\psi_z,\psi_z) &= \varepsilon c'^2+f \\ g_f^{\varepsilon}(\psi_x,\psi_z) &= 1 \end{aligned}$$
The matrice is given by
$$\begin{pmatrix} 0 & 1 \\ 1 & \varepsilon c'^2+f \end{pmatrix}$$
This shows that the surfaces ψ is non-degenerate and is an isometric immersion $(\mathbb{R}\times\mathbb{R}, \psi^*(g_f^{\varepsilon})$ into (M, g_f^{ε}).
We denote by $G=\varepsilon c'^2+f$ and the matrice induced by the surface $\Sigma_2(c)$ is given by
$$\begin{pmatrix} 0 & 1 \\ 1 & G \end{pmatrix}. \tag{3.11}$$
and $X=\psi_x=\partial_x$, $Y=\psi_z=c'\partial_y+\partial_z$. By using the formula (1.8) of vector product we have
$$\psi_x\times\psi_z = \begin{pmatrix} c' \\ -\varepsilon \\ 0 \end{pmatrix}$$
that is
$$\psi_x\times\psi_z = c'\partial_x - \varepsilon\partial_y \tag{3.12}$$
and $g(\psi_x\times\psi_z, \psi_x\times\psi_z)=\varepsilon_1$. Thus the unit vector is
$$\xi = c'\partial_x - \varepsilon\partial_y. \tag{3.13}$$

Put $X=\partial_x$ and $Y=c'\partial_y+\partial_z$. The same calculus give $\psi_{xx}=0$, $\psi_{xz}=0$ and $\psi_{zz}=(c'f_y+\frac{1}{2}f_z)\partial_x+(c''-\frac{\varepsilon}{2}f_y)\partial_y$. Now the shape operator is given by the matrice

$$S=\begin{pmatrix} 0 & \frac{-c''-\frac{\varepsilon}{2}f_y}{DG} \\ 0 & 0 \end{pmatrix} \tag{3.14}$$

Now let compute the curvature. Since $\det S=0$, then by the Gauss equation (2.12), we have $K(X,Y)=K^M(X,Y)$ where $X=\partial_x$ and $Y=c'\partial_y+\partial_z$. Then

$$\begin{aligned} R^M(X,Y)X &= R^M(\partial_x, c'\partial_y+\partial_z)\partial_x \\ R^M(X,Y)X &= 0 \end{aligned}$$

Then the surface $\Sigma_2(c)$ is flat and minimal.

□

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