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OPTIMIZATION OF DIFFERENTIAL INCLUSIONS OF PARABOLIC TYPE

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ABSTRACT

Sufficient conditions for optimality are derived for the problems under consideration on the basis of the apparatues of locally conjugate mappings

Keyword: *Multivalued mappings, subdifferential ,Locally conjugate mappings. Sufficient conditions.*

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Introduction

A great many problems in economic dynamics, as well as classical problems on optimal control, differential games, and soon, can be reduced to investigations described by multivalued mappings with discrete and continuous time and with concentrated or lumped parameters [1]-[4].

The present article is devoted to an investigation of problems with convex differential inclusions of parabolic type. For such problems we use constructions of convex analysis in terms of locally conjugate mappings [1]-[4] to get sufficient conditions. At the end of we consider an optimal control problem described by the heat equation. This example shows that in known problems the conjugate equation which is traditionally obtained with the help of the Hamiltonian function. Since the continuous problems posed are described by multivalued mappings, it is obvious that many problems involving optimal control of chemical engineering, heat, and diffusion processes (see, for example [2], [5], [6]) can be reduced to this formulation.

The results obtained we generalize to the multi-dimensional case with a second-order elliptic operator for bounded cylindrical domains.

In [7]-[10] necessary conditions for an extremum are obtained for some control problems with distributed parameters in abstract Hilbert spaces. As a rule, the methods of these papers require the introduction of operators with a maximal monotone graph in $R \times R$.

It must be pointed out that in differential inclusions the solution must is taken in the space $C^{1,2}$ [11]. However, as will be seen from the context, the definition below of the concept of a solution in this or that sense is introduced only for simplicity and does not in any way restrict the class of problems under consideration. Therefore, at the end of the paper we indicate ways of extending the results to the case of generalized solutions [11].

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1. Necessary Conditions and Problem Statement

The basic concepts and definitions given below can be found in [1], [2]-[4]. Let R^n be the n-dimensional Euelidian space; (x_1, x_2) is a pair of elements $x_1, x_2 \in R^n$; and $<x_1, x_2>$ is their inner product. We say that multivalued mapping a: $R^{2n} \to 2^{R^n}$ is convex if its graph $\text{gf}a = \{(x_1,x_2,v) : v \in a(x_1,x_2)\}$ is a convex subset of R^{3n}.
For such mappings we introduce the notation

$$W_a(x_1,x_2,v^*) = \inf\{< v,v^* >: v \in a(x_1,x_2)\}, v^* \in R^n,$$

$$a(x_1,x_2,v^*) = \{v \in a(x_1,x_2) :< v,v^* >= W_a(x_1,x_2,v^*)\}$$

For convex a we let $W_a(x_1,x_2,v^*) = +\infty$ of $a(x_1,x_2) = \varnothing$.
The cone of tangent directions at a point $(x_1,x_2,v) \in \text{gf}a$ will be denoted by $K_a(x_1,x_2,v)$. For a convex mapping a

$$K_a(x_1,x_2,v) = \text{con}(\text{gf}a - x_1,x_2,v)) = \{(\bar{x}_1,\bar{x}_2,\bar{v}) : \bar{x}_1 = \lambda(x_1^0 - x_1), \bar{x}_2 = \lambda(x_2^0 - x_2), \bar{v} = \lambda(v^0 - v),$$
$$\lambda > 0, (x_1^0 x_2^0, v^0 \in \text{gf}a\}$$

A mapping

$$a^*(v^*;(x_1,x_2,v)) = \{(x_1^*,x_2^*) : (-x_1^*,-x_2^*,v^*) \in K_a^*(x_1,x_2,v)\}$$

is called the locally conjugate mapping(LCM) [2]-[4] to a at the point (x_1,x_2,v), where
$K_a^*(x_1,x_2,v)$ is the cone dual to the cone $K_a(x_1,x_2,v)$.
A function g is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.
In §2 we formulate the sufficient conditions for convex parabolic differential inclusions with distributed parameters:

$$\text{(1.1)} \qquad I(x(.,.)) = \iint_Q g(x(t,\tau),t,\tau)dtd\tau$$

$$\text{(1.2)} \qquad \frac{\partial^2 x(t,\tau)}{\partial\tau^2} - \frac{\partial^2 x(t,\tau)}{\partial t} \in a\left(\frac{\partial^2 x(t,\tau)}{\partial\tau}, x(t,\tau)\right), \ 0 < t \le 1, \ \ 0 < \tau < 1$$

$$x(0,\tau) = \beta(\tau), \ x(t,0) = \alpha_0(t), \ x(t,1) = \alpha_1(t), \ Q = [0,1] \times [0,1]$$

Here a: $R^{2n} \to 2^{R^n}$ is a convex multivalued mapping, $g(x,t,\tau)$ is continuous function that is convex with the respect to x, $g : R^n \times Q \to R$, β and α_i(i=1,2) are continuous functions, $\beta : [0,1] \to R^n$, $\alpha_i : [0,1] \to R^n$. The problem is to find a solution $\tilde{x}(t,\tau)$ of the first boundary value problem (1.1), (1.2) that minimizes *I(x(.,.))*. Here a solution is understood to be a classical solution only for simplicity of the exposition(see below). In what follows we introduce the concept of a generalized solution and show that it is possible to carry over the results obtained to this case.

The subject of the next investigation is the multidimensional convex optimal control problem for partial differential inclusions of

(1.3) $\mathrm{I}(x(.,.)) = \int_0^1 \int_G g(x(t,\tau),t,\tau)dtd\tau \to \inf$

(1.4) $£x(t,\tau) \in \frac{\partial x(t,\tau)}{\partial t} + a\left(\frac{\partial x(t,\tau)}{\partial \tau}, x(t,\tau)\right)$

(1.5) $x(0,\tau) = \beta(\tau),\ \tau \in G \subset R^n$

(1.6) $x(t,\tau) = \alpha(t,\tau),\ \ (t,\tau) \in L$

where a: $\mathrm{R}^{2n} \to 2^{R^n}$ and G is the domain of change arguments $\tau=(\tau_1, ..., \tau_n)$ in the differential inclusion (1.4). Thus, the domain in which (1.4) given is a cylinder $D = \{\tau \in G, 0 < t < 1\}(D \subset R^{n+1})$ of height 1 and with base G, L is the lateral surface of $D = \{\tau \in S, 0 < t < 1\}$, and $G \times \{0\}$ and $G \times \{1\}$ are the lower and upper bases, respectively.
Further,

$$£x = \sum_{i,j=1}^{n} \frac{\partial}{\partial \tau_i}\left(d_{ij}(\tau)\frac{\partial x}{\partial \tau_j}\right) + \sum_{i=1}^{n} b_i(\tau)\frac{\partial x}{\partial \tau_i} + (\tau)x$$

is a second order elliptic operator, $d_{ij}(\tau) \in C^1(\overline{D})(d_{ij} = d_{ji})$, $b_i(\tau) \in C^1(\overline{D})$, $C(\tau) \in C(\overline{D})$. Here $C(\overline{D})$ and $C^1(\overline{D})$ are the spaces of continuous functions and the functions having a continuous derivative in D, respectively. A function $x(t,\tau)$ in $C^{1,2}(D) \bigcup C[D \bigcup L \bigcup (G \times \{0\}]$, that satisfies the inclusion (1.4) in D, the initial condition (1.5) on $(G \times \{0\}$, and the boundary condition (1.6) on L is called a classical solution of the problem posed, where $C^{1,2}(D)$ is the space for functions $u(t,\tau)$ having contiuous derivatives $\partial u / \partial t$, $\partial^2 u / \partial \tau_i \partial \tau_j$, $i, j = 1,...,n$ (See the definition of the spaces $C^{2s,s}, s \geq 1, C^{r,0}, r \geq 1$ for example, in [11]).

2. Sufficient Conditions for Optimality for Differential Inclusions of Parabolic Type

At first we consider the convex problem (1.1), (1.2), We know Theorem 2.1. Suppose that g(x,t,τ) is continuous function convex with respect to x, and a is a convex closed mapping. Then for the optimality of the solution $\widetilde{x}(t,\tau)$ among all admissible solutions it is sufficient that there exist classical solutions $u^*(t,\tau), x^*(t,\tau))$ such that the conditions 1) – 2) hold:

1) $(u^*(t,\tau), \partial^2 x^*(t,\tau)/\partial\tau^2) \in a^*\left(x^*(t,\tau); \partial\tilde{x}(t,\tau)/\partial\tau, \tilde{x}(t,\tau), \partial^2\tilde{x}(t,\tau)/\partial\tau^2 - \partial\tilde{x}(t,\tau)/\partial t)\right)$

$+\{0\}\times\left(\partial g(\tilde{x}(t,\tau),t,\tau) - \partial x^*(t,\tau)/\partial t - \partial u^*(t,\tau)/\partial\tau\right); \quad x^*(t,0)=0,\ x^*(t,1)=0,\ x^*(1,\tau)=0$

2) $\partial^2\tilde{x}(t,\tau)/\partial\tau^2 - \partial\tilde{x}(t,\tau)/\partial t \in a\left(\partial\tilde{x}(t,\tau)/\partial\tau, \tilde{x}(t,\tau), x^*(t,\tau)\right)$

Proof. By Theorem 2.1.III in [1]

$$a^*(v^*;(x_1,x_2,v)) = \partial_{(X_1,X_2)}W_a(x_1,x_2,v^*), v^* \in a(x_1,x_2,v^*)$$

Then by using the Moreau-Rockafeller theorem [1], from condition 1) we obtain the inclusion

$$\left(u^*(t,\tau), \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} + \frac{\partial u^*(t,\tau)}{\partial\tau}\right) \in \partial_{(X_1,X_2)}\Big[W_a\Big(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), x^*(t,\tau)\Big)$$
$$+ g_1\Big(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), t,\tau\Big)\Big], (t,\tau)\in Q,\ g_1(x_1,x_2,t,\tau) \equiv g(x,t,\tau)$$

using the definitions of subdifferential and *Wa* , we rewrite the last relation in the form

$$\left\langle \frac{\partial^2 x(t,\tau)}{\partial\tau^2} - \frac{\partial x(t,\tau)}{\partial t}, x^*(t,\tau)\right\rangle - \left\langle \frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2} - \frac{D\tilde{x}_{(t_{1i})}}{\partial t}, x^*(t,\tau)\right\rangle$$
$$+ g(x(t,\tau),\tau) - g(\tilde{x}(t,\tau),t,\tau) \ge \left\langle u^*(t,\tau), \frac{\partial x(t,\tau)}{\partial\tau} - \frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle$$
$$+ \left\langle \frac{\partial x^*(t,\tau)}{\partial\tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} + \frac{\partial u^*(t,\tau)}{\partial\tau},\ x(t,\tau) - \tilde{x}(t,\tau)\right\rangle$$

Integrating the preceding relation over the domain *Q*, we get

$$(2.1) \quad \iint_Q \left[g(x(t,\tau),t,\tau) - g(\tilde{x}(t,\tau),t,\tau)\right]dtd\tau \ge \iint_Q \left\langle u^*(t,\tau), \frac{\partial x(t,\tau)}{\partial\tau} - \frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle dtd\tau$$
$$+ \iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} - \frac{\partial u^*(t,\tau)}{\partial t}, x(t,\tau) - \tilde{x}(t,\tau)\right\rangle dtd\tau - \iint_Q \left\langle \frac{\partial^2 x(t,\tau)}{\partial\tau^2} - \frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2}, x^*(t,\tau)\right\rangle dtd\tau$$
$$+ \iint_Q \frac{\partial}{\partial t}\left\langle x^*(t,\tau), x(t,\tau) - \tilde{x}(t,\tau)\right\rangle dtd\tau$$

It is clear that

(2.2) $$\iint_Q \left\langle u^*(t,\tau), \frac{\partial}{\partial\tau}\left(x(t,\tau)-\tilde{x}(t,\tau)\right)\right\rangle dtd\tau = \iint_Q \left\langle \frac{\partial}{\partial\tau}\left(u^*(t,\tau)-x(t,\tau)-\tilde{x}(t,\tau)\right)\right\rangle dtd\tau$$

$$-\iint_Q \left\langle \frac{\partial u^*(t,\tau)}{\partial\tau}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle dtd\tau$$

where, since $x(t,1)=\tilde{x}(t,1)=\alpha_1(t),\, x(t,0)=\tilde{x}(t,0)=\alpha_0(t)$

(2.3) $$\iint_Q \frac{\partial}{\partial\tau}\left\langle u^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle .dtd\tau = \int_0^1 \left\langle u^*(t,1), x(t,1)-\tilde{x}(t,1)\right\rangle dt$$

$$-\int_0^1 \left\langle u^*(t,0), x(t,0)-\tilde{x}(t,0)\right\rangle dt = 0$$

Analogously ($x^*(1,\tau)=0$)

(2.4) $$\iint_Q \frac{\partial}{\partial t}\left\langle x^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle dtd\tau = \int_0^1 \left\langle x^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle d\tau$$

$$-\int_0^1 \left\langle x^*(0,\tau), x(0,\tau)-\tilde{x}(0,\tau)\right\rangle d\tau = \int_0^1 \left\langle x^*(t,\tau), x(t,\tau)-\tilde{x}(t,\tau)\right\rangle d\tau$$

For brevity of notation we demote the right-hand side of (2.1) by P. Then by (2.2)-(2.4) we get from (2.1)

$$P = \iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle dtd\tau - \iint_Q \left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} - \frac{\partial^2\tilde{x}(t,\tau)}{\partial\tau^2}, x\ \ (t,\tau)\right\rangle dt$$

After simple transformations, we obtain

$$P = \iint_Q \frac{\partial}{\partial\tau}\left\langle \frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}, x(t,\tau)-\tilde{x}(t,\tau)\right\rangle dtd\tau - \iint_Q \frac{\partial}{\partial\tau}\left\langle x^*(t,\tau), \frac{\partial x^*(t,\tau)}{\partial\tau} t,\tau) - \frac{\partial\tilde{x}(t,\tau)}{\partial\tau}\right\rangle dtd\tau$$

$$= \int_0^1 \left[\left\langle \frac{\partial^2 x^*(t,1)}{\partial\tau^2}, x(t,1)-\tilde{x}(t,1)\right\rangle - \left\langle \frac{\partial^2 x^*(t,0)}{\partial\tau^2}, x(t,0)-\tilde{x}(t,0)\right\rangle\right] dt$$

$$-\int_0^1 \left[\left\langle x^*(t,1), \frac{\partial}{\partial\tau}\left(x(t,1)-\tilde{x}(t,1)\right)\right\rangle - \left\langle x^*(t,0), \frac{\partial}{\partial\tau}\left(x(t,0)-\tilde{x}(t,0)\right)\right\rangle\right] dt = 0$$

where it is taken into account that $x^*(t,0)0x^*(t,1)=0$ by the condition of the theorem. Thus, we have finally

$$\iint_Q g(x(t,\tau),t,\tau)dtd\tau \geq \iint_Q g(\tilde{x}(t,\tau),t,\tau)dtd\tau$$

The theorem is proved.

We now try to apply the results in this section to the problem (1.3)-(1.6); here the use of theorem 2.1 plays a decisive role in the investigation of this last problem.

Theorem 2.2 If $g(x,t,\tau)$ is continuous functions convex with respect to x, and *a* is a convex closed mapping, then $\tilde{x}(t,\tau)$ minimizes the functional (1.3) among all admissible solutions of the problem (1.3)-(1.6) if there exists a classical solution $(u_i^*(t,\tau), x^*(t,i), i=1,......,n)$ of the following boundary value problem:

$$(u_1^*(t,\tau), u_2^*(t,\tau),....,u_n^*(t,\tau), £^* x\ (t,\tau)) \in a^*\left(x^*(t,\tau); \left(\frac{\partial\tilde{x}(t,\tau)}{\partial\tau}, \tilde{x}(t,\tau), £\tilde{x}(t,\tau) - \frac{\partial\tilde{x}(t,\tau)}{\partial t} \right)\right)$$

$$+\{0\}\times\left\{ \partial g(\tilde{x}(t,\tau),t,\tau) - \frac{\partial x^*(t,\tau)}{\partial\tau} - div((u_1^*(t,\tau),....,u_n^*(t,\tau)) \right\}$$

$$x^*(1,\tau)=0, \quad x^*(t,\tau)=0, \quad \tau\in S$$

where $£^*$ is the operator adjoint to £, and 0^n is the n-dimensional zero vector.
Proof. By arguments analogous to those in the proof of the preceding theorem 2.1 it is not hard to see that

$$(2.5)\quad \int_0^1\int_G [g(x(t,\tau),t,\tau) - g(\tilde{x}(t,\tau),t,\tau)]dtd\tau \geq \int_0^1\int_G \sum_{i=1}^n u_i^*(t,\tau)\frac{\partial}{\partial\tau_i}(x(t,\tau)-\tilde{x}(t,\tau))dtd\tau$$

$$+\int_0^1\int_G \left(£^* x^*(t,\tau) + \frac{\partial x^*(t,\tau)}{\partial\tau} + \sum_{i=1}^n \frac{\partial u_i^*(t,\tau)}{\partial\tau_i} \right)(x(t,\tau)-\tilde{x}(t,\tau))dtd\tau$$

$$-\int_0^1\int_G \left(£x(t,\tau) - £\tilde{x}(t,\tau) - \frac{\partial x(t,\tau)}{\partial\tau} + \frac{\partial\tilde{x}(t,\tau)}{\partial\tau} \right)x^*(t,\tau)dtd\tau$$

Using the boundary conditions (1.6) and the fact that $x^*(t,\tau)=0,\ \tau\in S,$ we get from the familiar Green's formula that

$$(2.6)\quad \int_G \left[x^*(t,\tau)£(x(t,\tau)-\tilde{x}(t,\tau)) - (x(t,\tau)-\tilde{x}(t,\tau)£^* x^*(t,\tau)\right]d\tau$$

$$=\int_S \sum_{i=1}^n \left\{ \sum_{j=1}^n d_{ij}\left[(x^*(t,\tau)-\tilde{x}(t,\tau)) - (x(t,\tau)-\tilde{x}(t,\tau))\frac{\partial x^*(t,\tau)}{\partial\tau_j} \right] \right.$$

$$\left. + b_i(\tau)(x(t,\tau)-\tilde{x}(t,\tau))x^*(t,\tau) \right\}\cos(n_0,\tau)d\tau = 0$$

where n_0 is the outher normal to the surface *S* with respect to *G*. Then, by (2.5) and (2.6)

$$(2.7)\quad I(x(t,\tau)) - I(\tilde{x}(t,\tau)) \geq \int_0^1\int_G \sum_{i=1}^n \frac{\partial}{\partial\tau_i}\left[u_i^*(t,\tau)(x(t,\tau)-(\tilde{x}(t,\tau))\right]dtd\tau$$

$$+\int_0^1\int_G \frac{\partial}{\partial\tau}\left[x_i^*(t,\tau)(x(t,\tau)-(\tilde{x}(t,\tau))\right]dtd\tau$$

But with the Gauss-Ostrogradskii formula it can be shown by simple computations that

(2.8) $$\int_G \sum_{i=1}^{n} \frac{\partial}{\partial \tau_i} \left[u_i^*(t,\tau)(x(t,\tau) - \tilde{x}(t,\tau)) \right] d\tau = = \int_S \sum_{i=1}^{n} \left[u_i^*(t,\tau)(x(t,\tau) - \tilde{x}(t,\tau)) \right] \cos(n_0, \tau_i) d\tau = 0$$

Moreover, by the initial condition (1.5)

(2.9) $$\int_0^1 \int_G \frac{\partial}{\partial \tau} \left[x^*(t,\tau)(x(t,\tau) - \tilde{x}(t,\tau)) \right] dt d\tau = 0 = \int_G \left[x^*(t,\tau)(x(t,\tau) - \tilde{x}(t,\tau)) \right] d\tau$$

Using the fact that x*(1, τ)=0, $\tau \in S$, we get from (2.9)

(2.10) $$\int_0^1 \int_G \frac{\partial}{\partial \tau} \left[x^*(t,\tau)(x(t,\tau) - \tilde{x}(t,\tau)) \right] dt d\tau = 0$$

Then it is clear from (2.7), (2.8), (2.10) that

$$I(x(t,\tau)) \geq I(\tilde{x}(t,\tau)$$

The theorem is proved.
In the conclusion of this section we consider an example.

$$I(x(t,\tau)) \to \inf,$$

(2.11) $$\frac{\partial^2 x(t,\tau)}{\partial \tau^2} - \frac{\partial x(t,\tau)}{\partial t} = A_1 \frac{\partial x(t,\tau)}{\partial \tau} + A_2 x(t,\tau) + Bu(t,\tau), \quad u(t,\tau) \in U$$

$$x(0,\tau) = \varphi(\tau), \ x(t,0) = \alpha_0(t), \ x(t,1) = \alpha_\tau(t)$$

where A_1 and A_2 are $n \times n$ matrices, B is a rectangular $n \times r$ matrix, $U \subset R^r$ is a convex closed set, and g is continuously differentiable function of x. It is required to find a controlling parameter $\tilde{u}(t,\tau) \in U$ such that the solution $\tilde{x}(t,\tau)$ corresponding to it minimizes *I(x(.,.))* .

In this case

$$a(x_1, x_2) = A_1 x_1 + A_2 x_2 + BU$$

By elementary computations we find that

$$a^*(v^*; (x_1, x_2, v)) = \begin{cases} (A_1^* v^*, A_2^* v^*), & B^* v^* \in [con(U-u)]^* \\ \varnothing, & B^* v^* \notin [con(U-u)]^* \end{cases}$$

where $v = A_1 x_1 + A_2 x_2 + Bu$, and [con*M*]* is the cone dual to the cone con*M*.
Then, using Theorem 2.1, we get the relations

(2.12) $u^*(t,i) = A_1^* x^*(t,\tau)$

(2.13) $\dfrac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \dfrac{\partial x^*(t,\tau)}{\partial t} + \dfrac{\partial u^*(t,\tau)}{\partial \tau} - g^{|}(\tilde{x}(t,\tau),t,\tau) = A_2^* x^*(t,\tau)$

(2.14) $\left\langle u - \tilde{u}(t,\tau), B^* x^*(t,\tau) \right\rangle, \quad u \in U$

(2.15) $x^*(1, \tau)=0$, $x^*(t,0=$, $x^*(t,1)=0$

Obviously, (2.13) can be written in the form

(2.16) $\left\langle B\tilde{u}(t,\tau), x^*(t,\tau) \right\rangle = \inf\limits_{u \in U} \left\langle Bu, x^*(t,\tau) \right\rangle$

(2.17) $\dfrac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \dfrac{\partial x^*(t,\tau)}{\partial t} = A_1^* \dfrac{\partial x^*(t,\tau)}{\partial \tau} + A_2^* x^*(t,\tau) + g^{|}(\tilde{x}(t,\tau),t,\tau)$

Thus, we have obtained the following result.
Theorem 2.3. The trajectory $\tilde{x}(t,\tau)$ corresponding to the control $\tilde{u}(t,\tau)$ minimizes I(x(.,.)) in the problem (2.10) if there exists a function $x^*(t,\tau)$ satisfying the conditions (2.15)-(2.17).

In conclusion we discuss the possibility of passing to more general function spaces os solutions in the problems under consideration. It is known that for the theory of partial differential equations the concept of a generalized solution is important both from the theoretical and from the practical point of view [11], [12]. As a rule, the definition of such solutions associates with a given equation a certain integral identity that uses, in turn, the class of generalized derivatives and compactly supported functions. Therefore, on this path the most natural approach for differential inclusions with distributed parameters is apparently the use of single-valued branches selections of a multivalued mapping.

Thus, suppose that we have the problem (1.3)-(1.6) with homogeneous boundary conditions, where $\varphi(x) \in L_2(G)$, $H^{1,0}(D)$ is the Hilbert space (for a more detailed study see, for example, [11] or [12]) consisting of the elements $x(t,\tau) \in L_2(D)^*$ having square integrable generalized derivatives on D , where the inner product and norm are defined by the respective expressions

$$\left\langle x_1, x_2 \right\rangle_{H^{1,0}(D)} = \int_D \left(x_1, x_2 + \frac{\partial x_1 \partial x_2}{\partial \tau \partial \tau} \right) dt d\tau, \qquad \|x\|_{H^{1,0}(D)} = \sqrt{\left\langle x, x \right\rangle_{H^{1,0}(D)}}$$

By analogy with the classical theory of the first boundary value problem for partial differential equations of parabolic type, a function $x(t,\tau) \in H^{1,0}(D)$ is called a generalized solution of our problem id it satisfies the boundary conditions $x(t,\tau)=0$, $\tau \in S$, $0<t<1$, and the identity

$$\int_D \left(x\eta_t - \sum_{i=1}^{n} d_{ij} \frac{\partial x}{\partial \tau_j} \frac{\partial \eta}{\partial \tau_i} - \sum_{i=1}^{n} b_i \frac{\partial x}{\partial \tau_i} \eta - ex\eta \right) dtd\tau = \int_G \varphi\eta d\tau + \int_D f\eta dtd\tau$$

for all $\eta(t,\tau) \in H^1(D)$*[11]* with the conditions $\eta(t,\tau)=0$, $\tau \in G$, $\eta(t,\tau)=0$, $0<t<1$, $t \in S$. Here $f=f(p,x)$ is an arbitrary measurable selection of the multivalued mapping $a(p,x)$.

It is easy to see that the concept of a solution a. e. Can be introduced in addition to see concepts of a classical solution and a generalized solution.

A function $x(t,\tau) \in H^{2,1}(D)$*([11], [12])* is said to be a solution a. e. for the problem (1.3)-(1.6) with homogeneous boundary conditions if it satisfies for almost all $(t,\tau) \in D$ in inclusion (1.4), the initial condition (1.5), and the homogenous boundary conditions. A generalized solution is defined analogously for the adjoint boundary value problem. We now recall that all the results obtained have used the integration by parts formula and the Green-Ostrogradskii formula that follows from it, but the latter can be used for getting the indicated classes of generalized solutions. Therefore, it is not difficult to verify the validity of all the assertions in this general case.

Conclusions

For considered problems is used constructions of convex analysis in terms of locally conjugate mappings to get sufficientconditions for optimality. At the and of paper is considered an optical control problem described by the heat equation. This example shows that in known problems the conjugate inclusion coincides with the conjugate equation which is traditionally obtained with the help of the Hamiltonian function.

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