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EMBEDDING THEOREMS IN BANACH -VALUED SOBOLEV-LIOUVILLE SPACES AND THEIR APPLICATIONS

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ABSTRACT

In this paper we introduce a Banach- valued Sobolev-Liouville spaces associated with Banach spaces E_1,E and some parameters and proved continuity and compacness of embedding operators in these spaces in terms of theory interpolations of Banach spaces uniformly with respect to these parameters and proved estimate of semigroup operator in weighted spaces. This problem arises in the investigation of boundary value problems for differential-operator equations with parameters.Further we consider certain class of partial differential -operator equation with parameters in Lp spaces and establish coercive solvability of this problem uniformly with respect to these parameters.

Keywords: *Banach spaces, Sobolev spaces, positive operators, differential operator equation embedding theorems, interpolation spaces, semi-group of operators.*

1. INTRODUCTION

Embedding theorems in function spaces were studied in a series of books and papers(see, for example, [1] , [2] , [3] , [4] , [5]).In abstract function spaces embedding theorems have been considered by Sobolev [6] , Lions-Peetre [7] , Yakubov-Shakhmurov ,[8] ,Shakhmurov [9 - 11] , Lizorkin-Shakhmurov [12] for instance. Lions-Peetre [7] showed that, if

$$u \in L_2(0,T,H_0)\ ,\ u^{(m)} \in L_2(0,T,H\ ),$$

then

$$u^{(i)} \in L_2(0,T,[H,H_0]_{\frac{i}{m}}),\quad ,i=1,2,...m-1,$$

where H_0 , H are Hilbert spaces, H_0 is continuously and densely embedded in H and $[H_0,H]$ are interpolation spaces between H_0 and H for $0 \leq \theta \leq 1$ Yakubov-Shakhmurov [8] investigated similar questions in anisotropic spaces $W_2^l(\Omega,H_0,H)\ ,\ \Omega \subset R^n$.Later Shakhmurov [9 - 11] and Lizorkin-Shakhmurov [12] considered these questions for the spaces $W_p^l(\Omega.H_0,H)$ and their corresponding weighted spaces.

In this paper , we prove theorems on continuity and compactness of embedding operators in anisotropic, Banach-valued function spaces $W_p^l(\Omega,E_0,E),$ where E_0 and E are Banach spaces such that E_0 is continuously and

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densely embedded in E..Here $l = (l_1, l_2, \ldots l_n)$ and l_k, $k = 1, 2, \ldots, n$ are positive numbers $W_p^l(\Omega, E_0, E)$ consists of functions $u \in L_p(\Omega, E_0)$ such that the derivatives are

$$D_k^{l_k} u = \frac{\partial^{L_k}}{\partial x_k^{l_k}} u \in L_p(\Omega, .E) \ , k = 1, 2, \ldots n.$$

Let $r_1, r_2, \ldots, r_n$ be nonnegative numbers , p and q be real numbers,

$$1 \le p \le q, \ \varkappa = \sum_{k=1}^{n} \frac{r_k + \frac{1}{p} - \frac{1}{q}}{l_k}$$

and

$$D^r = D_1^{r_1} D_2^{r_2} \ldots D_n^{r} = \frac{\partial^r}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_n^{r_n}}$$

.

Let A be a positive operator on E, then there are fractional powers of operator A (see [15] §1.15.1) and for each fractional powers A^θ of A , let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm. Under certain assumptions to be stated later, we prove that the operators $u \to D^r u$ are bounded from space $W_p^l(\Omega, E(A), E)$ to space $L_q(\Omega, E(A^{1-\varkappa}))$, i.e embedding

$$D^r W_p^l(\Omega, E(A), E) \subset L_q\left(\Omega, E\left(A^{1-\varkappa}\right)\right)$$

is continuous.More precisely for $0 \le \mu \le 1 - \varkappa$ we prove the estimate

$$\|D^r u\|_{L_q(\Omega, E(A^{1-\varkappa}))} \le C_\mu \left(h^\mu \|u\|_{W_p^l(\Omega, E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega, E)}\right)$$

for all $u \in W_p^l(\Omega; E(A), E)$ and for all $h \ge 0.$ The constant Cμ in the above equation is independent of $u \in W_p^l(\Omega; E(A), E)$ and of the chooice of $h \ge 0.$.Further we prove compactness of this embedding operator. Furthermore we consider certain applications of this theorems. This kind of embedding theorems arise in the investigation of boundary value problems for anisotropic partial differential-operator equations

$$\sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{l_k} u + Au \sum_{|\alpha:2l|<1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} A_\alpha(x) D^\alpha u = f$$

depend on parameters $t = (t_1, t_2, \ldots, t_n)$, where A is a positive operator on the Banach space E , $A_\alpha(x)$ is an operator such that $A_\alpha(x) A^{-(1-|\alpha:l|)}$ is bounded on E , where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), l = (l_1, l_2, \ldots l_n), |\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}, D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$$

We proof coersive solvobility of this differential-operator equation in the spaces $L_p(R^n, E)$ uniformly with respect to parameter t .In this direction we should mention the works presented in [9 - 11] and [13 - 14] .

2.Notations and Definitons

Let R be the set of real numbers ,C be the set of complex numbers. Let E and E_0 be Banach spaces and $L(E_0, E)$ denotes the spaces of bounded linear operators acting from E_0 to E .For $E_0 = E$ we denote L(E,E) by L(E) , I denotes the identity operator in the Banach space E.We will sometimes write $A + \xi$ or A_ξ instead of $A + \xi I$ for a scalar ξ . $(A - \xi I)^{-1}$ will denote the inverse operator of the operator $A - \xi I$ or the resolvent of operator A . Let $S_\varphi = \{\xi, \ \xi \in C, |\arg \xi - \pi| \le \pi - \varphi\} \cup \{0\}, 0 < \varphi \le \pi.$

Definition 1. A linear operator A is said to be $\varphi-$ positive in a Banach space E, if D(A) is dense on E and

$$\left\|(A - \xi I)^{-1}\right\|_{L(E)} \le L(1 + |\xi|)^{-1}$$

with $\xi \in S_{\varphi},$ where L is a positive constant.

Definition 2.

$$E\left(A^{\theta}\right)=\left\{\, u,\, u\in D\left(A^{\theta}\right),\|u\|_{E(A^{\theta})}=\left\|A^{\theta}u\right\|_{E}+\|u\|<\infty,\quad -\infty<\theta<\infty\right\}.$$

Let be g = g (x) measurable positive function in $\Omega \subset R^{n}$

Definition 3. We denote by $L_{pg}(\Omega, E)$ the space of strongly measurable functions such that are defined on $\Omega \subset R^{n}$ and assume values in E, with the norm

$$\|u\|_{L_{pg}(\Omega,E)}=\left(\int_{\Omega}\|u\|_{E}^{p}\, g(x)\, dx\right)^{1/p},\ 1\le p<\infty.$$

For $g=g(x)\equiv 1$ we will denote $L_{pg}(\Omega, E)$ by $L_{p}(\Omega; E)$. Suppose that $S=S(R^{n})$ is Schwartzs space of test functions and $S'(E)=S(R^{n},E)$ is the space of linear continued mapping from S into E and is called E- valued Schwartzs distrubutions.For $\varphi \in S$ the Fourier transform $\hat{\varphi}$ and inverse Fourier transform $\check{\varphi}$ are defined by the relations

$$\hat{\varphi}(\xi)=(F\varphi)(\xi)=(2\pi)^{-n/2}\int_{R^{n}}\varphi(x)\, e^{-i\xi x}\, dx,$$

$$\check{\varphi}(x)=\left(F^{-1}\varphi\right)(x)=(2\pi)^{-n/2}\int_{R^{n}}\varphi(\xi)\, e^{i\xi x}\, d\xi,$$

where

$$\xi=(\xi_{1},\xi_{2},\dots,\xi_{n}),\ x=(x_{1},x_{2},\dots,x_{n}),\ \ \xi x=\xi_{1}x_{1}+\xi_{2}x_{2}+\dots+\xi_{n}x_{n}.$$

The Fourier transformation and the inverse Fourier transformation of Banach valued generalized functions $f\in S'(R^{n},E)$ are defined by the relations.

$$<\hat{f},\varphi>=<f,\hat{\varphi}> \text{ and } <\check{f},\varphi>=<f,\check{\varphi}>. \qquad (1)$$

where $<f,\varphi>$ means the value of generalized function $f\in S'(R^{n},E)$ on the $\varphi\in S(R^{n})$.

Definition 4. Let r = (r1, r2, ..., rn) , ri are positive integers. The E -values generalized functions $D^{r}f$ is called the genralized derivative in the sense of Schwarts distributions of the generalized function $f\in S'(R^{n},E)$, if the relation

$$<D^{r}f,\varphi>=(-1)^{|r|}<f,D^{r}\varphi>$$

holds for all $\varphi\in S.$ It is known for all $\varphi\in S$ the relations

$$F(D_{x}^{\alpha}\varphi) = (i\xi_{1})^{\alpha_{1}}\dots(i\xi_{n})^{\alpha_{n}}\hat{\varphi}, \qquad (2)$$

$$D_{\xi}^{\alpha}F[\varphi] = F[(-ix_{1})^{\alpha_{1}}\dots(-ix_{n})^{\alpha_{n}}\varphi]$$

holds. Let λ is infinitely differentiable function with polinomial structure. Then for $f\in S'(R^{n},E)$ $\lambda f\in S'(R^{n},E)$ is generalised function defined by the relation

$$<\lambda f,\varphi>=<f,\lambda f>\quad \forall\varphi\in S(R^{n}).$$

By using definition 4 , and relations (2) it proves that for all $f\in S'(R^{n},E)$ the relations

$$F(D_{x}^{r}f)=(i\xi_{1})^{r_{1}}\dots(i\xi_{n})^{r_{n}}\hat{f}\qquad D_{\xi}^{r}(F(f))=F[(-ix_{n})^{r_{1}}\dots(-ix_{n})^{r_{n}}f\ ]$$

(3) holds.

We can see that direct generalization of the definition of the Banach valued generalized derivative to nonintegral- valued vector (r1,r2, ...rn) is , in general impossible since,in this case,the function $(i\xi)^{r}=(i\xi_{1})^{r_{1}},\dots(i\xi_{n})^{r_{n}}$ can be even nondifferentiable .The way out of this situation consists in considering not the whoule S but his subspace S_{0} which is invariant with respect to the multiplication by $(i\xi)^{r}$ for any r = (r1, r2, ..rn) , $r_{i}\in[0,\infty)$ (see [4]). Let N be set of natural numbers and zero.

Let

$$S_{0}=S_{0}(R^{n})=\left\{\varphi,\varphi\in S(R^{n}),\int_{R}x_{j}^{s}\varphi(x_{1},\dots,x_{j},\dots,x_{n})\,dt_{j}=0,\ j=1,\dots,n,s\in N\right\}.$$

We denote by $\hat{S}_0 = \hat{S}_0(R^n)$ the image S_0 under a Fourier transformation and by $S_0'(R^n, E)$ and by $\hat{S_0}'(R^n, E)$ the E valued conjugate spaces in S_0 and $\hat{S}_0$ respectively. The Fourier transformation $F: \hat{S}_0'(R^n, E) \longrightarrow S_0'(R^n, E)$ is defined by the relation

$$\langle Ff, \varphi\rangle = \langle f, F\varphi\rangle \quad , \forall \varphi \in S_0(R^n).$$

We denote its inverse also by F^{-1}.For any $r = (r_1, r_2, ... r_n)$, $r_i \in [0, \infty)$ the function $(i\xi)^r$ will be defined such that

$$(i\xi)^r = \begin{cases} (i\xi_1)^{r_1} ... (i\xi_n)^{r_n} & , \xi_1, \xi_2, ... \xi_n \neq 0 \\ 0 \quad , & \xi_1, \xi_2, ..., \xi_n = 0 \end{cases}$$

where

$$(i\xi_k)^{r_k} = \exp[r_k \ln|\xi_k| + \pi i/2], \quad k = 1, 2, ..., n,$$

but

$$\xi^r = \begin{cases} \xi_1^{r_1} ... \xi_n^{r_n}, & \xi_1, ..., \xi_n \neq 0 \\ 0 & , \xi_1, ..., \xi_n = 0. \end{cases}$$

For $r = (r_1, r_2, ..., r_n)$, $r_i \in [0, \infty)$ we set $D^r\varphi = F^{-1}(i\xi)^r \hat{\varphi}$ for $\varphi \in S_0(R^n)$ r-th Liuville derivative of the E- valued generalized function $f \in S_0'$ is defined by the relation

$$\langle D^r f, \varphi\rangle = \langle f, D^r\varphi\rangle \quad , \forall \varphi \in S_0$$

The Banach space E is said to be $\xi-$convex (see [16]) (or convex in the sence of Burkholder) if there exists on E × E a symmetric function $\xi(u, v)$ which is convex with respect to every one of the variables and satisfies the condition

$$\xi(0,0) > 0, \xi(u, v) \leq \|u + v\| \text{ for } \|u\|_E = \|v\|_E = 1.$$

It is shown in Burkholders work [17] that Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{f(y)}{x - y} dy$$

is bounded in the space $L_p(R.E)$, $p \in (1, \infty)$, for those and only those Bahach spaces E which posses the property of $\xi-$convexity. In literature the $\xi-$convex Banach spaces is offen called UMD spaces. UMD spaces very broad , it contains Lp, lp spaces ,the Lorentz spaces L_{pq}, $p, q \in (1, \infty)$ for instance. Let $C^{(l)}(\Omega, E)$ the space of continuously differentiable functions l-th order with values in E .Let E , E_1 are Banach spaces .

Definition 5. A function $\Psi \in C^{(l)}(R^n, L(E, E_1))$ is called a multiplier from $L_{pg}(R^n, E)$ to $L_{qg}(R^n, E_1)$ if there exists a constant M > 0 such that

$$\left\|F^{-1}\Psi(\xi) Fu\right\|_{L_{qg}(R^n, E_1)} \leq C\|u\|_{L_{pg}(R^n, E)} \tag{4}$$

for all $u \in L_p(R^n, E)$. We denote the set of all multipliers from $L_p(R^n, E)$ to $L_q(R^n, E_1)$ by $M_{p,g}^{q,g}(E, E_1)$. For E = E1 we denote $M_{p,g}^{q,g}(E, E_1)$ by $M_p^q(E)$ and for $g(x) \equiv 1$ we denote $M_{p,g}^{q,g}(E, E_1)$ by $M_p^q(E, E_1)$.

Example 1.We note that if $\delta \in C^\infty(R)$ with $\delta(y) \geq 0$ for all $y \geq 0$,

$\delta(y) = 0 \qquad |y| \leq \frac{1}{2}$ and $\delta(-y) = -\delta(y)$ for all y , then $\delta \in M_p^p(R)$.

Let $H_k = \{\Psi_h \in M_p^q(E, E_1), h = (h_1 h_2 ..., h_n) \in K\}$ be a collection of multipliers in $M_p^q(E, E_1)$. We say that Hk is a uniform collection of multipliers if there is a constant M0 > 0, independent of $h \in K$, such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_q(R^n, E_1)} \leq M_0 \|u\|_{L_p(R^n, E)} \qquad (5)$$

for all $h \in K$ and $u \in L_p(R^n, E)$. Let,

$r = (r_1, r_2, ... r_n)$, $r + a = (r_1 + a, r_2 + a, ..., r_n + a)$, $l = (l_1, l_2, ..., l_n)$

$|(r+a) : l| = \sum_{k=1}^{n} \frac{r_k + a}{l_k}$, $\xi^r = \xi_1^{r_1} \xi_2^{r_2} ... \xi_n^{r_n}$, $|\xi|^r = |\xi|^{r_1} |\xi|^{r_2} ... |\xi|^{r_n}$.

We also define

$$U_n = \{\beta = (\beta_1, \beta_2, ..., \beta_n), \beta_i \in (0, 1), \forall i = 1, 2, ..., n\},$$

$$V_n = \{\xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \xi_i \neq 0, \forall i = 1, 2, ..., n\}.$$

We say that the Banach space E satisfies the multiplier condition with respect to p and (or with respect to p in the case of p = q) and with respect to weighted function g (x) if for all $\Psi \in C^{(n)}(R^n, L(E))$ and for all $\beta \in U_n$, and $\xi \in V_n$, $\exists\ C \in R_+$ inequality

$$\left\|D^\beta \Psi(\xi)\right\|_{L(E)} \leq C |\xi|^{-\left(\beta + \frac{1}{p} - \frac{1}{q}\right)} \qquad (6)$$

implies

$$\Psi \in M_{p,g}^{q,g}(E).$$

For $g(x) \equiv 1$ in similar way we define the Banach spaces satisfying multiplier condition with respect to p and q. It is well known (see [4]) that any Hilbert space satisfies the multiplier condition with respect to any p and q with $1 < p \leq q < \infty$. There are however Banach spaces which are not Hilbert spaces but satisify the multiplier condition , for example $\xi-$convex Banach lattice spaces (see [16 - 18]). Let E and E_1 be Banach spaces and $E_1 \subset E$, E_1 continuously and densely embeded in E and l = (l_1, l_2, ..., l_n) , l_i, i = 1, 2, ..., n positive real numbers.

Definition6.

$W_p^l(R^n, E) = \{ u,\ u \in S_0(R^n, E), F^{-1}(i\xi_k)^{l_k} \hat{u} \in L_p(R^n, E), k = 1, 2, ..., n \},$

$\|u\|_{W_p^l(R^n, E)} = \|u\|_{L_p(R^n, E)} + \sum_{k=1}^{n} \left\|F^{-1}(i\xi_k)^{l_k} \hat{u}\right\|_{L_p(R^n, E)} < \infty, 1 \leq p < \infty$

Let

$W_p^l(R^n, E_1, E) = \{ u, u \in W_p^l(R^n, E) \cap L_p(R^n, E_1), \|u\|_{W_p^l(R^n, E_1, E)}$

$= \|u\|_{L_p(R^n, E_1)} + \sum_{k=1}^{n} \left\|F^{-1}\left[(i\xi_k)^{l_k} \hat{u}\right]\right\|_{L_p(R^n, E)} \}.$

Let be t = (t_1, t_2, ...,t_n) , where t_k, k = 1, 2, ..., n are nonnegative parameters. Let us define in the space $W_p^l(R^n, E_1, E)$ with parametrics norm

$\|u\|_{W_{p,t}^l(R^n, E(A), E)} = \|u\|_{L_p(R^n, E_1)} + \sum_{k=1}^{n} \left\|t_k F^{-1}\left[(i\xi_k)^{l_k} \hat{u}\right]\right\|_{L_p(R^n, E)}.$

For

$\Omega \subset R^n, W_p^l(\Omega, E_1, E) =$

$\{ u, u \in L_p(\Omega, E_1), u = \phi$ a.e.in $\Omega, \forall \phi \in W_p^l(R^n, E_1, E),$

$\|u\|_{W_p^l(\Omega, E_1, E)} = \inf_\phi \|\phi\|_{W_p^l(R^n, E_1, E)} \}.$

3.Embedding theorems

In this section we prove that the generalised derivative operator D^r gives a continuous embedding of some Sobolev-Liuville spaces of vector-functions.

Lemma 1.Let A be a positive linear operator on a Banach space E , b be a nonnegative real number and $r = (r_1, r_2, ..., r_n)$ where $r_k \in \{0, b\}$ .Let be $t = (t_1, t_2, ..., t_n)$, where t_k , k = 1, 2, ..., n are nonnegative parameters,

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$

$l = (l_1, l_2, ..., l_n)$, $l_k \geq 0$

such that

$\varkappa = |(\alpha + r) : l| \leq 1.$

Let δ be a multiplier of the form described in example1. For any h>0 and $,0 \leq \mu \leq 1 - \varkappa$ the operator-function

$$\Psi_t(\xi) = \Psi_{t,h,\mu}(\xi) = \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} \xi^r (i\xi)^\alpha A^{1-\varkappa-\mu} h^{-\mu} \left[A + \sum_{k=1}^{n} t_k (\delta(\xi_k))^{l_k} + h^{-1}\right]^{-1}$$

is bounded operator in E uniformly with respect to $\xi \in R^n, h > 0$ and parameters i.e there exists a constant C_μ such that

$$\left\|\Psi_{t,h,\mu}(\xi)\right\|_{L(E)} \leq C_\mu \qquad (7)$$

for all $\xi \in R^n$ and h > 0.

Proof; Since

$$-\left[\sum_{k=1}^{n} t_k (\delta(\xi)\xi_k)^{l_k} + h^{-1}\right] \in S(\varphi)$$

for all $\varphi \in [0, \pi)$

then by the definition1 of positive operator

$$A \ , \ A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-1}$$

is invertiable in the space E. Let

$$u = h^{-\mu}\left[A + \sum_{k=1}^{n} t_k (\rho(\xi_k)\xi_k)^{l_k} + h^{-1}\right]^{-1} f.$$

Then

$$\left\|\Psi_t(\xi) f\right\|_E = \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} |\xi|^{r+\alpha} \left\|A^{1-\varkappa-\mu} u\right\|_E =$$

$$\left\|(hA)^{1-\varkappa-\mu} u\right\|_E h^{-(1-\mu)} \left|(ht_1)^{\frac{1}{l_1}} \xi_1\right|^{\alpha_1+r_1} \dots \left|(ht_n)^{\frac{1}{l_n}} \xi_n\right|^{\alpha_n+r_n}.$$

Using the Moment inequality for powers of a positive operators , we get a constant C_μ depending only on μ such that

$$\left\|\Psi_t(\xi)\right\|_E \leq C_\mu h^{(1-\mu)} \|hAu\|^{1-\varkappa-\mu} \|u\|^{\varkappa+\mu} \left|(ht_1)^{\frac{1}{l_1}} \xi_1\right|^{\alpha_1+r_1} \dots \left|(ht_n)^{\frac{1}{l_n}} \xi_n\right|^{\alpha_n+r_n}.$$

Now , we apply theYoung inequality ,which states that $ab \leq \frac{a^{r_1}}{r_1} + \frac{b^{r_2}}{r_2}$ for any positive real numbers a, b and r1, r2 with $\frac{1}{r_1} + \frac{1}{r_2} = 1,$ to the product

$$\|hAu\|^{1-\varkappa-\mu}\left[\|u\|^{\varkappa+\mu} \left|(ht_1)^{\frac{1}{l_1}} \xi_1\right|^{\alpha_1+r_1} \dots \left|(ht_n)^{\frac{1}{l_n}} \xi_n\right|^{\alpha_n+r_n}\right]$$

with $r_1 = \frac{1}{1-\varkappa-\mu}, r_2 = \frac{1}{\varkappa+\mu}$ to get

$$\begin{aligned}\left\|\Psi_t(\xi) f\right\|_E \leq\ & C_\mu h^{-(1-\mu)} (1-\varkappa-\mu) \|hAu\| \\ & + (\varkappa+\mu)(ht_1|\xi_1|)^{\frac{\alpha_1+r_1}{\varkappa+\mu}} \dots (ht_n|\xi_n|)^{\frac{\alpha_n+r_n}{\varkappa+\mu}} \qquad (8)\end{aligned}$$

Since

$$\sum_{i=1}^{n} \frac{\alpha_i + r_i}{(\varkappa+\mu)} = \frac{1}{\varkappa+\mu} \sum_{i=1}^{n} \frac{\alpha_i + r_i}{l_i} = \frac{\varkappa}{\varkappa+\mu} \leq 1$$

there is a constant M_0 independent of $\xi,$ such that

$$|\xi_1|^{\frac{\alpha_1+r_1}{\varkappa+\mu}} \dots |\xi_n|^{\frac{\alpha_n+r_n}{\varkappa+\mu}} \leq M_0 \left(1 + \sum_{k=1}^{n} |\xi_k|^{l_k}\right)$$

for all $\zeta \in R^n.$ It is clear that $|y|^l \leq (\delta(y)\,y)^l$ for all $|y| > \frac{1}{2}.$

Thus

$$|\xi_1|^{\frac{\alpha_1+r_1}{\varkappa+\mu}} \dots |\xi_n|^{\frac{\alpha_n+r_n}{\varkappa+\mu}} \leq M_1 \left[1 + \sum_{k=1}^{n} (\delta(\xi_k)\xi_k)^{l_k}\right]$$

for a suitable $M_1 > 0$ and all $\zeta \in R^n.$ Supstituting this on the inequality (8) and absorbing the constant coefficients in C_μ , we obtain

$$\left\|\psi_t(\xi) f\right\| \leq C_\mu h^\mu \left[\|Au\| + \left(\sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right)\|u\|\right].$$

Substituting the value of u, we get

$$\|\psi_1(\xi)f\| \leq C_\mu \left(\left\| A\left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} f \right\| + \right. \quad (9)$$
$$\left[\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right] \times$$
$$\times \left\| \left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} f \right\|.$$

Since A is positive operator in the space E

$$A\left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right] = \left[I-\left(\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right)\right] \times \left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1}$$

there is a constant M > 0 such that

$$\left\| \left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} f \right\| \leq M \left[1+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} \|f\|$$

for all $f \in E.$ Combing those with the inequality (9) we obtain

$$\|\Psi_1(\xi)f\|_E \leq C_\mu \left(\left| 1-\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1} \right| \left[1+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} \|f\|_E + \right. \quad (10)$$
$$\left. +\left(\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right)\left[1+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}+h^{-1}\right]^{-1} \right) \|f\| \leq C_\mu \|f\|$$

for all $f \in E$, h > 0. The inequality (10) implise the estimate (7) .

Theorem 1. Let E be a Banach space satisfying the multiplier condition with respect to p and q, where $1 < p \leq q < \infty.$ Let $t = (t_1, t_2, ..., t_n)$, where t_k, k = 1, 2, ..., n are nonnegative parameters and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $l = (l_1, l_2, ..., l_n)$, where lk nonnegative real numbers such that

$$\varkappa = \left| \left(\alpha + \frac{1}{p} - \frac{1}{q}\right) : .l \right| \leq 1,$$

and let $0 \leq \mu \leq 1 - \varkappa$. Assume further that A is a positive operator on E.Then

$$D^\alpha W^l_{p,t}(R^n, E(A), E) \subset L_q\left(R^n, E\left(A^{1-\varkappa-\mu}\right)\right)$$

is a continuous embedding, and there is a constant $C\mu > 0,$ depending only on μ, such that

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} \|D^\alpha u\|_{L_p(R^n, E(A^{1-\varkappa-\mu}))} \quad (11)$$
$$\leq C_\mu \left[h^\mu \|u\|_{W^l_{p,t}(R^n, E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(R^n, E)}\right]$$

for all $u \in W^l_p(R^n, E(A), E)$ and h > 0.

Proof. We have

$$\|D^\alpha u\|_{L_q(R^n, E(A^{1-\varkappa-\mu}))} = \left(\int_{R^n} \|D^\alpha u\|^q_{E(A^{1-\varkappa-\mu})} dx\right)^{\frac{1}{q}} \backsim \quad (12)$$
$$\left(\int_{R^n} \|A^{1-\varkappa-\mu} D^\alpha u\|^q_E dx\right)^{\frac{1}{q}} \backsim \|A^{1-\varkappa-\mu} D^\alpha u\|_{L_q(R^n, E)}$$

for all u such that

$$\|D^\alpha u\|_{L_q(R^n, E(A^{1-\varkappa-\mu}))} < \infty.$$

On the other hand in view of generalised Liouville derivative

$$A^{1-\alpha-\mu} D^\alpha u = F^{-1} F A^{1-\varkappa-\mu} D^\alpha u = F^{-1} A^{1-\varkappa-\mu} F D^\alpha u =$$
$$F^{-1} A^{1-\varkappa-\mu} (i\xi)^\alpha F u = F^{-1} (i\xi)^\alpha A^{1-\varkappa-\mu} F u. \quad (13)$$

Hence denoting Fu by $\hat{u}$ we get from relations (12) , (13)

$$\|D^\alpha u\|_{L_q(R^n, E(A^{1-\varkappa-\mu}))} \backsim \left\|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu} \hat{u}\right\|_{L_q(R^n, E)}.$$

Similarly, in view of definition 6

$$\|u\|_{W^l_{p,t}(R^n,E(A),E)} = \|u\|_{L_p(R^n,E(A))} + \sum_{k=1}^{n}\left\|t_k D_k^{l_k} u\right\|_{L_p(R^n,E)} = \left\|F^{-1}\hat{u}\right\|_{L_p(R^n,E(A))} +$$

$$\sum_{k=1}^{n}\left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n,E)}$$

$$\backsim \left\|F^{-1}A\hat{u}\right\|_{L_p(R^n,E)} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n,E)}$$

for all $u \in W_p^l(R^n, E(A), E)$.

Thus pruving the inequality (11) for some constants Cμ is equalent to proving

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}}\left\|F^{-1}(i\xi)^{\alpha}A^{1-\varkappa-\mu}\hat{u}\right\|_{L_q(R^n,E)}$$

$$\leq C_\mu\Big(h^{\mu}\left\|F^{-1}A\hat{u}\right\|_{L_p(R^n,E)} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n,E)} + h^{-(1-\mu)}\left\|F^{-1}\hat{u}\right\|_{L_p(R^n,E)}\Big)$$

for a suitable Cμ .Now if _ is a multiplier of the form described as in example1, by virtue of multiplier there is a constant $C_k > 0$ for each k = 1, 2..., n such that

$$\left\|F^{-1}\frac{1}{i}\delta(\xi_k)(i\xi_k)^{l_k}\hat{u}\right\|_{L_p(R^n,E)} \leq C_k\left\|F^{-1}(i\xi_k)^{l_k}\hat{u}\right\|_{L_p(R^n,E)}$$

for all $\xi \in R^n$. Thus the inequality (11) will follow if we prove the following inequality

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}}\left\|F^{-1}\left[(i\xi)^{\alpha}A^{1-\varkappa-\mu}\hat{u}\right]\right\|_{L_q(R^n,E)}$$

$$\leq C_\mu\left\|F^{-1}\left[h^{\mu}\Big(A+\sum_{k=1}^{n}t_k\big(\delta(\xi_k)\xi_k\big)^{l_k}\Big)+h^{-(1-\mu)}\right]\hat{u}\right\|_{L_p(R^n,E)} \quad (14)$$

for a suitable Cμ > 0, and for all $u \in W_p^l(R^n, E(A), E)$.

Let us express the left hand side of (14) as follows

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}}\left\|F^{-1}\left[(i\xi)^{\alpha}A^{1-\varkappa-\mu}\hat{u}\right]\right\|_{L_q(R^n,E)} = \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} F^{-1}(i\xi)^{\alpha}A^{1-\varkappa-\mu}$$

$$\left[h^{\mu}\Big(A+\sum_{k=1}^{n}t_k\big(\delta(\xi_k)\xi_k\big)^{l_k}\Big)+h^{-(1-\mu)}\right]^{-1} \quad (15)$$

$$\times\left[h^{\mu}\Big(A+\sum_{k=1}^{n}t_k\big(\delta(\xi_k)\xi_k\big)^{l_k}\Big)+h^{-(1-\mu)}\right]_{L_q(R^n,E)}$$

(Since A is the positive operator in E so it is possible).By virtue of definition of multiplier it is clear that the inequality (14) will follow immediately from (15) if we can prove that the operator-function

$$\Psi_{t,h,\mu} = (i\xi)^{\alpha}A^{1-\varkappa-\mu}\left[h^{\mu}\Big(A+\sum_{k=1}^{n}t_k\big(\delta(\xi_k)\xi_k\big)^{l_k}\Big)+h^{-(1-\mu)}\right]^{-1}$$

is a multiplier in $M_p^q(E)$, which is uniform with respect to h > 0 and paremeters. Since E satis fies the multiplier condition with respect to p and q, in order to show that $\Psi_{t,h,\mu} \in M_p^q(E)$, it suffcies to show that there is a constant Mμ > 0 with

$$\left\|D_\xi^{\beta}\Psi_{t,h,\mu}(\xi)\right\|_{L(E)} \leq M_\mu|\xi|^{-\left(\beta+\frac{1}{p}-\frac{1}{q}\right)} \quad (16)$$

for all $\beta \in U_n$ and $\xi \in V_n$. To see this, we apply lemma1 and get a constant Mμ > 0 depending only on μ such that

$$\left\|\Psi_{t,h,\mu}(\xi)\right\|_{L(E)} \leq M_\mu|\xi|^{-\eta} \quad (17)$$

for all $\xi \in R^n$ and $\eta = \frac{1}{p} - \frac{1}{q}$.

This shows that the inequality (16) is satisfied for $\beta = (0, ..., 0)$. From now we drop the subscripts h, μ and we write Ψ_t instead of $\Psi_{t,h,\mu}$. We next consider (16) for $\beta = (\beta_1, ...\beta_n)$ where $\beta_k = 1$ and $\beta = 0$ for $j \neq k$ Then

$$D_\xi^\beta \Psi_t(\xi) = D_k^1 \Psi_t(\xi) = \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} (i)^\beta \beta_k \xi_1^{\beta_1} \ldots \xi_{k-1}^{\beta_{k-1}} \xi_k^{\beta_k - 1} \ldots \xi_n^{\beta_n} A^{1-\varkappa-\mu}$$
$$\left[h^\mu \left(A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k}\right) + h^{-(1-\mu)}\right]$$
$$+ (i\xi)^\beta A^{1-\varkappa-\mu} \left[h^\mu \left(A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k}\right) + h^{-(1-\mu)}\right]^{-2}$$
$$h\left[l_k t_k \delta^{l_k - 1}(\xi_k) D_k^1 \delta(\xi_k) \xi_k^{l_k} + l_k t_k \delta^{l_k}(\xi_k) \xi_k^{l_k - 1}\right]$$
$$= \frac{1}{\xi_k} \{ \beta_k (i\xi)^\beta \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} A^{1-\varkappa-\mu} \left[h^\mu \left(A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-(1-\mu)}\right)\right]^{-1}$$
$$+ l_k h^\mu t_k \left[\delta^{l_k - 1}(\xi_k) \xi_k^{l_k + 1} D_k^1 \delta(\xi_k) + \delta^{l_k}(\xi_k) \xi_k^{l_k}\right] \times$$

$$\left[h^\mu \left(A + \sum_{k=1}^{n} (\delta(\xi_k)\xi_k)^{l_k} + h^{-(1-\mu)}\right)\right]^{-1}$$
$$(i\xi)^\alpha A^{1-\varkappa-\mu} \left[h^\mu \left(A + \sum_{k=1}^{n} (\delta(\xi_k)\xi_k)^{l_k} + h^{-(1-\mu)}\right)\right]^{-1} \}$$

We get from here by using (17) that

$$\left\|D_k^1 \Psi_t(\xi)\right\|_{L(E)} \leq \frac{M_\mu |\xi|^{-\eta}}{|\xi_k|}$$
$$\left\{1 + h^\mu l_k t_k \left[\delta^{l_k - 1}(\xi_k) D_k^1 \delta(\xi_k) \xi_k^{l_k + 1} + (\delta(\xi_k)\xi_k) l_k\right]\right\}$$
$$\left\| h^\mu \left[A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-(1-\mu)}\right]^{-1} \right\|_{L(E)}.$$

Now , using the fact that A is a positive operator, we can write

$$\left\|D_k^1 \Psi_t(\xi)\right\|_{L(E)} \leq M_\mu |\xi|^{-\eta} |\xi_k|^{-1} 1 + t_k \left[\delta^{l_k - 1}(\xi_k) D_k^1 \delta(\xi_k) + (\delta(\xi_k)\xi_k)^{l_k}\right] \times$$
$$\times \left[1 + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right]^{-1}$$

for a suitable Mμ depending only on μ .Since $D_k \delta(\xi_k) = 0$ for $|\xi_k| > 1,$ we conclude that the expressin inside the bracked is bounded above by a scalar depending only on μ. Thus, we have a constant Mμ depending only on μ such that

$$\left\|D_k^1 \Psi_t(\xi)\right\|_{L(E)} \leq M_\mu |\xi|^{-\eta} |\xi|^{-1}, k = 1, 2 \ldots n$$

Repeating the above process, we see that there is a constant Mμ > 0 depending only μ such that

$$\left\|D^\beta \Psi_t(\xi)\right\|_{L(E)} \leq M_\mu |\xi|^{-\left(\beta + \frac{1}{p} - \frac{1}{q}\right)}$$

for all

$\beta \in U_n,\ \xi \in V_n.$ Thus the operator-function $\Psi_{t,h,\mu}(\xi)$ is a uniform multiplier with respect to h > 0 and t i.e

$$\Psi_{t,h,\mu} \in H_K \subset M_p^q(E), K = R_+.$$

This completes the proof of the theorem1. It is possible to state theorem1 in a more general setting . For this , weuse the consept of extension operator.

Condition 1.Let A be positive operator in Banach spaces E satisfaying multiplier condition with respect to p and q .Let be region $\Omega \subset R^n$ such that there is linear bounded extension operator B acting from $L_q(\Omega, E)$ to $L_q(R^n, E)$ also from $W_p^l(\Omega, E(A), E)$ to $W_p^l(R^n, E(A), E),$ for $1 < p \leq q < \infty.$

Remark 1. If $\Omega \subset R^n$ is a region satisfying the strong l-horn condition (see [2] ,p.117) and $l = (l_1, \ldots l_n)$, l_i, i = 1, 2, ...n are nonnegative integers numbers, E = R,A = I , then there is linear bounded extension operator from

$$W_p^l(\Omega) = W_p^l(\Omega, R, R) \quad \text{to} \quad W_p^l(R^n) = W_p^l(R^n, R, R).$$

Theorem 2. Let condition1 is holds. Then for

$$\varkappa = \sum_{k=1}^{n} \frac{\alpha_k + \left(\frac{1}{p} - \frac{1}{q}\right)}{l_k} \leq 1$$

and

$$0 \leq \mu \leq 1 - \varkappa, 1 < p \leq q < \infty$$
$$D^\alpha : W_p^l(\Omega, E(A), E) \subset L_q\left(\Omega, E\left(A^{1-\varkappa-\mu}\right)\right)$$

gives a continuous embedding , and there is a constant Cμ , depending only on μ , such that

$$\|D^\alpha u\|_{L_q(\Omega, E(A^{1-\varkappa-\mu}))} \leq C_\mu \left[h^\mu \|u\|_{W_p^l(\Omega, E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega, E)}\right] \quad (18)$$

for all $u \in W_p^l(\Omega, E(A), E)$ and $h > 0$.

Proof. It is suffces to prove the estimate (18) .Let B is linear bounded extension operator from $L_p(\Omega, E)$ to $L_p(R^n, E)$ and from $W_p^l(\Omega, E(A), E)$ to $W_p^l(R^n, E(A), E)$, and let B_Ω be the restriction operator from R^n to Ω. Then for any $u \in W_p^l(\Omega; E(A), E)\big|$ we have

$$\|D^\alpha u\|_{L_q(\Omega, E(A^{1-\varkappa-\mu}))} = \|D^\alpha B_\Omega B u\|_{L_q(\Omega, E(A^{1-\varkappa-\mu}))} \leq C\|D^\alpha B u\|_{L_q(R^n, E(A^{1-\varkappa-\mu}))} \leq$$

$$\leq C_\mu\left[h^\mu \|Bu\|_{W_p^l(R^n, E(A), E)} + h^{-(1-\mu)}\|Bu\|_{L_p(R^n, E)}\right]$$
$$\leq C\mu\left[h^\mu \|u\|_{W_p^l(\Omega, E(A) E)} + h^{-(1-\mu)}\|u\|_{L_p(\Omega, E)}\right].$$

Result 1. Let all conditions of theorem2 holds. Then for all $u \in W_p^l(\Omega, E(A), E)$ we have multiplicative estimate

$$\|D^\alpha u\|_{L_q(\Omega, E(A^{1-\varkappa-\mu}))} \leq C_\mu \|u\|^{1-\mu}_{W_p^l(\Omega, E(A), E)} \|u\|^\mu_{L_p(\Omega, E)}.$$

Indeed setting

$$h = \|u\|_{L_p(\Omega, E)} \cdot \|u\|^{-1}_{W_p^l(\Omega, E(A)}$$

in estimate (18) we obtain (19) .

Theorem 3. Assume that all conditions of theorem2 are satisfied and let be $l_1, l_2, ..., l_n$ positive integer numbers, be bounded in R^n and A^{-1} be compact operator in the space E.Then for $0 \leq \mu \leq 1 - \varkappa$ the embedding

$$D^\alpha W_p^l(\Omega; E(A), E) \subset L_q\left(\Omega; E, A^{1-x-\mu}\right)$$

is compact.

Proof.By the virtue of [9] the embedding $W_p^l(\Omega; E(A), E) \subset L_q(\Omega; E)$ is compact.Then by the estimate (19) we obtain the assertion of theorem3.

Result 2 If $l_1 = l_2 = \ldots = l_n = l$ then we obtain continiuty of embedding operators in isotropic class $W_p^l(\Omega, E(A) E)$. Remark 2. If E = H and $p = q = 2$, $\Omega = (0, T)$, $l_1 = l_2 = \ldots = l_n = l$, $A = A^\times \geq cI$ then we obtain the result of Lions-Peetre [7] and even in the one dimensional case the rezult of Lions-Peetre are improving for in generall nonselfedjoint positive operators A. If E = R, A = I then we obtain embedding theorems $D^\alpha W_p^l(\Omega) \subset L_q(\Omega)$ proved in [2] for numerical Sobolev spaces $W_p^l(\Omega)$. Let be $\gamma(x)$ measurable function in $\Omega \subset R^n$, $l = (l_1 l_2, ..., l_n)$ n-tuples integer numbers and

$$W^l_{p,\gamma}(\Omega; E(A), E) = \Big\{u;\ u \in L_{p,\gamma}(\Omega; E(A)),\ D_k^{l_k} u \in L_{p,\gamma}(\Omega; E),$$
$$\|u\|_{W^l_{p,\gamma}(\Omega; E(A), E)} = \|u\|_{L_{p,\gamma}(\Omega; E(A))} + \sum_{k=1}^n \left\|D_k^{l_k} u\right\|_{L_{p,\gamma}(\Omega; E)} < \infty\Big\}$$

Using similar techniques as in theorem2 we prove;

Theorem 4 Let the following conditions be satisfied;

1) $\gamma(x)$ is positive measurable function in $\Omega \subset R^n$,such that $\int_\Omega \gamma^{-1}(x)\,dx < \infty;$

2) E is aBanach space satisfying the multiplier condition with respect to p and q,where $1 < p \leq q < \infty,$ and with respect to weighted function $\gamma(x)$.

3) $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $l = (l_1, l_2, ..., l_n)$ are n-tuples of nonnegative integer numbers such that

$$\varkappa = \left|\left(\alpha + \frac{1}{p} - \frac{1}{q}\right) : l\right| = \sum_{k=1}^n \frac{\alpha_k + \frac{1}{p} - \frac{1}{q}}{l_k} \leq 1,\ 0 \leq \mu \leq 1 - \varkappa;$$

4) A is $\varphi-$ positive operator in E for $\varphi \in (0, \pi]$;

5) $\Omega.$ is region such that there exists linear bounded extension operator acting from $L_{p,\gamma}(\Omega; E)$ to $L_{p,\gamma}(R^n; E)$ also from $W^l_{p,\gamma}(\Omega; E(A), E)$ to $W^l_{p,\gamma}(R^n; E(A), E)$.

Then the embedding $D^\alpha W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{q,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))$ is contiuous and there exists a positive constant Cµ such that

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega;E)} \le C_\mu \left[h^\mu \|u\|_{W^l_{p,\gamma}(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(\Omega;E)}\right]$$

for all $u \in W^l_{p,\gamma}(\Omega; E(A), E)$ and all h > 0.

Theorem 5. Suppose all conditions of theorem 4 are satisfied and suppose Ω is bounded region in R^n, A^{-1} is compact operator in E.then for $0 < \mu \le 1 - \varkappa$ the embedding

$$D^\alpha W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{q,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))$$

is compact.

Proof. Indeed putting in preceding inequality $h = \frac{\|u\|_{L_{p,\gamma}(\Omega;E)}}{\|u\|_{W^l_{p,\gamma}(\Omega;E(A),E)}}$ we obtain multiplicative inequality

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega;E)} \le C_\mu \|u\|^{1-\mu}_{W^l_{p,\gamma}(\Omega;E(A),E)} \|u\|^\mu_{L_{p,\gamma}(\Omega;E)}.$$

By virtue [9] embedding $W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{q,\gamma}(\Omega; E)$ is compact Then from this embedding theorem and preceding multiplicative inequality we obtain assertion of theorem 5. A region $\Omega \subset R^n$ satifies l-horn conditions (see[2] ,p.117) i.e.there exists domains Ω_k and Gk ,k = 1, 2, ..,N for some N such that

$$\Omega = \bigcup_{k=1}^N \Omega_k = \bigcup_{k=1}^N \Omega_k + G_k.$$

Let $\bar{\Omega}$ be denote aclosure of region $\Omega.$. From [19] we obtain

Theorem 6. Let the following conditions be hold:

1) E is Banach space;

2) $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), l = (l_1, l_2, ..., l_n), \varkappa = \sum_{k=1}^n \frac{\alpha_k + \frac{1}{p}}{l_k} < 1, 1 \le p \le \infty;$

3) $\Omega \subset R^n$ is region satisfying l-horn conditions;

4) Let be $\gamma(x)$ positive mesurable function in Ω and

$$\int_\Omega \gamma^{-\frac{1}{p-1}}(x)\, dx < \infty;$$

5) there exists constant C > 0 such that $\gamma(x) \le C\gamma(x+y)$, a.e. for all $x \in \Omega_k$ and $y \in G_k$, $k = 1, 2, .., N.$

Then the embedding

$$D^\alpha W^l_{p,\gamma}(\Omega; E) \subset C(\bar{\Omega}; E)$$

is contiuous and there exists a positive constant M such that

$$\|D^\alpha u\|_{C(\bar{\Omega};E)} \le M\left[h^{1-\mu} \|u\|_{W^l_{p,\gamma}(\Omega;E)} + h^{-\mu} \|u\|_{L_{p,\gamma}(\Omega;E)}\right].$$

Theorem 7.Let be E Banach space and A be a linear operator in E of type $\varphi, \varphi \in (0, \pi]$. Moreover let m be a positive integer, $1 \le p < \infty$ and $\frac{1}{2p} < \alpha < m + \frac{1}{2p}$. Let $0 \le \gamma < 2pm.$

Then for $\lambda \in S(\varphi)$ the operator $-A_\lambda^{\frac{1}{2}}$ generates a semigroup $e^{A_\lambda^{\frac{1}{2}} x}$ which is golomorphic for x > 0 and strongly continuous for x > 0.

Moreover there exist constant C > 0 such that for every

$$u \in \left(E, E\left(A^m\right)\right)_{\frac{\alpha}{l}-\frac{1+\gamma}{2pm},p}$$

and

$$\lambda \in S_1\left(\varphi\right) = \left\{\lambda : \left|\arg\lambda\right| \le \varphi\right\}$$

$$\int_0^\infty \left\|A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}}u\right\|_E x^\gamma dx \le C\left(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1+\gamma}{2mp},p}} + |\lambda|^{p\alpha-\frac{1+\gamma}{2}}\|u\|_E^p\right)$$

For the proof of this theorem we need some lemmas

Lemma1.Let B be positive operator of type φ with $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ in the space E.Moreover let l be a positive integer and $\beta \in \left(\frac{1}{p}, l+\frac{1}{p}\right)$ and let $0 \le \gamma < 2pl.$

Then for every $u \in E$ such that

$$\int_0^\infty \left\|B^\alpha e^{-xB}u\right\|_E^p x^{\gamma-1}dx \le \infty$$

we have

$$\int_0^\infty \left\|B^\beta e^{-xB}u\right\|_E^p x^\gamma dx \le \int_0^\infty \left\|x^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+xI\right)^{-1}\right)^l u\right\|_E^p x^{\gamma-1}dx$$

Proof.By virtue of [7] and[8] we have

$$\left\|B^l e^{-xB}\right\| \le \frac{M(l-1)!}{\pi|\cos\varphi|^l}x^{-l}.$$

$$\left\|e^{-xB}\right\| \le \frac{M}{\pi}\left(\frac{e^{\cos\varphi}}{|\cos\varphi|} + \varphi e\right).$$

Therefore for $u \in E$ such that

$$\int_0^\infty \left\|B^\beta e^{-xB}u\right\|_E^p x^\gamma dx \le \infty$$

we have

$$\left(\int_0^\infty \left\|B^\beta e^{-xB}u\right\|_E^p x^\gamma dx\right)^{\frac{1}{p}} =$$

$$\left(\int_0^\infty \left\|\frac{\Gamma(2l)}{\Gamma(\beta)\Gamma(2l-\beta)}\int_0^\infty t^{\beta-1}\left(B\left(B+tI\right)^{-1}\right)^{2l} e^{-tB}u dt\right\|_E^p x^\gamma dx\right)^{\frac{1}{p}} \le$$

$$C(l,\beta)\left(\int_0^\infty \left\|\int_0^{x^{-1}} t^{\beta-1}\left(B\left(B+tI\right)^{-1}\right)^l e^{-xB}\left(B\left(B+tI\right)^{-1}\right)^l u dt\right\|_E^p x^\gamma dx\right)^{\frac{1}{p}} \le$$

$$C(l,\beta)\left(\int_0^\infty \left\|\int_{x^{-1}}^\infty t^{\beta-1}\left(B\left(B+tI\right)^{-1}\right)^l e^{-xB}\left(B\left(B+tI\right)^{-1}\right)^l u dt\right\|_E^p x^\gamma dx\right)^{\frac{1}{p}} \le$$

$$C(l,\beta)\left(\int_0^\infty \left(\int_0^{x^{-1}} t^{\beta-1}(M+1)^l\left\|\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E dt\right)^p x^\gamma dx\right)^{\frac{1}{p}} +$$

$$C(l,\beta)\left(\int_0^\infty x^{\frac{\gamma}{p}-1}\left(\int_{x^{-1}}^\infty t^{\beta-1}\frac{M^l}{t^l}\left\|\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E dt\right)^p dx\right)^{\frac{1}{p}} \le$$

$$C(M,\varphi,l,\beta)\left\{\left(\int_0^\infty \left(\int_0^{x^{-1}} (tx)^{\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}dt\right)^p \frac{dx}{x}\right)^{\frac{1}{p}} +\right.$$

$$\left.\left(\int_0^\infty \left(\int_{x^{-1}}^\infty (tx)^{\frac{1+\gamma}{p}-l}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}\right)^p \frac{dx}{x}\right)^{\frac{1}{p}}\right\} =$$

$$C(M,\varphi,l,\beta)\left\{\left(\int_0^\infty \left(\int_0^y \left(\frac{y}{t}\right)^{-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}dt\right)^p \frac{dy}{y}\right)^{\frac{1}{p}} +\right.$$

$$\left.\left(\int_0^\infty \left(\int_y^\infty \left(\frac{y}{t}\right)^{l-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}\right)^p \frac{dy}{y}\right)^{\frac{1}{p}}\right\}$$

The function

$$\int_0^y \left(\frac{y}{t}\right)^{-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}$$

is multiplicative convolution with respet to the Haar measure of the group $\frac{dt}{t}$ of the functions

$$t \to \left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)\right)^l u\right\|$$

and $t \to t^{-\frac{1+\gamma}{p}}\chi(1,\infty)(t)$, where $\chi\left(1,\infty\right)$ is the characteristic function of the interval $(1,\infty)$. Therefore

$$\left(\int_0^\infty \left(\int_0^y \left(\frac{y}{t}\right)^{-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B\left(B+tI\right)^{-1}\right)^l u\right\|_E \frac{dt}{t}dt\right)^p \frac{dy}{y}\right)^{\frac{1}{p}}$$

is the norm of this convolution in the space $L_p^*\left(0,\infty;E\right)$ (i.e $L_p\left(0,\infty;E\right)$

with respect to the measure $\frac{dt}{t}$). We now apply Young's inequality for multiplicative convolution that can be obtained from the analogous inequality for the ordinary convolution (see [21] , [15]) through the change of variable $t = e^{\xi}$ Therefore the $L_p^*(0,\infty;E)$ norm of the convoution is less or equal then the $L_1^*(0,\infty;E)$ norm of the first function times the $L_p^*(0,\infty;E)$ norm of the second one, so that

$$\left(\int_0^\infty\left(\int_0^y\left(\frac{y}{t}\right)^{-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E\frac{dt}{t}dt\right)^p\frac{dy}{y}\right)^{\frac{1}{p}}\leq$$

$$\int_1^\infty t^{-\frac{1+\gamma}{p}}\frac{dt}{t}\left(\int_0^\infty\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E^p\frac{dt}{t}dt\right)^{\frac{1}{p}}=$$

$$\frac{p}{1+\gamma}\left(\int_0^\infty\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E^p\frac{dt}{t}\right)^{\frac{1}{p}};$$

In a similar way one gets the inequality

$$\left(\int_0^\infty\left(\int_y^\infty\left(\frac{y}{t}\right)^{l-\frac{1+\gamma}{p}}\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E\frac{dt}{t}dt\right)^p\frac{dy}{y}\right)^{\frac{1}{p}}\leq$$

$$\int_1^\infty t^{l-\frac{1+\gamma}{p}}\frac{dt}{t}\left(\int_0^\infty\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E^p\frac{dt}{t}dt\right)^{\frac{1}{p}}=$$

$$\frac{p}{pl-1-\gamma}\left(\int_0^\infty\left\|t^{\beta-\frac{1+\gamma}{p}}\left(B(B+tI)^{-1}\right)^l u\right\|_E^p\frac{dt}{t}\right)^{\frac{1}{p}};$$

From these inequalites we obtain assertion of lemma1.

Proof of theorem7 : For $\lambda \in S_\varphi$ put $B = A_\lambda^{\frac{1}{2}}$. By the[8, lemma2.4] one can apply lemma1 to the operator B with $l = 2m, \beta = 2\alpha$. Then by [21, Th.10.6] it is

$$\left(A_\lambda^{\frac{1}{2}}\right)^{2\alpha} = A_\lambda^\alpha \text{ and } \left(A_\lambda^{\frac{1}{2}}\right)^{2m} = A_\lambda^m ,$$

therefore we have

$$\int_0^\infty\left\|A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}}u\right\|_E^p x^\gamma dx=\int_0^\infty\left\|\left(A_\lambda^{\frac{1}{2}}\right)^{2\alpha}e^{-xA_\lambda^{\frac{1}{2}}}u\right\|_E^p x^\gamma dx\leq C$$

$$\int_0^\infty\left\|t^{2\alpha-\frac{1+\gamma}{p}}A_\lambda^{\frac{1}{2}}\left(\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-1}\right)_E^{2mp}\right\|^p\frac{dt}{t}dx\leq$$

$$C\int_0^\infty\left\|\left(A+(\lambda+t^2)I\right)^m\left(\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-1}\right)^{2m}\right\|_E^p\times$$

$$\left\|t^{2\alpha-\frac{1+\gamma}{p}}A_\lambda^m\left(\left(A+(\lambda+t^2)I\right)^{-m}\right)_E\right\|^p\frac{dt}{t}.$$

For $t \in R_+$ it is

$$\left\|\left(A+(\lambda+t^2)I\right)^m\left(\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-1}\right)^{2m}\right\|\leq$$

$$\left\|\left(A+(\lambda+t^2)I\right)\left(A+(\lambda+t^2)I+2tA_\lambda^{\frac{1}{2}}\right)^{-1}\right\|^m=$$

$$\left\|I-2tA_\lambda^{\frac{1}{2}}\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-2}\right\|^m\leq\left(1+2\left\|A_\lambda^{\frac{1}{2}}\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-1}\right\|\times\right.$$

$$\left.\left\|tA_\lambda^{\frac{1}{2}}\left(A_\lambda^{\frac{1}{2}}+tI\right)^{-1}\right\|^m\right)\leq C$$

Moreover if j is a integer less then m the by the [21, Th.8.1] for every $v \in E(A^m)$ and $\lambda \in C$ we have

$$\left\|\lambda^j A^{m-j}v\right\|\leq C(L,m,j)\left(|\lambda|^m\|v\|+\|A^m v\|\right).$$

Therefore we have

$$\int_0^\infty\left\|A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}}u\right\|_E^p x^\gamma dx\leq C\left(\int_0^\infty\left\|t^{2\alpha-\frac{1+\gamma}{p}}A^m\left(A+(\lambda+t^2)I\right)_E^{-m}u\right\|_E^p\frac{dt}{t}+\right.$$

$$\left.\int_0^\infty\left\|t^{2\alpha-\frac{1+\gamma}{p}}\lambda^m\left(A+(\lambda+t^2)I\right)_E^{-m}u\right\|_E^p\frac{dt}{t}\right).$$

But, by viertue of [8, lemmas2.3 and 2.5] we get

$$\int_0^\infty\left\|t^{2\alpha-\frac{1+\gamma}{p}}A^m\left(A+(\lambda+t^2)I\right)_E^{-m}u\right\|_E^p\frac{dt}{t}\leq C\|u\|_{\frac{\alpha}{m}-\frac{1+\gamma}{2mp},p}$$

and

$$\int_0^\infty\left\|t^{2\alpha-\frac{1+\gamma}{p}}\lambda^m\left(A+(\lambda+t^2)I\right)_E^{-m}u\right\|_E^p\frac{dt}{t}\leq$$

$$|\lambda|^{mp}\int_0^\infty\left[t^{2\alpha-\frac{1+\gamma}{p}}\left(L\left(1+\left|(\lambda+t^2)\right|\right)^{-1}\right)^m\|u\|_E\right]^p\frac{dt}{t}\leq$$

$$C|\lambda|^{mp}\int_0^\infty\left[t^{2\alpha-\frac{1+\gamma}{p}}\left(|\lambda|+t^2\right)^{-m}\|u\|_E\right]^p\frac{dt}{t}=C|\lambda|^{mp}\times$$

$$\int_0^\infty\left[\left(s|\lambda|^{\frac{1}{2}}\right)^{2\alpha-\frac{1+\gamma}{p}}\left(|\lambda|+\left(s|\lambda|^{\frac{1}{2}}\right)\right)^{-m}\|u\|_E\right]^p\frac{ds}{s}=C|\lambda|^{p\alpha-\frac{1+\gamma}{2}}\times$$

$$\int_0^\infty \left[s^{2\alpha - \frac{1+\gamma}{p}} \left(1+s^2\right)^{-m} \|u\|_E \right]^p \frac{dt}{t} = C\,|\lambda|^{p\alpha - \frac{1+\gamma}{2}} \|u\|_E^p .$$

This completes the proof of thorem 7.

4. Applications

1. Let $s \in R,\ s > 0.$ Define $l_p^s = \left\{u;\ \ u = \{u_i\}, i = 1, 2, ..., \infty, u_i \in C\right\}$ with the norm

$$\|u\|_{l_p^s} = \left(\sum_{i=1}^{\infty} 2^{ips} |u_i|^p\right)^{1/p} < \infty$$

Note that $l_p^0 = l_p.$ Let A is infinite matrix defined in the space lp such that $D(A) = l_p^s,\ \ A = \left[\delta_{ij} 2^{si}\right],$ where $\delta_{ij} = 0$ when $i \neq j,\ \ \delta_{ij} = 1,$ when i = j, $i, j = 1, 2, ..., \infty.$ It is clear to see that,this operator A is positive in the space l_p.Then by the theorem2 we obtain the continuous embedding

$$D^\alpha W_p^l\left(\Omega, l_p^s, l_p\right) \subset L_p\left(\Omega, l_p^{s(1-\varkappa-\mu)}\right), \varkappa = \sum_{k=1}^n \frac{\alpha_k + \frac{1}{p} - \frac{1}{q}}{l_k}, \text{ where } 0 \leq \mu \leq$$

$$1 - \varkappa,$$

and also the estimate (18) . whose haven't been obtained with classical method until now.

5.Coercive solvability for differential-operator equations

Let us consider differential-operator equations

$$Lu = \sum_{k=1}^n (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u + \sum_{|\alpha:2l|<1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{2l_k}} A_\alpha(x) D^\alpha u = f \qquad (20)$$

in the space $L_p\left(R^n, E\right),\quad \text{where, } A_\lambda = A - \lambda I,\ A \text{ and } A_\alpha(x)$ are in general , unbounded operators in Banach space E, t_k, k = 1, 2, ..., n parameters, $l = (l_1, l_2, ..., l_n),\ l_i -$ positive integers .

Theorem 8. Let $t_k > 0$, k = 1, 2, ..., n, A be positive operator in Banach space E satisfying multiplier condition with respect to p, $p,\ 1 < p < \infty$ and let $A_\alpha(x) A^{-(1-|\alpha:2l|-\mu)} \in L_\infty\left(R^n, L(E)\right),\ \exists\, \mu,\ 0 < \mu < 1 - |\alpha:2l|.$

Then for all $f \in L_p\left(R^n, E\right)$, and for sufficiently large $|\lambda| > 0\ ,\ \lambda \in S(\pi)$ equation (20) has a unique solution u (x) that belongs to space $W_p^{2l}\left(R^n, E(A), E\right)$ and hold estimate

$$\sum_{k=1}^n t_k \left\|D_k^{2l_k} u\right\|_{L_p(R^n,E)} + \|Au\|_{L_p(R^n,E)} \leq C\|f\|_{L_p(R^n,E)}. \qquad (21)$$

Proof: First we will consider principial part of equation (20) i.e. differential operator equation

$$L_0 u = \sum_{k=1}^n (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u = f. \qquad (22)$$

Then we applly Fourier transform to equation (22) with respect to $x = (x_1, ..., x_n)$ and obtain

$$\sum_{k=1}^n t_k \xi_k^{2l_k} u^\wedge(\xi) + A_\lambda u^\wedge(\xi) = f^\wedge(\xi). \qquad (23)$$

In view of condition theorem4

$$\sum_{k=1}^n t_k \xi_k^{2l_k} \geq 0 \text{ for all } \xi = (\xi_1, ..., \xi_n) \in R^n ,$$

therefore

$$\lambda - \sum_{k=1}^n t_k \xi_k^{2l_k} \in S(\pi)$$

for all $\xi \in R^n;$ that is operator $A - \left[\lambda - \sum_{k=1}^n t_k \xi_k^{2l_k}\right] I$ is invertible in E .

Hence (23) implies that the solution of equation (22) can be represented in the form

$$u(x) = F^{-1}\left[A - \left(\lambda - \sum_{k=1}^n t_k \xi_k^{2l_k}\right) I\right]^{-1} f^\wedge \qquad (24)$$

It is clear to see that operator- function

$$\varphi_{\lambda,t}(\xi) = \left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right) I\right]^{-1}$$

is multiplier in the space $L_p(R^n, E)$ uniformly to $\lambda \in S(\pi)$. Actually, since $S(\pi) = R_-$ by the definition1 for all $\xi \in R^n$ and $\lambda < 0$ we get

$$\|\varphi_\lambda(\xi)\|_{L(E)} = \left\| \left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right)\right]^{-1} \right\| \leq M\left(1 + \left|\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right|\right)^{-1} \leq M_0$$

Moreover since

$$D_k \varphi_{\lambda,t}(\xi) = \left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right)\right]^{-2} .2l_k t_k \xi_k^{2l_k - 1}$$

then

$$\|\xi_k D_k \varphi_{\lambda,t}\|_{L(E)} \leq 2l_k t_k \xi_k^{2l_k} \left\| \left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right)\right]^{-2} \right\| \leq 2l_k t_k \xi_k^{2l_k} \left(1 + \left|\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right|\right)^{-2} \leq M \quad (25)$$

Using the estimate (25) we show for $\beta = \beta_1, ..., \beta_n) \in U_n$ and $\xi = (\xi_1, ..., \xi_n) \in V_n$ uniformly with respect to parameters t and λ;

$$|\xi|^\beta \left\| D_\xi^\beta \varphi_{\lambda,t}(\xi) \right\|_{L(E)} \leq C. \quad (26)$$

In similer way we prove that for operator-functions $\varphi_{k\lambda,t}(\xi) = \xi_k^{2l_k} \varphi_{\lambda,t}$, k =1, 2, .., n and $\varphi_{0\lambda,t} = A\varphi_{\lambda,t}$ holds the estimates [26]. Since Banach space E satisfyes multiplier condition with respect to p, then in view of estimates (26) and (27) we obtain that operator-function $\varphi_{\lambda,t}, \varphi_{k\lambda,t}, \varphi_{o\lambda,t}$ are multiplier in the space $L_p(R^n, E)$. As result of

$$\left\|D_k^{2l_k} u\right\|_{L_p(R^n,E)} = \left\|F^{-1}(i\xi_k)^{2l_k} \hat{u}\right\|_{L_p(R^n,E)} = \left\|F^{-1}(i\xi_k)^{2l_k}\left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right) I\right]^{-1} \hat{f}\right\|_{L_p(R^n;E)}$$

and $\|Au\|_{L_p(R^n,E)} = \left\|F^{-1} A\hat{u}\right\|_{L_p(R^n,E)} = \left\|F^{-1} A\left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right) I\right]^{-1} \hat{f}\right\|_{L_p(R^n,E)}$.

and by the definition of multiplier we obtain that for all

$f \in L_p(R^n, E)$

there is uniqe solution of equation (22) in the form

$$u(x) = F^{-1}\left[A - \left(\lambda - \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right)\right]^{-1} \hat{f}$$

and holds estimate

$$\sum_{k=1}^{n} t_k \left\|D_k^{2l_k} u\right\|_{L_p(R^n E)} + \|Au\|_{L_p(R^n,E)} \leq C\|f\|_{L_p(R^n E)}. \quad (27)$$

In the space $L_p(R^n, E)$, we consider the differential operator $L_0 - \lambda$ generated by the problem (22), that is

$$D(L_0 - \lambda) = W_p^{2l}(R^n, E(A), E) \text{ and } (L_0 - \lambda)u = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u.$$

The estimate (28) implies that the operator $L_0 - \lambda$ for all $\lambda \leq 0$ has a bounded invers acts from $L_p(R^n, E)$ into $W_p^{2l}(R^n, E(A), E)$. We denote by $L - \lambda$ the differential operator in the space $L_p(R^n, E)$ generated by the problem (20) .Namely

$$D(L - \lambda) = W_p^{2l}(R^n, E(A), E), (L - \lambda)u = (L_0 - \lambda)u + L_1 u \quad (28)$$

,where

$$L_1 u = \sum_{|\alpha:2l|<1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{2l_k}} A_\alpha(x) D^\alpha u.$$

In view of conditions theorem 8 and by virtue of theorem1 for all

$u \in W_p^{2l}(R^n, E(A), E)$

$$\|L_1 u\|_{L_p(R^n,E)} \le \sum_{|\alpha:2l|<1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{2l_k}} \|A_\alpha(x) D^\alpha u\|_{L_p(R^n,E)}$$

$$\le \sum_{|\alpha:2l|<1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{2l_k}} \left\|A^{1-|\alpha:2l|-\mu} D^\alpha u\right\|_{L_p(R^n,E)} \le \quad (29)$$

$$\le C\left[h^{\mu}\left(\sum_{k=1}^{n} t_k \left\|D_k^{2l_k} u\right\|_{L_p(R^n E)} + \|Au\|_{L_p(R^n,E)}\right) + h^{-(1-\mu)}\|u\|_{L_p(R^n,E)}\right].$$

Then from estimates (29) , (30) and for all $u \in W_p^{2l}(R^n, E(A), E)$ we obtain

$$\|L_1 u\|_{L_p(R^n,E)} \le C\left[h^{\mu}\|(L_0-\lambda)u\|_{L_p(R^n,E)} + h^{-(1-\mu)}\|u\|_{L_p(R^n,E)}.\right] \quad (30)$$

Since

$\|u\|_{L_p(R^n,E)} = \frac{1}{\lambda}\|(L_0-\lambda)u + L_0 u\|_{L_p(R^n,E)}$ for all $u \in W_p^{2l}(R^n, E(A), E)$

by the definition1 we get

$$\|u\|_{L_p(R^n,E)} \le \frac{1}{|\lambda|}\|(L_0-\lambda)u\|_{L_p(R^n,E)} +$$

$$\|L_0 u\|_{L_p(R^n,E)} \le \frac{1}{|\lambda|}\|(L_0-\lambda)u\|_{L_p(R^n,E)} + \quad (31)$$

$$+\frac{1}{|\lambda|}\left[\sum_{k=1}^{n} t_k \left\|D_k^{2l_k} u\right\|_{L_p(R^n E)} + \|Au\|_{L_p(R^n,E)}\right].$$

From estimates (28) , (30) - (32) for all $u \in W_p^{2l}(R^n, E(A), E)$ obtain

$$\|L_1 u\|_{L_p(R^n,E)} \le Ch^{\mu}\|(L_0-\lambda)u\|_{L_p(R^n,E)} +$$

$$C_1|\lambda|^{-1} h^{-(1-\mu)}\|(L_0-\lambda)u\|_{L_p(R^n,E)}. \quad (32)$$

Then choosing h and λ such that

$$Ch^{\mu} < 1, \quad C_1|\lambda|^{-1} h^{-(1-\mu)} < 1$$

from (33) obtain that

$$\left\|L_1(L_0-\lambda)^{-1}\right\|_{L(E)} < 1. \quad (33)$$

Using relation (29) estimates (28) and (34) and perturbation theory of linear operators, we establish that the differential operator $L - \lambda$ is invertiable from $L_p(R^n, E)$ into $W_p^{2l}(R^n, E(A), E)$.

This implise the estimate (21) .

Remark 3. There are a lot of positive operators in the different concrete Banach spaces. Therefore putting instead of E, concrete Banach spaces and instead of operator A, concrete positive differential, psedodifferential operators,or finite, infinite matrices, ets.on the differential-operator equations (20) by virtue of theorem 4,we can obtain coersive solvablity of different class of partial differential equations or system of equations.

5. Conclusion

In this paper we introduce a Banach- valued Sobolev-Liouville spaces associated with Banach spaces E_1,E and some parameters and proved continuity and compacness of embedding operators in these spaces in terms of theory interpolations of Banach spaces uniformly with respect to these parameters and proved estimate of semigroup operator in weighted spaces. This problem arises in the investigation of boundary value problems for differential-operator equations with parameters.Further we consider certain class of partial differential -operator equation with parameters in Lp spaces and establish coercive solvability of this problem uniformly with respect to these parameters. In turn this equation have many applications to partial differential equations and finite and infinite systems of equations.

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