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A CLASSIFICATION OF $(k,\mu)'$-ALMOST KENMOTSU MANIFOLDS ADMITTING COTTON TENSOR

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Abstract. The object of the present paper is to classify $(k,\mu)'$-almost Kenmotsu manifolds admitting Cotton tensors. We have characterized $(k,\mu)'$-almost Kenmotsu manifolds with vanishing and parallel Cotton tensors. Beside this, $(k,\mu)'$-almost Kenmotsu manifolds satisfying Cotton semisymmetry and $Q(g,C)=0$ are studied. Further, Cotton pseudo-symmetric $(k,\mu)'$-almost Kenmotsu manifolds are classified.

1. Introduction

On a $(2n+1)$-dimensional Riemannian manifold (M^{2n+1},g), the $(0,3)$-Cotton tensor C is defined by [9]

$$\begin{aligned} C(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &\quad -\frac{1}{4n}((Xr)g(Y,Z) - (Yr)g(X,Z)), \end{aligned} \tag{1.1}$$

where S and r denotes Ricci tensor and scalar curvature of M respectively. The Cotton tensor is skew-symmetric in the first two indices and totally trace free. As it is well known that a Riemannian manifold (M^n,g) is locally conformally flat if and only if (1) for $n \geq 4$ the Weyl tensor vanishes, (2) n = 3 the Cotton tensor vanishes. Moreover for $n \geq 4$, if the Weyl tensor vanishes, then the Cotton tensor vanishes. We also see that when n = 3, the Weyl tensor always vanishes, but the Cotton tensor does not vanish in general. In [20], Wang studied Cotton flat almost coKähler 3-manifolds. In [5], the authors characterize two classes of almost Kenmotsu manifolds admitting quasi-conformal curvature tensor and extended quasi-conformal curvature tensor, which are generalization of the conformal

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curvature tensor.

We now define an endomorphism $X \wedge_A Y$ of the vector fields of M by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \tag{1.2}$$

where A is a symmetric $(0, 2)$-tensor. Also for a $(0, k)$-tensor field T, $k \geq 1$ and a $(0, 2)$-tensor field A on M we define the tensor $Q(A, T)$ by

$$\begin{aligned} Q(A, T)(X_1, X_2, .., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, .., X_k) \\ &\quad -.. - T(X_1, X_2, .., (X \wedge_A Y)X_k). \end{aligned} \tag{1.3}$$

A Riemannian manifold M is said to be Ricci pseudo-symmetric [17] if the tensor fields $R \cdot S$ and $Q(g, S)$ are linearly dependent, i.e., there exist a function $L_S : M \to \mathbb{R}$ such that $R \cdot S = L_S Q(g, S)$ holds on M. In particular, a Ricci pseudo-symmetric manifold with $L_S = 0$ reduces to a Ricci semisymmetric manifold. The notion of pseudo-symmetry also appears in the theory of plane gravitational waves. In [1], pseudo-symmetric contact metric manifolds were studied by Arslan et. al. Also Chaki type pseudo-symmetric lightlike hypersurfaces were studied by Sahin and Yildiz [16]. Further, pseudo-symmetric Riemannian spaces were studied by Özen and Altay [13]. Also Suh et. al. [15] studied Reeb parallel Ricci tensor on real hypersurfaces in complex two-plane Grassmannians.

ξ-conformally flat K-contact manifolds have been studied by Zhen et al. [21]. Since at each point $p \in M^{2n+1}$ the tangent space $T_p(M^{2n+1})$ can be decomposed into the direct sum $T_p(M^{2n+1}) = \phi(T_p(M^{2n+1})) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^{2n+1})$ generated by ξ_p, the conformal curvature tensor $\mathcal{C}$ is a map

$$\mathcal{C} : T_p(M^{2n+1}) \times T_p(M^{2n+1}) \times T_p(M^{2n+1}) \text{â†}' \phi(T_p(M^{2n+1})) \oplus \{\xi_p\}.$$

An almost contact metric manifold M^{2n+1} is called ξ-conformally flat if the projection of the image of $\mathcal{C}$ in $\{\xi_p\}$ is zero.

In 1978, Gray [10] presented a new classes of manifold, namely, manifolds of Codazzi type Ricci tensor, lies between the class of Ricci symmetric manifolds and the class of manifolds of constant scalar curvature.

Definition 1.1. *A semi-Riemannian manifold M is said to be of Codazzi type Ricci tensor if,* $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$ *for any vector fields X, Y and Z holds on M.*

The paper is organized as follows:
In Section 2, we give some preliminary ideas on almost Kenmotsu manifolds. Section 3 is devoted to study $(k, \mu)'$-almost Kenmotsu manifolds satisfying Cotton flatness($C = 0$), Cotton parallelity($\nabla C = 0$), Cotton semisymmetry($R \cdot C = 0$), $Q(g, C) = 0$ and Cotton pseudo-symmetry($R \cdot C = f_C Q(g, C)$).

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily.

If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, Y on M. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ [2]. Recently in ([6], [7], [8], [14]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold [12]. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X,Y. It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [14]:

$$h\xi = 0,\; l\xi = 0,\; tr(h) = 0,\; tr(h\phi) = 0,\; h\phi + \phi h = 0,$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.3}$$

for any vector fields X,Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([6])

$$h = 0 \Leftrightarrow h' = 0, \;\; h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.4}$$

In [6], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p \in M$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' \;=\; & \{Z \in T_p(M) : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \\ & + \mu(g(Y,Z)h'X - g(X,Z)h'Y)\}. \end{aligned} \tag{2.5}$$

The above notion is called generalized nullity distributions when one allows k,μ to be smooth functions.
Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.4) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigen spaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. In [6], it is proved that in a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, $k < -1$, $\mu = -2$ and $\mathrm{Spec}(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given by the following:

(a) $K(X,\xi) = k - 2\lambda$ if $X \in [\lambda]'$ and
$K(X,\xi) = k + 2\lambda$ if $X \in [-\lambda]'$,
(b) $K(X,Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ if $X, Y \in [-\lambda]'$ and
$K(X,Y) = -(k+2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,
(c) M^{2n+1} has constant negative scalar curvature $r = 2n(k-2n)$.

Also
$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.6}$$
In [18], Wang and Liu proved that for a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, the Ricci operator Q of M^{2n+1} is given by
$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{2.7}$$
From (2.5), we have
$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{2.8}$$
where $k,\mu \in \mathbb{R}$. Also we get from (2.8)
$$R(\xi,X)Y = k(g(X,Y)\xi - \eta(Y)X) + \mu(g(h'X,Y)\xi - \eta(Y)h'X). \tag{2.9}$$
Using (2.2), we have
$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y). \tag{2.10}$$

3. Cotton tensor on $(k,\mu)'$-almost Kenmotsu manifolds

In this section, we study Cotton tensor on $(k,\mu)'$-almost Kenmotsu manifolds. Before discussing our main results, we first state the following Lemma:

Lemma 3.1. *(Prop. 4.2 of [6]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belonging to the $(k,-2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemann curvature tensor satisfies:*
$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0,$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0,$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)(g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda),$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)(g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}).$$

Since the scalar curvature $r = 2n(k-2n) = \textit{constant}$ on M^{2n+1}, then the Cotton tensor defined in (1.1) reduces to

$$C(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{3.1}$$

Now from above we can state the following:

Proposition 3.1. *The Cotton tensor of a $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1} vanishes if and only if the Ricci tensor is of Codazzi type.*

Analogous to the definition of ξ-conformally flat almost contact metric manifold, we define ξ-Cotton flat $(k,\mu)'$-almost Kenmotsu manifold as follows:

Definition 3.1. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} is said to be ξ-Cotton flat if the Cotton tensor C satisfies $C(X,Y)\xi = 0$ holds for any vector fields X, Y on M^{2n+1}.*

We now further investigate this as follows:
From (2.7), we have

$$S(X,Y) = -2ng(X,Y) + 2n(k+1)\eta(X)\eta(Y) - 2ng(h'X,Y) \tag{3.2}$$

for any vector fields X, Y on M^{2n+1}.
Taking covariant derivative of (3.2) along any vector field Z we have

$$\begin{aligned}\nabla_Z S(X,Y) &= -2n\nabla_Z g(X,Y) + 2n(k+1)(\nabla_Z \eta(X))\eta(Y)\\ &\quad +2n(k+1)\eta(X)(\nabla_Z \eta(Y)) - 2n\nabla_Z g(h'X,Y).\end{aligned} \tag{3.3}$$

Now, we have

$$(\nabla_Z S)(X,Y) = \nabla_Z S(X,Y) - S(\nabla_Z X, Y) - S(X, \nabla_Z Y).$$

Using (3.2) and (3.3) in the foregoing equation, we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2n(k+1)(\nabla_Z \eta)X)\eta(Y) + 2n(k+1)\eta(X)(\nabla_Z \eta)Y\\ &\quad -2ng((\nabla_Z h')X,Y).\end{aligned} \tag{3.4}$$

Now, using (2.6) and (2.10) in (3.4) we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2n(k+1)\eta(Y)(g(X,Z) - \eta(X)\eta(Z)\\ &\quad +g(h'X,Z)) + 2n(k+1)\eta(X)(g(Y,Z) - \eta(Y)\eta(Z)\\ &\quad +g(h'Y,Z)) + 2ng(h'Z + h'^2 Z, X)\eta(Y)\\ &\quad +2n\eta(X)g(h'Z + h'^2 Z, Y).\end{aligned} \tag{3.5}$$

Making use of (3.5) in (3.1) we get after simplification

$$C(X,Y)Z = 2n(k+2)(g(h'X,Z)\eta(Y) - g(h'Y,Z)\eta(X)) \tag{3.6}$$

Now from (3.6), we observe that in a $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1}, the Cotton tensor C satisfies $C(X,Y)\xi = 0$ for all vector fields X, Y on M^{2n+1}. Thus we state the following:

Proposition 3.2. *A $(k,\mu)'$-almost Kenmotsu manifold is always ξ-Cotton flat.*

Now if the Cotton tensor C vanishes identically on M^{2n+1}, then from (1.1) we can say that the conformal curvature tensor is harmonic and therefore, from Corollary 3.3 of [19] we get the following:

Proposition 3.3. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} is Cotton flat if and only if it is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

We now discuss about $(k,\mu)'$-almost Kenmotsu manifolds admitting parallel Cotton tensor, i.e., $\nabla C = 0$ holds on M^{2n+1}.

Differentiating (3.6) covariantly along any vector field W, we get

$$\begin{aligned} \nabla_W C(X,Y)Z &= 2n(k+2)((\nabla_W g(h'X,Z))\eta(Y) + g(h'X,Z)\nabla_W \eta(Y) \\ &\quad -(\nabla_W g(h'Y,Z))\eta(X) - g(h'Y,Z)\nabla_W \eta(X)). \end{aligned}$$

Now, using (2.4), (2.6), (2.10) and (3.6) in the above equation, we infer that

$$\begin{aligned} (\nabla_W C)(X,Y)Z &= \nabla_W C(X,Y)Z - C(\nabla_W X,Y)Z - C(X,\nabla_W Y)Z \\ &\quad -C(X,Y)\nabla_W Z \\ &= 2n(k+2)\{-\eta(Y)\eta(Z)g(h'W,X) + g(h'X,Z)(g(W,Y) \\ &\quad -\eta(W)\eta(Y) + g(h'W,Y)) + \eta(X)\eta(Z)g(h'W,Y) \\ &\quad -g(h'Y,Z)(g(W,X) - \eta(W)\eta(X) + g(h'W,X)) \\ &\quad +(k+1)(\eta(Y)\eta(Z)(g(W,X) - \eta(W)\eta(X)) \\ &\quad +\eta(X)\eta(Z)(-g(W,Y) + \eta(W)\eta(Y)))\}. \end{aligned}$$

Consider $\nabla C = 0$ and substituting $X = Z = \xi$ in the foregoing equation yields

$$2n(k+2)\{g(h'W,Y) - (k+1)(g(W,Y) - \eta(W)\eta(Y))\} = 0,$$

which implies either $k = -2$ or

$$g(h'W,Y) - (k+1)(g(W,Y) - \eta(W)\eta(Y)) = 0.$$

Case 1. If $k = -2$, then from $\lambda^2 = -k-1$ we get $\lambda^2 = 1$. Without loss of generality we assume that $\lambda = -1$.

Now letting $X,\ Y,\ Z \in [\lambda]'$ and noticing that $k = -2$, $\lambda = -1$, from Lemma 3.1 we have

$$R(X_\lambda, Y_\lambda)Z_\lambda = 0,$$

and

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = -4(g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}),$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$ it follows that $K(X, \xi) = -4$ for any $X \in [-\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [\lambda]'$. Again we see that $K(X, Y) = -4$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X, Y \in [\lambda]'$. As is shown in [6] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature tensor field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = -1$, then the two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case 2. If $g(h'W, Y) - (k+1)(g(W, Y) - \eta(W)\eta(Y)) = 0$, then substituting the value of $g(h'W, Y)$ obtained from (2.7) we get

$$S(W, Y) = -2n(k+2)g(W, Y) + 4n(k+1)\eta(W)\eta(Y). \tag{3.7}$$

Tracing (3.7) we get $r = 2n(k - 4n - 2nk)$ and equating it with the given value of $r = 2n(k - 2n)$ yields $k = -1$ which is a contradiction to the fact that $k < -1$ for a $(k, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$.
Hence we state the following:

Theorem 3.1. *A $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$ is Cotton parallel if and only if M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

We now define

Definition 3.2. *A $(k, \mu)'$-almost Kenmotsu manifolds M^{2n+1} is said to be Cotton semisymmetric if the Cotton tensor C satisfies $R \cdot C = 0$ on M^{2n+1}, where R is the Riemann curvature tensor.*

Let M^{2n+1} be Cotton semisymmetric. Therefore, $(R(X, Y) \cdot C)(U, V)W = 0$ for any vector fields $X,\ Y,\ U,\ V$ and W. Then we have

$$C(R(X, Y)U, V)W + C(U, R(X, Y)V)W + C(U, V)R(X, Y)W = 0. \tag{3.8}$$

Using (3.6) in (3.8) we obtain

$$\begin{aligned}
&2n(k+2)(g(h'R(X, Y)U, W)\eta(V) - g(h'V, W)\eta(R(X, Y)U)) \\
&+2n(k+2)(g(h'U, W)\eta(R(X, Y)V) - g(h'R(X, Y)V, W)\eta(U))
\end{aligned}$$

$$+2n(k+2)(g(h'U,R(X,Y)W)\eta(V)-g(h'V,R(X,Y)W)\eta(U))$$
$$=0. \tag{3.9}$$

Substituting $U=\xi$ in the foregoing equation and using (2.8), we obtain

$$2n(k+2)g(k\{\eta(Y)h'X-\eta(X)h'Y\}-2\{\eta(Y)h'^2X-\eta(X)h'^2Y\},W)\eta(V)$$
$$-2n(k+2)g(h'R(X,Y)V,W)-2n(k+2)g(h'V,R(X,Y)W)=0.$$

Now replacing W by ξ in the above equation and using (2.8), we infer

$$2n(k+2)g(h'V,k\{\eta(Y)X-\eta(X)Y\}-2\{\eta(Y)h'X-\eta(X)h'Y\})=0. \tag{3.10}$$

Using (2.4) in (3.10) and then substituting $Y=\xi$, after simplification we have

$$2n(k+2)(2(k+1)\{g(X,V)-\eta(X)\eta(V)\}+kg(h'V,X))=0. \tag{3.11}$$

We now obtain the value of $g(h'V,X)$ from (2.7) and then using it in (3.11) we get

$$2n(k+2)(-\frac{k}{2n}S(V,X)+(k+2)g(V,X)+(k+1)(k-2)\eta(V)\eta(X))=0, \tag{3.12}$$

which implies that either $k=-2$ or

$$S(V,X)=\frac{2n(k+2)}{k}g(V,X)+\frac{2n(k+1)(k-2)}{k}\eta(V)\eta(X).$$

In the first case as discussed earlier in Case 1 of Theorem 3.1, M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

In the second case, tracing the (3.12) we obtain $r=\frac{2n}{k}(k^2+2nk+2n-1)$. Also, in a $(k,\mu)'$-almost Kenmotsu manifold the scalar curvature r is given by $r=2n(k-2n)$. Equating these two value of r, we get $k=\frac{1-2n}{4n}$. For $n=1$, $k=-\frac{1}{4}$ and as n increases, the value of k is approaching towards $-\frac{1}{2}$ and hence $-\frac{1}{2}<k\leq-\frac{1}{4}$. This contradicts the fact that $k\leq-1$.

Hence we can state the following:

Theorem 3.2. *A $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1} is Cotton semisymmetric if and only if M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Now if the Cotton tensor C satisfies the condition $Q(g,C)=0$, then we have $Q(g,C)(U,V,W;X,Y)=0$ for all vector fields $U,\ V,\ W,\ X$ and Y on M^{2n+1}. Thus we have from (1.3)

$$C((X\wedge_g Y)U,V)W+C(U,(X\wedge_g Y)V)W+C(U,V)(X\wedge_g Y)W=0.$$

Now using (1.2) in the foregoing equation yields

$$g(Y,U)C(X,V)W-g(X,U)C(Y,V)W$$
$$+g(Y,V)C(U,X)W-g(X,V)C(U,Y)W$$
$$+g(Y,W)C(U,V)X-g(X,W)C(U,V)Y=0. \tag{3.13}$$

Putting $Y = W = \xi$ in the foregoing equation and using (3.6) yields that $C(U,V)X = 0$ and hence from Prop. 3.2 we get M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. From the above discussion we have the following:

Theorem 3.3. *In a $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1}, the Cotton tensor C satisfies the condition $Q(g,C) = 0$ if and only if M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Now as a generalization of the notion of Cotton semisymmetry, we define

Definition 3.3. *A $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1} is said to be Cotton pseudo-symmetric if there exist a smooth function $f_C : M \to \mathbb{R}$ such that $R \cdot C = f_C Q(g,C)$ holds on M^{2n+1}.*

In particular, a Cotton pseudo-symmetric manifold with $f_C = 0$ reduces to a Cotton semisymmetric manifold. We now characterize Cotton pseudo-symmetric $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1}, i.e., M^{2n+1} satisfies

$$(R(X,Y)\cdot C)(U,V)W = f_C Q(g,C)(U,V,W;X,Y)$$

for any vector fields X, Y, U, V and W on M^{2n+1}.
In view of (3.9) and (3.13), it follows from that

$$\begin{aligned} & 2n(k+2)(g(h'R(X,Y)U,W)\eta(V) - g(h'V,W)\eta(R(X,Y)U)) \\ & +2n(k+2)(g(h'U,W)\eta(R(X,Y)V) - g(h'R(X,Y)V,W)\eta(U)) \\ & +2n(k+2)(g(h'U,R(X,Y)W)\eta(V) - g(h'V,R(X,Y)W)\eta(U)) \\ = \; & f_C(g(Y,U)C(X,V)W - g(X,U)C(Y,V)W \\ & +g(Y,V)C(U,X)W - g(X,V)C(U,Y)W \\ & +g(Y,W)C(U,V)X - g(X,W)C(U,V)Y). \end{aligned}$$

Substituting $W = \xi$ in the above equation and using Prop. 3.1, we obtain

$$\begin{aligned} & 2n(k+2)(g(h'U,R(X,Y)\xi)\eta(V) - g(h'V,R(X,Y)\xi)\eta(U)) \\ = \; & f_C(\eta(Y)C(U,V)X - \eta(X)C(U,V)Y). \end{aligned}$$

Now using (2.8) and (3.6) in the foregoing equation we get

$$\begin{aligned} & 2n(k+2)(g(h'U, k\{\eta(Y)X - \eta(X)Y\} - 2\{\eta(Y)h'X - \eta(X)h'Y\})\eta(V) \\ & -g(h'V, k\{\eta(Y)X - \eta(X)Y\} - 2\{\eta(Y)h'X - \eta(X)h'Y\})\eta(U)) \\ = \; & f_C(2n(k+2)\eta(Y)\{g(h'U,X)\eta(V) - g(h'V,X)\eta(U)\} \\ & -2n(k+2)\eta(X)\{g(h'U,Y)\eta(V) - g(h'V,Y)\eta(U)\}). \end{aligned} \tag{3.14}$$

Setting $U = \xi$ in (3.14), we obtain

$$\begin{aligned} & 2n(k+2)(-k\{\eta(Y)g(h'V,X) - \eta(X)g(h'V,Y)\} \\ & +2\{\eta(Y)g(h'V,h'X) - \eta(X)g(h'V,h'Y)\}) \\ = \; & f_C(-2n(k+2)\eta(Y)g(h'V,X) + 2n(k+2)\eta(X)g(h'V,Y)). \end{aligned} \tag{3.15}$$

Now using (2.4) in (3.15), we have

$$\begin{aligned} &2n(k+2)(k-f_C)(\eta(X)g(h'V,Y)-\eta(Y)g(h'V,X)) \\ &+4n(k+1)(k+2)(\eta(X)g(V,Y)-\eta(Y)g(V,X))=0. \end{aligned} \tag{3.16}$$

Replacing X by ξ in (3.16), we get

$$2n(k+2)(k-f_C)g(h'V,Y)+4n(k+1)(k+2)(g(V,Y)-\eta(Y)\eta(V))=0.$$

Now substituting the value of $2ng(h'V,Y)$ from (2.7), we obtain

$$\begin{aligned} &(k+2)(-(k-f_C)S(V,Y)-\{2n(k-f_C)-4n(k+1)\}g(Y,V) \\ &+\{2n(k+1)(k-f_C)-4n(k+1)\}\eta(Y)\eta(V))=0. \end{aligned}$$

We now discuss it in the following cases.

Case 1. If $k=-2$, then as discussed earlier, M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

Case 2. If

$$\begin{aligned} &-(k-f_C)S(V,Y)-\{2n(k-f_C)-4n(k+1)\}g(Y,V) \\ &+\{2n(k+1)(k-f_C)-4n(k+1)\}\eta(Y)\eta(V)=0, \end{aligned} \tag{3.17}$$

then we consider the following two subcases:

(i). If $f_C=k$, then from the above equation we see that

$$4n(k+1)(g(Y,V)-\eta(Y)\eta(V))=0,$$

which implies $k=-1$, a contradiction.

(ii). If $f_C\neq k$, then from (3.17) we can write

$$\begin{aligned} S(V,Y) &= \frac{-2n(k-f_C)+4n(k+1)}{k-f_C}g(V,Y) \\ &\quad+\frac{2n(k+1)(k-f_C)-4n(k+1)}{k-f_C}\eta(V)\eta(Y). \end{aligned}$$

Tracing the previous equation yields $r=\frac{2n}{k-f_C}(k^2+2nk+4n-kf_C+2nf_C)$. Now equating it with $r=2n(k-2n)$ we obtain $k=-1$, a contradiction. Hence, we are in a position to state the following:

Theorem 3.4. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} is Cotton pseudo-symmetric if and only if M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Remark 3.1. *If we consider $f_C=0$ in the above theorem, then we obtain Theorem 3.2. So, Theorem 3.4 generalizes Theorem 3.2.*

Example 3.1. *In [4], the authors presented an example of a 5-dimensional $(k,\mu)'$-almost Kenmotsu manifold with $k=-2$ and $\mu=-2$. Since $k=-2$, from (3.6) we can say that the Cotton tensor C vanishes and M^5 is locally isometric to $\mathbb{H}^3(-4)\times\mathbb{R}^2$. Hence, all the Theorems are trivially satisfied by this example.*

4. Conclusion

In this paper, we have studied $(k,\mu)'$-almost Kenmotsu manifolds with Cotton flatness, Cotton Parallelity, Cotton semisymmetry, $Q(g,C)=0$ and Cotton pseudo-symmetry. Finally, we conclude from all the Propositions and Theorems proved here and Corollary 3.3 of [19] that

In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, the following conditions are equivalent:

(1) M^{2n+1} is Cotton flat,
(2) The Ricci tensor is of Codazzi type,
(3) The conformal curvature tensor is harmonic,
(4) M^{2n+1} is Cotton parallel,
(5) M^{2n+1} is Cotton semisymmetric,
(6) M^{2n+1} satisfies $Q(g,C)=0$,
(7) M^{2n+1} is Cotton pseudo-symmetric.

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