HSJG
Hagia Sophia Journal of Geometry
Vol. 2, No. 1, 1-8, 2020

RELATIONS BETWEEN AREAS OF LORENTZIAN SPHERICAL REGIONS

Gülay KORU YÜCEKAYA

Abstract

In this study, during the one-parameter closed spherical motion $B^{'} = K/K^{'}$ in 3-dimensional Lorentzian space L_1^3, the unit time-like Steiner vector of the motion; the end points of the orthonormal frame $\{\overrightarrow{e_1},\ \overrightarrow{e_2},\ \overrightarrow{e_3}\ \}$ of the K moving Lorentzian sphere, where $\overrightarrow{e_1},\ \overrightarrow{e_2}$ are the space-like vectors and $\overrightarrow{e_3}$ is the time-like vector, are expressed in terms of field vectors of the regions that are limited by the spherical orbits on the fixed unit Lorentzian sphere $K^{'}$ during the one-parameter closed spherical motion $B^{'} = K/K^{'}$.

Furthermore, for one-parameter closed spherical motion $B^{'} = K/K^{'}$, relations and results between the areas obtained by field vector, F_X, of the spherical region bounded by the closed spherical space-like curve (X) drawn by a fixed point X, which selected from the moving Lorentzian sphere $K^{'}$ on the fixed unit Lorentzian sphere K and in a closed spherical motion $B^{'} = K/K^{'}$; the orthonormal vectors $\{0;\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}\}$, which selected in the moving unit Lorentzian sphere K, the spherical regions of the end points on the sphere that the spherical orbits of the fixed unit Lorentzian sphere are constrained.

In addition, the correlations and results obtained were analyzed using the new expression of the unit time-like Steiner vector of the motion and the same results were obtained.

Keywords and 2010 Mathematics Subject Classification

Keywords: One-parameter closed Lorentzian spherical motion— Lorentzian spherical curve— time-like vector— time-like Steiner vector— Area of a Lorentzian spherical region.

MSC: 53A17, 53A35, 53B30.

Selçuk University, The Faculty of Sciences, Department of Mathematics, Selçuklu, Konya, 42003, Turkey.

Corresponding author: gulay.koru@selcuk.edu.tr

Article History: Received 6 August 2019 ; Accepted 10 September 2019

1. Introduction

We give the basic theory about differential geometry in Lorentzian 3-space. More details and background were published previously [1], [2], [3], [4].

Let V be a vector space. The scalar product

$$\begin{aligned} <,>_L : V\times V &\rightarrow \mathbb{R} \\ \left(\overrightarrow{X},\overrightarrow{Y}\right) &\rightarrow \ <\overrightarrow{X},\overrightarrow{Y}>_L = \sum_{i=1}^{n-1} x_i y_i - x_n y_n \end{aligned}$$

is called a scalar product with index 1 in the sense of Lorentzian, and the pair $\{V,<,>_L\}$ is called a Lorentzian vector space. In particular, if $V=\mathbb{R}^3$, the pair, $\{\mathbb{R}^3,<,>_L\}$ is called the 3-dimensional Lorentzian vector space and it is denoted by L_1^3.

For $\overrightarrow{X}=(x_1,x_2,x_3)$ and $\overrightarrow{Y}=(y_1,y_2,y_3)$ in L_1^3, the Lorentzian inner product is given by

$$<\overrightarrow{X},\overrightarrow{Y}>_L = x_1y_1 + x_2y_2 - x_3y_3.$$

For $\overrightarrow{X} \in L_1^3$, in the norm of the vector $\overrightarrow{X}$ is defined by $\left\|\overrightarrow{X}\right\|_L = \sqrt{\left|<\overrightarrow{X},\overrightarrow{X}>_L\right|}$. A non-zero vector $\overrightarrow{X}$ in is called a

time-like vector if $<\overrightarrow{X},\overrightarrow{X}>_L < 0$,
null-like or light-like vector if $<\overrightarrow{X},\overrightarrow{X}>_L = 0$,
space-like vector if $<\overrightarrow{X},\overrightarrow{X}>_L > 0$ [5].

Let $\overrightarrow{e_i} = (\delta_{i1}, \delta_{i2}, \delta_{i3})$. The vector product in is given by

$$\overrightarrow{X} \wedge_L \overrightarrow{Y} = -\det \begin{bmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & -\overrightarrow{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} \overrightarrow{e_1} \wedge_L \overrightarrow{e_2} &= \overrightarrow{e_3},\ \overrightarrow{e_2} \wedge_L \overrightarrow{e_3} = -\overrightarrow{e_1},\ \overrightarrow{e_3} \wedge_L \overrightarrow{e_1} = -\overrightarrow{e_2}, \\ <\overrightarrow{e_1},\overrightarrow{e_1}>_L &= 1,\ <\overrightarrow{e_2},\overrightarrow{e_2}>_L = 1, <\overrightarrow{e_3},\overrightarrow{e_3}>_L = -1 \end{aligned}$$

and

$$<\overrightarrow{e_1},\overrightarrow{e_2}>_L = <\overrightarrow{e_2},\overrightarrow{e_3}>_L = <\overrightarrow{e_3},\overrightarrow{e_1}>_L = 0.$$

For $\overrightarrow{X},\ \overrightarrow{Y},\ \overrightarrow{Z}$ in L_1^3 we have

$$\overrightarrow{X} \wedge_L \left(\overrightarrow{Y} \wedge_L \overrightarrow{Z}\right) = -<\overrightarrow{X},\overrightarrow{Z}>_L \overrightarrow{Y} + <\overrightarrow{X},\overrightarrow{Y}>_L \overrightarrow{Z}.$$

Between two vectors X and Y on the same time cone, there is a unique angle $\varphi \geq 0$, called the hyperbolic angle, and it satisfies

$$<\overrightarrow{X},\overrightarrow{Y}>_L = -\left\|\overrightarrow{X}\right\|_L \left\|\overrightarrow{Y}\right\|_L \cosh\varphi. \tag{2}$$

If the tangent vector $\dot{\overrightarrow{X}}(t)$ of a differentiable curve $X(t)$, t in $I \subseteq \mathbb{R}$, is space-like (time-like, null), then the curve $X(t)$ is called space-like (time-like, null) [6]. In this paper, we will work with regular space-like curves on the space-like sphere $x_1^2 + x_2^2 - x_3^2 = -1^2$ with time-like normal vector.

A Lorentzian motion $B' = K/K^{'}$ of the moving unit Lorentzian sphere K with fixed center O on the unit Lorentzian sphere $K^{'}$ with the same center, defines a direct Lorentzian motion about the fixed point O. So, a Lorentzian spherical motion is a space motion on the 3-dimensional Lorentzian space.

The two positively oriented, orthonormal coordinate systems

$$\{O,\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}\}, \left\{O,\overrightarrow{e_1}',\overrightarrow{e_2}',\overrightarrow{e_3}'\right\}$$

represent the moving K and the fixed $K^{'}$ Lorentzian spheres, respectively, during the Lorentzian motion $B^{'} = K/K'$, which fixes the initial point O. These two coordinate systems depend on the unit Lorentzian spheres K and $K^{'}$ in a fixed way. Let us denote these orthonormal systems by

$$E = \begin{bmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{e_3} \end{bmatrix} \text{ and } E^{'} = \begin{bmatrix} \overrightarrow{e_1}' \\ \overrightarrow{e_2}' \\ \overrightarrow{e_3}' \end{bmatrix}.$$

Then, $E = AE^{'}$ for an orthogonal Lorentzian matrix A. Here $A^{-1} = SA^TS$ and $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are signature matrices in L_1^3. Let the matrices E, $E^{'}$ and A be differentiable functions of sufficiently high order, of a parameter t in $\mathbb{R}$. Here the matrix A is not periodic, but the spherical curve (X) traced in $K^{'}$ by the point X taken from the moving sphere K, is periodic. Therefore, we get the motion $B^{'} = K/K^{'}$ of the moving Lorentzian sphere K with respect to the fixed Lorentzian sphere $K^{'}$, which is called a one-parameter closed spherical motion [7].

In this paper, we will take $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_1}', \overrightarrow{e_2}'$ to be space-like vectors, and $\overrightarrow{e_3}$, $\overrightarrow{e_3}'$ to be time-like vectors.

For an orthogonal Lorentzian matrix A we have

$$A\left(SA^TS\right) = I_3$$

or

$$dA\left(SA^TS\right) = -Ad\left(SA^TS\right).$$

The matrix $W = dA\left(SA^TS\right)$ is a skew-ajdoint matrix. Hence, we get

$$W = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \tag{3}$$

If we differentiate both sides of $E = AE^{'}$, we get

$$dE = WE. \tag{4}$$

If we denote the position vector with respect to K of the point X in K by

$$X = \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix}^T,$$

then we write

$$X = X^TE.$$

Change of X with respect to K is

$$dX = dX^TE + X^TWE$$

by (4). If X is a fixed point in K, then we have

$$dX = X^TWE. \tag{5}$$

Let the vector

$$\overrightarrow{W} = \omega_1\overrightarrow{e_1} + \omega_2\overrightarrow{e_2} - \omega_3\overrightarrow{e_3} \tag{6}$$

consists of the non-zero components of W, be given. Since $X^TW = \overrightarrow{W} \wedge_L \overrightarrow{X}$, we can write

$$d\overrightarrow{X} = \overrightarrow{W} \wedge_L \overrightarrow{X}$$

instead of (5). The expression $dE = WE$ becomes

$$d\begin{bmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{e_3} \end{bmatrix} = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{e_3} \end{bmatrix} \tag{7}$$

by (3). So, we can write $d\overrightarrow{e_i} = \overrightarrow{W} \wedge_L \overrightarrow{e_i}$. This equality is equivalent to the expression $dE = WE$. The vector $\overrightarrow{W}$ in (6) is called the instantaneous rotation vector of the closed Lorentzian motion $B^{'} = K/K^{'}$. The pfaffian vector $\overrightarrow{W}$ at time t of a one-parameter motion on a Lorentzian sphere, plays the role of the Darboux rotation vector in the differential geometry of space curves. The line of direction of $\overrightarrow{W}$ passes through the poles P and P' on the Lorentzian spheres K and $K^{'}$. Let $\overrightarrow{p}$ and $\overrightarrow{p}^{'}$ be the position vectors of the poles P and P'. Then

$$\overrightarrow{W} = \left\|\overrightarrow{W}\right\|_L \overrightarrow{p}.$$

Where $\left\|\overrightarrow{W}\right\|_L$ is the angular velocity of the motion $B^{'}=K/K^{'}$. The vector

$$\overrightarrow{s}=\oint\left\|\overrightarrow{W}\right\|_L\overrightarrow{p}=\oint\overrightarrow{W}$$

is called the Steiner vector of the motion. Therefore, the Steiner vector has the form

$$\overrightarrow{s}=\oint\left(\omega_1\overrightarrow{e_1}+\omega_2\overrightarrow{e_2}-\omega_3\overrightarrow{e_3}\right) \tag{8}$$

where

$$s_1=\oint\omega_1,\ s_2=\oint\omega_2,\ s_3=\oint\omega_3$$

are the components of $\overrightarrow{s}$.

2. THEOREMS ABOUT AREAS OF LORENTZIAN SPHERICAL REGIONS

In this section; for the one-parameter closed Lorentzian spherical motion in 3-dimensional Lorentzian space, correlations with spherical fields are recalled. Besides, the unit time-like Steiner vector of the motion is expressed in terms of field vectors of the orbits indicated by the end points on the sphere of the fixed unit sphere of the orthonormal frame of the moving Lorentzian sphere.

Let the orthonormal system $\{0:\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}\}$ represent the moving unit sphere K in a one-parameter closed Lorentzian spherical motion. Let E_1,E_2,E_3 be the endpoints of this triad on the sphere. Let c_1,c_2,c_3 be the closed Lorentzian spherical curves traced on the fixed unit Lorentzian sphere $K^{'}$ by the points E_1,E_2,E_3, respectively, during the one-parameter closed Lorentzian spherical motion $B^{'}=K/K^{'}$.

Areas of the spherical regions on the unit Lorentzian sphere bounded by these Lorentzian spherical curves have the form

$$F_{E_i}=2\pi(1-n)-<\overrightarrow{s},\overrightarrow{e_i}>_L,\ i=1,2,3 \tag{9}$$

where $\overrightarrow{s}$ is the unit time-like Steiner vector of the motion [8]. Position vector of a fixed point X on the moving Lorentzian sphere K is

$$\overrightarrow{X}=x_1\overrightarrow{e_1}+x_2\overrightarrow{e_2}+x_3\overrightarrow{e_3} \tag{10}$$

where $\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}$ are orthonormal basis vectors and x_1,x_2,x_3 are the coordinates of X. Let (X) be the closed spherical curve traced on the unit Lorentzian sphere by X during the one-parameter closed Lorentzian spherical motion $B'=K/K^{'}$. In this case, by [8], the area of the spherical region bounded by the closed spherical curve (X) is given by

$$F_X=2\pi(1-n)-<\overrightarrow{s},\overrightarrow{x}>_L \tag{11}$$

or

$$f((X))=2x_1x_3\overrightarrow{f}(c_1,c_3)+\left(x_1^2+x_2^2\right)\overrightarrow{f}(c_1)+\left(x_2^2+x_3^2\right)\overrightarrow{f}(c_3)$$

where

$$\begin{aligned}\overrightarrow{f}(c_i) &= \oint\overrightarrow{e_i}\wedge_L d\overrightarrow{e_i},\ i=1,2,3\\ \overrightarrow{f}(c_i,c_k) &= \frac{1}{2}\oint\left(\overrightarrow{e_i}\wedge_L d\overrightarrow{e_k}+\overrightarrow{e_k}\wedge_L d\overrightarrow{e_i}\right)\end{aligned} \tag{12}$$

and

$$\oint de_i=0$$

[7].

Theorem 1. *Let $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$ be an orthonormal triad for the moving sphere K, where $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ are space-like vectors and $\overrightarrow{e_3}$ is time-like vector. Let E_1, E_2, E_3 be the endpoints of these vectors on the sphere, and c_1, c_2, c_3 be the orbits of these points on the fixed sphere during the one-parameter closed spherical motion $B' = K/K'$. Then*

$$\overrightarrow{s} = \frac{1}{2}\left[\overrightarrow{f}(c_3) - \overrightarrow{f}(c_2) - \overrightarrow{f}(c_1)\right] \tag{13}$$

where $\overrightarrow{f}(c_1)$, $\overrightarrow{f}(c_2)$, $\overrightarrow{f}(c_3)$ are the area vectors of the regions bounded by these spherical curves and $\overrightarrow{s}$ is the time-like Steiner vector of the motion.

Proof. From (12), (7) and (1) we have

$$\begin{aligned}
\overrightarrow{f}(c_1) &= \oint(\omega_3\overrightarrow{e_3} - \omega_2\overrightarrow{e_2}), \\
\overrightarrow{f}(c_2) &= \oint(\omega_3\overrightarrow{e_3} - \omega_1\overrightarrow{e_1}), \\
\overrightarrow{f}(c_1) &= \oint(\omega_2\overrightarrow{e_2} + \omega_1\overrightarrow{e_1}).
\end{aligned}$$

Thus

$$\overrightarrow{f}(c_1) + \overrightarrow{f}(c_2) - \overrightarrow{f}(c_3) = 2\oint(\omega_3\overrightarrow{e_3} - \omega_2\overrightarrow{e_2} - \omega_1\overrightarrow{e_1})$$

or

$$\overrightarrow{s} = \frac{1}{2}\left[\overrightarrow{f}(c_3) - \overrightarrow{f}(c_2) - \overrightarrow{f}(c_1)\right]$$

from (8). ■

3. RELATIONS BETWEEN AREAS OF LORENTZIAN SPHERICAL REGIONS

In this section; during the one-parameter closed Lorentzian spherical motion in the 3-dimensional Lorentzian space, the area of the Lorentzian spherical region bounded by the closed Lorentzian spherical curve which drawn on the fixed unit Lorentzian sphere by a fixed point X taking on the moving unit Lorentzian sphere; the relation between the coordinates of the fixed point taken on the moving Lorentzian sphere and the fields of spherical curves are given. If the moving unit sphere is represented by the orthonormal system $\{0; \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$, the end points of the orthonormal frames on the sphere indicate closed Lorentzian spherical curves on the fixed unit Lorentzian sphere during movement. In addition, the obtained correlation and results are given with the new expression of the unit time-like Steiner vector of the motion.

Theorem 2. *Let X be a fixed point of K during the closed Lorentzian spherical motion $B' = K/K'$, and E_1, E_2, E_3 be the endpoints of $\overrightarrow{e_1}, \overrightarrow{e_2}$ are space-like and $\overrightarrow{e_3}$ is time-like orthonormal vectors. Let c_1, c_2, c_3 be the orbits of these points on the fixed unit Lorentzian sphere K'. Let (X) be the closed spherical curve traced on the fixed Lorentzian sphere K' by X in K, during the motion $B' = K/K'$. The relationship between the area of the spherical region bounded by (X), and the areas $F_{E_1}, F_{E_2}, F_{E_3}$ of the spherical regions bounded by the spherical curves c_1, c_2, c_3 is given by*

$$F_X = x_1F_{E_1} + x_2F_{E_2} + (1 - x_1 - x_2)F_{E_3} - (1 - x_1 - x_2 - x_3)\cosh\varphi. \tag{14}$$

Proof. Let X be a fixed point on the moving Lorentzian sphere K. We have

$$\overrightarrow{X} = x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3}$$

by (10). For the area of the spherical region bounded by the closed spherical curve (X), we have

$$F_X = 2\pi(1-n) - \langle \overrightarrow{s}, \overrightarrow{x} \rangle_L$$

by (11), or

$$F_X = 2\pi(1-n) - \langle \overrightarrow{s}, x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \rangle_L$$

by (10), or

$$F_X = 2\pi(1-n) - x_1(2\pi(1-n) - F_{E_1}) - x_2(2\pi(1-n) - F_{E_2}) - x_3\langle \overrightarrow{s}, \overrightarrow{e_3} \rangle_L$$

by (9), or

$$F_X = 2\pi(1-n) - x_1(2\pi(1-n) - F_{E_1}) - x_2(2\pi(1-n) - F_{E_2}) - x_3\cosh\varphi$$

by (2). For the unit time-like vectors $\overrightarrow{s}$ and $\overrightarrow{e_3}$, (2) and (9) give

$$F_{E_3} = 2\pi(1-n) + \cosh\varphi.$$

Thus

$$2\pi(1-n) = F_{E_3} - \cosh\varphi$$

and

$$F_x = x_1F_{E_1} + x_2F_{E_2} + (1-x_1-x_2)F_{E_3} - (1-x_1-x_2-x_3)\cosh\varphi.$$

■

Corollary 3. *During a one-parameter closed Lorentzian spherical motion $B' = K/K'$, the relationship between the coordinates of a chosen fixed point X on the moving Lorentzian sphere K, and the spherical areas F_X, F_{E_1}, F_{E_2}, F_{E_3} is given by*

$$F_x - x_1F_{E_1} - x_2F_{E_2} - (1-x_1-x_2)F_{E_3} \leq 1-x_1-x_2-x_3, \ \textit{for}\ x_1+x_2+x_3 < 1$$

and

$$F_x - x_1F_{E_1} - x_2F_{E_2} - (1-x_1-x_2)F_{E_3} \geq 1-x_1-x_2-x_3, \ \textit{for}\ x_1+x_2+x_3 > 1$$

since $\cosh\varphi \geq 1$.

Corollary 4. *For the unit time-like vectors $\overrightarrow{s}$ and $\overrightarrow{e_3}$, in the special case $\overrightarrow{s} = \overrightarrow{e_3}$, $\varphi = 0$. Thus, the relationship between the areas in question becomes*

$$F_x - x_1F_{E_1} - x_2F_{E_2} - (1-x_1-x_2)F_{E_3} = 1-x_1-x_2-x_3$$

by (14).

Theorem 5. *Let X be a fixed point of K during the closed Lorentzian spherical motion $B' = K/K'$, and E_1, E_2, E_3 be the endpoints of $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ space-like and time-like orthonormal vectors. Let c_1, c_2, c_3 be the orbits of these points on the fixed unit Lorentzian sphere. Let (X) be the closed spherical curve traced on the fixed Lorentzian sphere by X in K, during the motion $B' = K/K'$. The relationship between the area of the spherical region bounded by (X), and the areas F_{E_1}, F_{E_2}, F_{E_3} of the spherical regions bounded by the spherical curves c_1, c_2, c_3 are given by*

$$F_X = x_1F_{E_1} + x_2F_{E_2} + (1-x_1-x_2)F_{E_3} - (1-x_1-x_2-x_3)\cosh\varphi$$

where $\overrightarrow{s} = \frac{1}{2}\left[\overrightarrow{f}(c_3) - \overrightarrow{f}(c_2) - \overrightarrow{f}(c_1)\right]$ is the unit time-like Steiner vector of the motion.

Proof.

$$F_X = 2\pi(1-n) - <\overrightarrow{s}, \overrightarrow{x}>_L$$

or

$$F_X = 2\pi(1-n) - \frac{1}{2} < \left[\overrightarrow{f}(c_3) - \overrightarrow{f}(c_2) - \overrightarrow{f}(c_1)\right], x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} >_L$$

by (10) and (13), or

$$\begin{aligned} 2F_X &= 4\pi(1-n) - x_1 < \overrightarrow{f}(c_3), \overrightarrow{e_1} >_L - x_2 < \overrightarrow{f}(c_3), \overrightarrow{e_2} >_L - x_3 < \overrightarrow{f}(c_3), \overrightarrow{e_3} >_L \\ +x_1 &< \overrightarrow{f}(c_2), \overrightarrow{e_1} >_L + x_2 < \overrightarrow{f}(c_2), \overrightarrow{e_2} >_L + x_3 < \overrightarrow{f}(c_2), \overrightarrow{e_3} >_L \\ +x_1 &< \overrightarrow{f}(c_1), \overrightarrow{e_1} >_L + x_2 < \overrightarrow{f}(c_1), \overrightarrow{e_2} >_L + x_3 < \overrightarrow{f}(c_1), \overrightarrow{e_3} >_L . \end{aligned} \tag{15}$$

From (9) and (13) we get

$$2F_{E_1} = 4\pi(1-n) - < \overrightarrow{f}(c_3), \overrightarrow{e_1} >_L + < \overrightarrow{f}(c_2), \overrightarrow{e_1} >_L + < \overrightarrow{f}(c_1), \overrightarrow{e_1} >_L \tag{16}$$

and

$$2F_{E_2} = 4\pi(1-n) - < \overrightarrow{f}(c_3), \overrightarrow{e_2} >_L + < \overrightarrow{f}(c_2), \overrightarrow{e_2} >_L + < \overrightarrow{f}(c_1), \overrightarrow{e_2} >_L \tag{17}$$

where $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ space-like vectors, $\overrightarrow{e_3}$ is time-like vector and $\overrightarrow{s}$ is the unit time-like Steiner vector. Then, (16) and (17) give

$$< \overrightarrow{f}(c_3), \overrightarrow{e_1} >_L = 4\pi(1-n) + < \overrightarrow{f}(c_2), \overrightarrow{e_1} >_L + < \overrightarrow{f}(c_1), \overrightarrow{e_1} >_L - 2F_{E_1} \tag{18}$$

and

$$< \overrightarrow{f}(c_3), \overrightarrow{e_2} >_L = 4\pi(1-n) + < \overrightarrow{f}(c_2), \overrightarrow{e_2} >_L + < \overrightarrow{f}(c_1), \overrightarrow{e_2} >_L - 2F_{E_2} \tag{19}$$

If we substitute (18) and (19) in (15), we get

$$2F_X = 4\pi(1-x_1-x_2) + 2(x_1F_{E_1} + x_2F_{E_2}) - 2x_3 < \overrightarrow{s}, \overrightarrow{e_3} >_L$$

or from (2)

$$2F_X = 4\pi(1-x_1-x_2) + 2(x_1F_{E_1} + x_2F_{E_2}) + 2x_3 \cosh\varphi$$

or from (2) and (9)

$$F_X = x_1F_{E_1} + x_2F_{E_2} + (1-x_1-x_2)F_{E_3} - (1-x_1-x_2-x_3)\cosh\varphi.$$

■

Corollary 6. *During a one-parameter closed Lorentzian spherical motion $B' = K/K'$, the relationship between the coordinates of a chosen fixed point X on the moving Lorentzian sphere K, and the spherical areas F_X, F_{E_1}, F_{E_2}, F_{E_3} are given by*

$$F_x - x_1F_{E_1} - x_2F_{E_2} - (1-x_1-x_2)F_{E_3} \leq 1 - x_1 - x_2 - x_3, \ \textit{for}\ x_1 + x_2 + x_3 < 1$$

and

$$F_x - x_1F_{E_1} - x_2F_{E_2} - (1-x_1-x_2)F_{E_3} \geq 1 - x_1 - x_2 - x_3, \ \textit{for}\ x_1 + x_2 + x_3 > 1$$

since $\cosh\varphi \geq 1$ *and* $\overrightarrow{s} = \frac{1}{2}\left[\overrightarrow{f}(c_1) - \overrightarrow{f}(c_2) - \overrightarrow{f}(c_1)\right]$ *is the unit time-like Steiner vector.*

4. Conclusion

In this work, compared to [9], further analysis has been conducted, and some additional results were obtained.

During the one-parameter closed spherical motion $B' = K/K'$ in 3-dimensional Lorentzian space, the unit time-like Steiner vector of the motion; the end points of the orthonormal triad of the K moving Lorentzian sphere are expressed in terms of field vectors of the regions that are limited by the spherical orbits on the fixed unit Lorentzian sphere K' during the closed spherical motion.

Furthermore, for closed spherical motion, relations and results between the areas obtained by field vector, F_X, of the spherical region bounded by the closed spherical space-like curve drawn by a fixed point selected from the moving Lorentzian sphere on the fixed unit Lorentzian sphere and in a closed spherical motion; the orthonormal vectors selected in the moving unit Lorentzian sphere, the spherical regions of the end points on the sphere that the spherical orbits of the fixed unit Lorentzian sphere are constrained.

In addition, the correlations and results obtained were analyzed using the new expression of the unit time-like Steiner vector of the motion and the same results were obtained.

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