Araştırma Makalesi / Research Article

A computational approach for solving second-order nonlinear ordinary differential equations by means of Laguerre series

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Abstract
In this work, a novel efficient numeric procedure for obtaining the approximate solution of a class of second-order nonlinear ordinary differential equations is presented which play a significant part in science and engineering branches. The technique is based on matrix equations and collocation points with truncated Laguerre series. The acquired approximate solutions subject to initial conditions are obtained in terms of Laguerre polynomials. Also, some examples together with error analysis techniques are acquired to demonstrate the efficacy of the present method, and the comparisons are made with current studies.

Keywords: Laguerre series, nonlinear ordinary differential equations, collocation method.

İkinci mertebeden lineer olmayan adi diferansiyel denklemlerin Laguerre serileri ile çözümü için hesaplamalı bir yaklaşım

Öz
Bu çalışmada, fen ve mühendislik dallarında önemli bir rol oynayan ikinci dereceden doğrusal olmayan adi diferansiyel denklemlerin bir sınıfının yaklaşık çözümünü elde etmek için yeni ve etkili bir sayısal prosedür sunulmuştur. Teknik, matris denklemlerine ve kesilmiş Laguerre serileri ile sıralama noktalarına dayanmaktadır. Başlangıç koşullarına tabi olarak elde edilen yaklaşık çözümler, Laguerre polinomları tarafından elde edilir. Ayrıca, mevcut yöntemin etkinliğini ortaya koymak için hata analizi teknikleri ile birlikte bazı örnekler alınmış ve güncel çalışmalar ile karşılaştırmalar yapılmıştır.

Anahtar kelimeler: Laguerre serileri, doğrusal olmayan adi diferansiyel denklemler, sıralama yöntemi.

1. Introduction

Nonlinear differential equations play important role in many fields of engineering, science and even in mathematical models in social sciences. Some real phenomena examples are given for mathematical models for urban growth, modeling learning theories in education and psychology, reaction rates in chemistry, optional pricing in economics so on. Moreover, nonlinear models in biology are important to explain by mathematical models. In order to solve these type of equations

Nonlinear differential equations with initial and boundary value conditions are significant problems and they are important in many fields such as engineering, astrophysics, physical sciences. In recent years, in order to get the solutions of these problems both analytically and numerically some techniques have been introduced. These type of equations are of great importance on applied sciences [3]. Due to this reason, the second-order nonlinear ordinary differential equations are considered:

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Geliş Tarihi: 11.06.2019, Kabul Tarihi: 19.09.2019

$$\sum_{k=0}^{2} P_k(x)y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{N} Q_{pq}(x)y^{(p)}(x)\,y^{(q)}(x) = g(x), \qquad 0 \le x \le b < \infty \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{1}\left(a_{kj}(x)y^{(k)}(0) + b_{kj}(x)y^{(k)}(b)\right) = \lambda_j, \qquad j = 0,1 \tag{2}$$

where the functions $P_k(x), Q_{pq}(x)$, and $g(x)$ are defined $0 \le x \le b < \infty$; a_{kj}, b_{kj}, and λ_j are appropriate and real constants; $y(x)$ is the unknown function to be computed. For this purpose, approximate solution of the problem (1)-(2) is assumed as

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_N(x)L_N(x)\,, \qquad 0 \le x \le b < \infty \tag{3}$$

where $L_N(x)$ delineates the Laguerre polynomials;

$$L_N(x) = \sum_{k=0}^{N} \frac{(-1)^k}{k!}\binom{n}{k}x^k\,, \qquad n \in \mathbb{N}, \qquad 0 \le x < \infty \tag{4}$$

and $a_N, (n = 0,1,\dots,N)$ are unknown coefficients with regard to the Laguerre polynomials, and $N \in \mathbb{Z}^+$ and $N \ge 2$.

2. Operational Matrix Relations

In this section, operational matrix relations are given for finding the approximate solution with regards to Laguerre polynomials in the form (3). For this purpose, Eq. (1) is considered in the form of two parts: the linear ordinary part and the nonlinear quadratic part, respectively as

$$L[y(x)] + N_2[y(x)] = g(x) \tag{5}$$

where

$$L[y(x)] = \sum_{k=0}^{2} P_k(x)y^{(k)}(x)$$

and

$$N_2[y(x)] = \sum_{p=0}^{2}\sum_{q=0}^{N} Q_{pq}(x)y^{(p)}(x)\,y^{(q)}(x) \quad .$$

Now, each terms of the Eq.(1) are presented by the matrix forms [2]. So, the linear ordinary part in Eq.(5) in the matrix form is shown as

$$\begin{aligned} [y(x)] &= \mathbf{L}(x)\mathbf{A}, \\ \left[y^{(1)}(x)\right] &= \mathbf{L}(x)\mathbf{C}\mathbf{A}, \\ \left[y^{(2)}(x)\right] &= \mathbf{L}(x)\mathbf{C}^2\mathbf{A} \end{aligned} \tag{6}$$

where

$\mathbf{L}(x) = [L_0(x) \quad L_1(x) \quad \dots \quad L_N(x)]$,

$\mathbf{C} = [c_{mn}]$,

$c_{mn} = \begin{cases} -1, & m < n \\ 0, & m \geq n \end{cases}$,

$\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$.

Then we define matrix representations of nonlinear quadratic part as

$$
\begin{aligned}
&[(y(x))^2] = \mathbf{L}(x)\bar{\mathbf{L}}(x)\bar{\mathbf{A}},\\
&\left[y^{(1)}(x)y(x)\right] = \mathbf{L}(x)\mathbf{C}\bar{\mathbf{L}}(x)\bar{\mathbf{A}},\\
&\left[\left(y^{(1)}(x)\right)^2\right] = \mathbf{L}(x)\mathbf{C}\bar{\mathbf{L}}(x)\bar{\mathbf{C}}\bar{\mathbf{A}},\\
&\left[y^{(2)}(x)y^{(1)}(x)\right] = \mathbf{L}(x)\mathbf{C}^2\bar{\mathbf{L}}(x)\bar{\mathbf{C}}\bar{\mathbf{A}},\\
&\left[y^{(2)}(x)y(x)\right] = \mathbf{L}(x)\mathbf{C}^2\bar{\mathbf{L}}(x)\bar{\mathbf{A}},\\
&\left[\left(y^{(2)}(x)\right)^2\right] = \mathbf{L}(x)\mathbf{C}^2\bar{\mathbf{L}}(x)\bar{\mathbf{C}}^2\bar{\mathbf{A}};
\end{aligned}
\tag{7}
$$

and

$\bar{\mathbf{L}}(x) = diag\,[\mathbf{L}(x) \quad \mathbf{L}(x) \quad \dots \quad \mathbf{L}(x)]$,

$\bar{\mathbf{C}} = diag\,[\mathbf{C} \quad \mathbf{C} \quad \dots \quad \mathbf{C}]$,

$\bar{\mathbf{A}} = [a_0\mathbf{A} \quad a_1\mathbf{A} \quad \dots \quad a_N\mathbf{A}]^T$.

3. Method of Solution

By putting the collocation points

$$x_i(x) = \frac{b}{N}i, \qquad i = 0,1,\dots,N, \qquad 0 \leq x_0 < x_1 < \cdots < x_N \leq b < \infty \tag{8}$$

into Eq.(1), for $i = 0,1,\dots,N$ thus the fundamental matrix equation is obtained as

$$\sum_{k=0}^{2} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_i)y^{(p)}(x_i)y^{(q)}(x_i) = g(x_i)$$

or

$$\sum_{k=0}^{2} \mathbf{P}_k\mathbf{Y}^{(k)} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{Y}^{(p,q)} = \mathbf{G}$$

where

$$\mathbf{P}_k = diag\,[P_k(x_0) \quad P_k(x_1) \quad \dots \quad P_k(x_N)],$$

$$\mathbf{Q}_{pq} = diag\left[Q_{pq}(x_0) \quad Q_{pq}(x_1) \quad \cdots \quad Q_{pq}(x_N)\right],$$

and

$$\mathbf{Y}_k = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix},\ \mathbf{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0) \\ y^{(p)}(x_1)y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N)y^{(q)}(x_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Also, we consider the ensuing matrix forms of the nonlinear quadratic part from (7)

$$\mathbf{Y}^{(0,0)} = \mathbf{L}^*_{00}\overline{\mathbf{A}},\ \mathbf{Y}^{(1,0)} = \mathbf{L}^*_{10}\overline{\mathbf{A}},\ \mathbf{Y}^{(1,1)} = \mathbf{L}^*_{11}\overline{\mathbf{A}},$$

$$\mathbf{Y}^{(2,0)} = \mathbf{L}^*_{20}\overline{\mathbf{A}},\ \mathbf{Y}^{(2,1)} = \mathbf{L}^*_{21}\overline{\mathbf{A}},\ \mathbf{Y}^{(2,2)} = \mathbf{L}^*_{22}\overline{\mathbf{A}};$$

$$\mathbf{L}^*_{00} = \begin{bmatrix} \mathbf{L}(x_0)\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\overline{\mathbf{L}}(x_N) \end{bmatrix},\ \mathbf{L}^*_{10} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N) \end{bmatrix}, \mathbf{L}^*_{11} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}} \\ \mathbf{L}(x_1)\mathbf{C}\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}} \end{bmatrix},$$

$$\mathbf{L}^*_{20} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0) \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1) \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N) \end{bmatrix},\ \mathbf{L}^*_{21} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}} \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}} \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}} \end{bmatrix}, \mathbf{L}^*_{22} = \begin{bmatrix} \mathbf{L}(x_0)\mathbf{C}^2\overline{\mathbf{L}}(x_0)\overline{\mathbf{C}}^2 \\ \mathbf{L}(x_1)\mathbf{C}^2\overline{\mathbf{L}}(x_1)\overline{\mathbf{C}}^2 \\ \vdots \\ \mathbf{L}(x_N)\mathbf{C}^2\overline{\mathbf{L}}(x_N)\overline{\mathbf{C}}^2 \end{bmatrix}.$$

Then the fundamental matrix equation can be shown as

$$\sum_{k=0}^{2} \mathbf{P}_k\mathbf{L}\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{L}^*_{pq}\overline{\mathbf{A}} = \mathbf{G}$$

or

$$\mathbf{W}\mathbf{A} + \mathbf{V}\overline{\mathbf{A}} = \mathbf{G} \tag{9}$$

where

$$\mathbf{W} = \sum_{k=0}^{2} \mathbf{P}_k\mathbf{L}\mathbf{A} = \left[w_{ij}\right]_{(N+1)\times(N+1)},$$

$$\mathbf{V} = \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{L}^*_{pq}\overline{\mathbf{A}} = \left[v_{ij}\right]_{(N+1)\times(N+1)^2}, \qquad i,j = 0,1,\dots,N.$$

Furthermore, fundamental matrix equation (9) is written in the augmented matrix form

$$[\mathbf{W};\mathbf{V}:\mathbf{G}] \tag{10}$$

If the identical procedure is used for the mixed conditions (2), then

for $j=0, \mathbf{U}_0=[y(0)]=\mathbf{L}(0)$,

for $j=1, \mathbf{U}_1=\left[y^{(1)}(0)\right]=\mathbf{L}(0)\mathbf{C}$.

Also, briefly

$$\mathbf{U}=\begin{bmatrix}\mathbf{U}_0\\ \mathbf{U}_1\end{bmatrix}_{2\times(N+1)},\ \mathbf{O}_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix}_{2\times(N+1)^2},\ \boldsymbol{\lambda}=\begin{bmatrix}\boldsymbol{\lambda}_1\\ \boldsymbol{\lambda}_2\end{bmatrix};\ [\mathbf{U};\quad \mathbf{O}_2:\quad \boldsymbol{\lambda}]. \tag{11}$$

Therefore, so as to get the solution of the problem (1)-(2) the row matrices (11) are replaced by the appropriate two rows of the augumented matrix (10). Thus, the new augumented matrix $\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{V}}:\widetilde{\mathbf{G}}\right]$ is constructed and by having the solution of the system, required Laguerre coefficients are calculated. Accordingly, requisite approximate solution is is occured in the Eq. (3).

4. Error Analysis

Here, a short introduction for the error analysis of the Laguerre polynomial solution (3) is held to demonstrate the exactitude of the method. Error function is defined by $x=x_\alpha,\ \alpha=0,1,\ldots$

$$E_N(x_\alpha)=\left|y(x_\alpha)-\sum_{k=0}^{2}P_k(x_\alpha)y_N^{(k)}(x_\alpha)-\sum_{p=0}^{2}\sum_{q=0}^{N}Q_{pq}(x_\alpha)y_N^{(p)}(x_\alpha)\,y_N^{(q)}(x_\alpha)+g(x_\alpha)\right|\cong 0$$

where $E_N(x_\alpha)\le 10^{-k_\alpha}=10^{-k}$, $(k\in\mathbb{Z}^+)$ is recommended, then the truncation limit N is escalated till having enough small prescribed 10^{-k} for the value of disparity $E_N(x_\alpha)$ at each of the point [3].

Step 0. Input initial data: $P_k(x)$, $Q_{pq}(x)$ and $g(x)$. Determine the mixed conditions

Step 1. Set N where $N\epsilon\mathbb{N}$.

Step 2. Construct the matrices $\mathbf{L}(x),\mathbf{C},\bar{\mathbf{L}}(x),\bar{\mathbf{C}}$ and $\mathbf{G}$ then $\mathbf{W}$ and $\mathbf{V}$.

Step 3. Define the collocation points $x_i=\frac{b}{N}i,\ \ i=0,1,\ldots,N$.

Step 4. Compute $[\mathbf{W};\mathbf{V}:\mathbf{G}]$.

Step 5. Compute $[\mathbf{U};\ \mathbf{O}_2:\boldsymbol{\lambda}]$.

Step 6. Construct the augmented matrix $[\widetilde{\mathbf{W}};\widetilde{\mathbf{V}}:\widetilde{\mathbf{G}}]$.

Step 7. Input: the augmented matrix arguments, forward elimination, back substitution. Output: $\mathbf{A}$ (Solve the system by Gaussian elimination method).

Step 8. Put arguments a_n in the truncated Laguerre series form.

Step 9. Output data: the approximate solution $y_N(x)$.

Step 10. Construct $y(x)$ is the exact solution of (1).

Step 11. Stop when $E_N(x)\le 10^{-k}$ where $k\epsilon\mathbb{Z}^+$. Otherwise, increase N and return to Step 1.

Figure 1. Algorithm of the present method.

5. Numerical Experiments

In this section, an example will be given to show applicability of our method. All the calculations and plots are done by using Maple18 and MatlabR2014b.

Example 5.1. [4]

Second-order nonlinear ordinary differential equation with quadratic terms with initial conditions is considered as an illustrative example:

$$y''(x) + 2y'(x) + y(x) + y^2(x) - y''(x)y'(x) = 12\exp(x) + 2\,; \quad y(0) = 3, y'(0) = 2. \qquad (12)$$

Problem’s exact solution is $y(x) = 2\exp(x) + 1$.

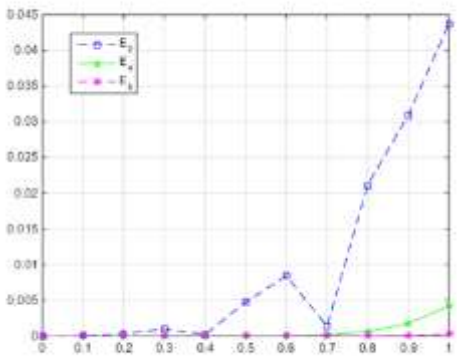

Figure 2. Comparison of the $x > 0$ of Example 5.1. for different N values

Table 1. $|E_N|$ comparisons of Example 5.1.

x	$\lvert E_2 \rvert$	$\lvert E_4 \rvert$	$\lvert E_5 \rvert$
0.0	0.000000	0.000000	0.000000
0.1	0.341836E-4	0.530766E-5	0.450128E-6
0.2	0.280551E-3	0.281048E-4	0.194988E-5
0.3	0.971761E-3	0.563571E-4	0.318105E-5
0.4	0.236493E-3	0.671672E-4	0.339969E-5
0.5	0.474425E-2	0.525094E-4	0.365512E-5
0.6	0.842376E-2	0.476701E-4	0.530256E-5
0.7	0.137505E-2	0.162679E-3	0.552450E-5
0.8	0.211081E-1	0.617047E-3	0.104534E-4
0.9	0.309206E-1	0.177815E-2	0.808003E-4
1.0	0.436563E-1	0.420369E-2	0.282554E-3

6. Conclusion

In this study, a computational procedure depending on Laguerre polynomials has been proposed in spite of solving a class of secod-order nonlinear ordinary differential equations having quadratic terms numerically. Furthermore, the error analysis is explained and applied to determine the reliability of the method. The technique has been tested on illustrative example which has been shown by figure and table. The method has significant importance such as; the present method has short and concise computing procedure by writing the algorithm in Maple18, has sufficient results when N is chosen large enough and the method also can be extended on other studies [5].

Author’s Contributions

The author developed the theoretical formalism, performed the numerical calculations and performed the simulations. Besides, the author made substantial contributions to conception and design, participated in drafting the article or revising it critically for important intellectual content. Finally, the author gave final approval of the version to be submitted and any revised version.

Statement of Conflicts of Interest

No potential conflict of interest was reported by the author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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