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Research Article

S-spectra and S-essential pseudospectra of the diagonal block operator matrices

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Abstract

In this article, the relationships between the S-spectra, the S-spectral radius, the ϵ-S-essential pseudospectra, and the ϵ-S-essential pseudospectral radius of the diagonal block operator matrices in the direct sum of Banach spaces and their block coordinate operators are studied. Then, the results are supported by applications.

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1. Introduction

Recently, there have been a lot of interest in the characterizing of the essential spectra, as there are plenty of practical applications that help scientists to deal with information overload.

The theory of the essential spectra of linear operators in Banach spaces, which has numerous applications in many parts of mathematics and physics including function theory, matrix theory, differential and integral equations, complex analysis and control theory is one of the modern parts of the spectral analysis.

The original definition of the essential spectrum has given by Weyl [16] around 1909. He defined the essential spectrum for a self-adjoint operator T on a Hilbert space as the set of all points of the spectrum of T that are not isolated eigenvalues of finite algebraic multiplicity. He proved that the addition of a compact operator to T does not affect the essential spectrum. Whether T is bounded or is not on a Banach space $\mathfrak{X}$, there are many ways to define the essential spectrum. Most of them are enlargement of the continuous spectrum. In the literature, we can find several definitions of the essential spectrum, which coincide the self-adjoint operators on Hilbert spaces (see, e.g [3, 13]).

Also, the concept of essential spectra was introduced and studied by many mathematicians. We can refer to the contributions of Weyl and his collaborators (see, e.g. [4, 8, 11, 15, 16]. Moreover, further important characterizations concerning essential spectra and their applications to transport operators are in [1, 4].

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It is very important to determine the spectra of linear bounded operators in mathematical physics in particular in quantum mechanics which is both relativistic and nonrelativistic. However, it is necessary to express the importance of non-selfadjoint operators and their spectra. The growing interest in non-Hermitian quantum mechanics, nonselfadjoint differential operators and in generally nonnormal phenomena has increased the importance of nonselfadjoint operators and pseudospectral theory. Our aim in this article is to show that there are ways to determine spectrum of some linear operators. Thus, we fill an important gap in the computational spectral theory. The arithmetic operations are not certain, when we compute the spectrum using a computer. So we can get the real solution of a lightly perturbed problem. The problem above does not occur, when we consider the pseudospectrum in bounded case.

In the mathematical literature, it is known that the spectral theory of linear operators in direct sum of Banach spaces should be examined in order to solve many physical problems in life sciences. These and other similar reasons led to the emergence of the topic examined in the current paper.

There are numerous physical problems arising in the modelling of proceses the physics of rigid bodies, multiparticle quantum mechanics and quantum field theory. These problems support to study the theory of linear operators in the direct sum of Banach spaces (see [6, 9, 14, 17] and references in them).

In this article, one of the basic questions consists in characterizing the S-spectra and the ϵ-S-essential pseudospectra of all the diagonal block operator matrices in the direct sum of Banach spaces. Namely, we show some relationships between the S-spectra, the S-spectral radius, the ϵ-S-essential pseudospectra, and the ϵ-S-essential pseudospectral radius of the diagonal block operator matrices in the direct sum of Banach spaces and their block coordinate operators (see, Theorem 3.1, 3.2 and 3.3). Finally, we give some remarkable examples as applications of our results.

2. Auxiliary definitions and results

In this section, we will give auxiliary definitions and results that we will need later.

Definition 2.1. [10] The infinite direct sum of Banach spaces $\mathfrak{B}_n,\ n \geq 1$ in the sense of $l_p,\ 1 \leq p < \infty$ and the infinite direct sum of linear densely defined closed operators A_n in $\mathfrak{B}_n,\ n \geq 1$ are defined as

$$\mathfrak{B} = \left(\bigoplus_{n=1}^{\infty} \mathfrak{B}_n\right)_p = \left\{ x = (x_n) : x_n \in \mathfrak{B}_n, n \geq 1, \|x\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|_{\mathfrak{B}_n}^p\right)^{1/p} < \infty \right\}$$

and

$$A = \bigoplus_{n=1}^{\infty} A_n,\ \ A : D(A) \subset \mathfrak{B} \to \mathfrak{B},$$

$$D(A) = \left\{x = (x_n) \in \mathfrak{B} : x_n \in D(A_n),\ n \geq 1,\ Ax = (A_n x_n) \in \mathfrak{B}\right\},$$

respectively.

Throughout the current paper, the norms $\|\cdot\|_p$ in $\mathfrak{B}$ and $\|\cdot\|_{\mathfrak{B}_n}$ in $\mathfrak{B}_n,\ n \geq 1$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_n,\ n \geq 1$, respectively. Also, the classes of linear bounded operators, compact operator, and linear closed densely defined operators from any Banach space $\mathfrak{X}_1$ to another Banach space $\mathfrak{X}_2$ are denoted by $\mathfrak{L}(\mathfrak{X}_1, \mathfrak{X}_2)$, $C(\mathfrak{X}_1, \mathfrak{X}_2)$, and $\mathfrak{C}(\mathfrak{X}_1, \mathfrak{X}_2)$, respectively. If $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}$, they are denoted by $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$, $C(\mathfrak{X}) = C(\mathfrak{X}, \mathfrak{X})$, and $\mathfrak{C}(\mathfrak{X}) = \mathfrak{C}(\mathfrak{X}, \mathfrak{X})$. The identity operator in a Banach space $\mathfrak{X}$ is denoted by I.

With the use of the techniques of the Banach spaces $l_p,\ 1 \leq p < \infty$ and the operator theory we can obtain the following proposition (see [7]).

Theorem 2.2. *Let $A_n \in \mathfrak{L}(\mathfrak{B}_n)$, $n \geq 1$ and $A = \bigoplus_{n=1}^{\infty} A_n : \mathfrak{B} \to \mathfrak{B}$. In order to $A \in \mathfrak{L}(\mathfrak{B})$ the necessary and sufficient condition is $\sup_{n\geq 1} \|A_n\| < \infty$. Moreover, in the case of $A \in \mathfrak{L}(\mathfrak{B})$, the norm of A is of the form $\|A\| = \sup_{n\geq 1} \|A_n\|$.*

By the definition of compactness of an operator in [2], we have that if $A \in C(\mathfrak{B})$, then $A_n \in C(\mathfrak{B}_n)$ for $n \geq 1$.

Now, let us give the following theorem about the compactness of the operator A.

Theorem 2.3. *Let $A_n \in C(\mathfrak{B}_n)$ for each $n \geq 1$ and $A = \bigoplus_{n=1}^{\infty} A_n : \mathfrak{B} \to \mathfrak{B}$. $A \in C(\mathfrak{B})$ if and only if*

$$\lim_{n\to\infty} \|A_n\| = 0.$$

Definition 2.4. [2] The spectrum and resolvent sets of an operator $T \in \mathfrak{C}(\mathcal{X})$ in any Banach space $\mathcal{X}$ are defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not have an inverse in } \mathfrak{L}(\mathcal{X})\}$$

and

$$\rho(T) = \mathbb{C} \setminus \sigma(T),$$

respectively.

Now, let us give some definitions from [5].

Definition 2.5. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces, $T \in \mathfrak{C}(\mathcal{X}_1, \mathcal{X}_2)$ and $S \in \mathfrak{L}(\mathcal{X}_1, \mathcal{X}_2)$ such that $T \neq S$ and $S \neq 0$. The S-resolvent set of the operator T is defined as

$$\rho_S(T) = \left\{\lambda \in \mathbb{C} : (\lambda S - T)^{-1} \in \mathfrak{L}(\mathcal{X}_2, \mathcal{X}_1)\right\}.$$

The S-resolvent operator of the operator T is defined as

$$R_S(\lambda, T) = (\lambda S - T)^{-1}.$$

The S-spectrum set of the operator T is denoted by

$$\sigma_S(T) = \mathbb{C} \setminus \rho_S(T).$$

In the case of $T \in \mathfrak{L}(\mathcal{X}_1, \mathcal{X}_2)$, the $S-$spectral radius of the operator T is defined as

$$r_S(T) = \sup\{|\lambda| : \lambda \in \sigma_S(T)\}.$$

Definition 2.6. Let $\mathcal{X}$ be a Banach space, $T \in \mathfrak{C}(\mathcal{X})$ and $S \in \mathfrak{L}(\mathcal{X})$ such that $T \neq S$ and $S \neq 0$. We define the following set:
(i) The S-point spectrum of T is denoted by

$$\sigma_{p,S}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ is not one-to-one}\}.$$

(ii) The S-continuous spectrum of T is denoted by

$$\sigma_{c,S}(T) = \left\{\lambda \in \mathbb{C} : \lambda S - T \text{ is one-to-one}, \ \overline{(\lambda S - T)(D(T))} = \mathcal{X}, \text{ and } (\lambda S - T)^{-1} \text{ is unbounded}\right\}.$$

(iii) The S-residual spectrum of T is denoted by

$$\sigma_{r,S}(T) = \left\{\lambda \in \mathbb{C} : \lambda S - T \text{ is one-to-one}, \ \overline{(\lambda S - T)(D(T))} \neq \mathcal{X}\right\}.$$

Definition 2.7. Let $\mathcal{X}$ be a Banach space, $T \in \mathfrak{C}(\mathcal{X})$ and $S \in \mathfrak{L}(\mathcal{X})$ such that $T \neq S,\ S \neq 0$ and $\epsilon > 0$. The ϵ-S-essential pseudospectrum set of the operator T is defined as

$$\sigma_{S,\epsilon}(T) = \sigma_S(T) \cup \left\{ \lambda \in \rho_S(T) : \|R_S(\lambda, T)\| > \frac{1}{\epsilon} \right\}.$$

In the case of $T \in \mathfrak{L}(\mathcal{X})$, the non-negative number

$$r_{S,\epsilon}(T) = \sup\{|\lambda| : \lambda \in \sigma_{S,\epsilon}(T)\}$$

is called the ϵ-S-essential pseudospectral radius of the operator T.

3. S-spectra and ϵ-S-essential pseudospectra of the diagonal block operator matrices

In this section, we will investigate the S-spectra, the S-spectral radius, the ϵ-S-essential pseudospectra, and the ϵ-S-essential pseudospectral radius of the diagonal block operator matrices in the infinite direct sum of Banach spaces.

Let us present our main results.

Theorem 3.1. *Let $\mathfrak{B}_n$ be a Banach space, $A_n \in \mathfrak{C}(\mathfrak{B}_n)$ and $S_n \in \mathfrak{L}(\mathfrak{B}_n)$ for $n \geq 1$. Moreover, let $\mathcal{B} = \bigoplus_{n=1}^{\infty} \mathfrak{B}_n$ be the direct sum of $\mathcal{B}_n$, $n \geq 1$, $A = \bigoplus_{n=1}^{\infty} A_n \in \mathfrak{C}(\mathcal{B})$, $\sup_{n\geq 1} \|S_n\| < \infty$ and $S := \bigoplus_{n=1}^{\infty} S_n$. Then, the parts of S-spectrum, the S-spectrum, and the S-resolvent sets of the operator A are of the forms*

$$\sigma_{p,S}(A) = \bigcup_{n=1}^{\infty} \sigma_{p,S_n}(A_n),$$

$$\begin{aligned}
\sigma_{c,S}(A) = &\left\{ \left(\bigcup_{n=1}^{\infty} \sigma_{p,S_n}(A_n) \right)^c \cap \left(\bigcup_{n=1}^{\infty} \sigma_{r,S_n}(A_n) \right)^c \cap \left(\bigcup_{n=1}^{\infty} \sigma_{c,S_n}(A_n) \right) \right\} \\
&\cup \left\{ \lambda \in \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n\geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty \right\}, \\
\sigma_{r,S}(A) = &\left(\bigcup_{n=1}^{\infty} \sigma_{p,S_n}(A_n) \right)^c \cap \left(\bigcup_{n=1}^{\infty} \sigma_{r,S_n}(A_n) \right), \\
\sigma_S(A) = &\bigcup_{n=1}^{\infty} \sigma_{S_n}(A_n) \cup \left\{ \lambda \in \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n\geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty \right\}, \\
\rho_S(A) = &\left\{ \lambda \in \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n\geq 1} \|R_{S_n}(\lambda, A_n)\| < \infty \right\},
\end{aligned}$$

respectively.

In the case of $A_n \in \mathfrak{L}(\mathfrak{B}_n)$, $n \geq 1$ and $A \in \mathfrak{L}(\mathfrak{B})$, the S-spectral radius of the operator A is of the form

$$r_S(A) = \sup_{n\geq 1} r_{S_n}(A_n).$$

In the special case, if the number of the operators S_n and A_n, $1 \leq n \leq m$, $m \in \mathbb{N}$ is finite, the following equalities hold

$$\sigma_S(A) = \bigcup_{n=1}^{m} \sigma_{S_n}(A_n),$$

$$\rho_S(A) = \bigcap_{n=1}^{m} \rho_{S_n}(A_n).$$

In the case of $A_n \in \mathfrak{L}(\mathfrak{B}_n)$, $1 \leq n \leq m$ *and* $A \in \mathfrak{L}(\mathfrak{B})$, *the* $S-$*spectral radius of the operator* A *is of the form*

$$r_S(A) = \max_{1 \leq n \leq m} r_{S_n}(A_n).$$

Proof. Note that the validity of the first relation is clear. Also, it is easy to prove the fifth equality using Theorem 2.2.

Now, let us prove the second relation on the $S-$continuous spectrum. By the definition of the $S-$continuous spectrum, $\lambda S - A$ is one-to-one operator, $Im(\lambda S - A) \neq \mathcal{B}$ and $Im(\lambda S - A)$ is dense in $\mathcal{B}$. Because for any $n \geq 1$ the operator $\lambda S_n - A_n$ is one-to-one operator in $\mathcal{B}_n$, there is $m \in \mathbb{N}$ such that $Im(\lambda S_m - A_m) \neq \mathcal{B}_m$ and for any $n \geq 1$ the linear manifold $Im(\lambda S_n - A_n)$ is dense in $\mathcal{B}_n$ or for each $n \geq 1$, $\lambda \in \rho_S(A_n)$ but $\sup_{n \geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty$. Consequently, we have

$$\lambda \in \left\{ \left(\bigcap_{n=1}^{\infty} [\sigma_{c,S_n}(A_n) \cup \rho_{S_n}(A_n)] \right) \cap \left(\bigcup_{n=1}^{\infty} \sigma_{c,S_n}(A_n) \right) \right\}$$
$$\cup \left\{ \lambda \in \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n \geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty \right\}.$$

Conversely, assume that the above relation is satisfied for the point $\lambda \in \mathbb{C}$. Consequently, for any $n \geq 1$, it is either

$$\lambda \in \sigma_{c,S_n}(A_n) \cup \rho_{S_n}(A_n)$$

or

$$\lambda \in \left\{ \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n \geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty \right\}$$

and there is $m \in \mathbb{N}$ such that $\lambda \in \sigma_{c,S_m}(A_m)$. Namely, for any $n \geq 1$, $\lambda S_n - A_n$ is one-to-one operator, $\overline{Im(\lambda S_n - A_n)} = \mathcal{B}_n$ and $Im(\lambda S_n - A_n) \neq \mathcal{B}_n$. Hence, we have that the operator $\lambda S - A$ is one-to-one operator, $\overline{Im(\lambda S - A)} = \mathcal{B}$ and $Im(\lambda S - A) \neq \mathcal{B}$. Therefore, we get $\lambda \in \sigma_{c,S}(A)$.

On the other hand, the simple calculations show that

$$\left(\bigcap_{n=1}^{\infty} [\sigma_{c,S_n}(A_n) \cup \rho_{S_n}(A_n)] \right) \cap \left(\bigcup_{n=1}^{\infty} \sigma_{c,S_n}(A_n) \right)$$
$$= \left(\bigcup_{n=1}^{\infty} \sigma_{p,S_n}(A_n) \right)^c \cap \left(\bigcup_{n=1}^{\infty} \sigma_{r,S_n}(A_n) \right)^c \cap \left(\bigcup_{n=1}^{\infty} \sigma_{c,S_n}(A_n) \right).$$

By using the same technique, we can prove the validity of the third relation of the theorem.

Now, let us prove that

$$r_S(A) = \sup_{n \geq 1} r_{S_n}(A_n).$$

We have already proved that the S-spectrum of the operator A is of the form

$$\sigma_S(A) = \bigcup_{n=1}^{\infty} \sigma_{S_n}(A_n) \cup \left\{ \lambda \in \bigcap_{n=1}^{\infty} \rho_{S_n}(A_n) : \sup_{n \geq 1} \|R_{S_n}(\lambda, A_n)\| = \infty \right\}.$$

Since $\sigma_{S_n}(A_n) \subset \sigma_S(A)$, $n \geq 1$, we have

$$r_{S_n}(A_n) \leq r_S(A).$$

Consequently,

$$\sup_{n \geq 1} r_{S_n}(A_n) \leq r_S(A).$$

On the contrary, in the case of $\sup\limits_{n\geq 1} r_{S_n}(A_n) < r_S(A)$, we must obtain at least one element $\lambda_* \in \sigma_S(A)$ such that

$$\sup_{n\geq 1} r_{S_n}(A_n) < |\lambda_*| \leq r_S(A).$$

Thus, there is an integer $n_* \geq 1$ such that

$$\lambda_* \in \sigma_{S_{n*}}(A_{n_*}).$$

Hence, we have

$$r_{S_{n*}}(A_{n_*}) < |\lambda_*|.$$

However, this is a contradiction. Thus, we have that

$$r_S(A) = \sup_{n\geq 1} r_{S_n}(A_n).$$

The similar results can be proved when the number of the operators S_n and A_n, $1 \leq n \leq m,\ m \in \mathbb{N}$ is finite. □

In the case of $S_n = I_n,\ n \geq 1$, the similar results have been obtained in [12].

Theorem 3.2. *Let $\mathfrak{B}_n$ be a Banach space, $A_n \in \mathfrak{C}(\mathfrak{B}_n)$ and $S_n \in \mathfrak{L}(\mathfrak{B}_n)$ for $n \geq 1$. Moreover, let $\mathcal{B} = \bigoplus\limits_{n=1}^{\infty} \mathfrak{B}_n$ be the direct sum of $\mathcal{B}_n$, $n \geq 1$, $A = \bigoplus\limits_{n=1}^{\infty} A_n \in \mathfrak{C}(\mathcal{B})$, $\sup\limits_{n\geq 1} \|S_n\| < \infty$ and $S := \bigoplus\limits_{n=1}^{\infty} S_n$. Then, for each $\epsilon > 0$ the ϵ-S-essential pseudospectrum set of the operator A is of the form*

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n).$$

In the case of $A_n \in \mathfrak{L}(\mathfrak{B}_n)$, $n \geq 1$ and $A \in \mathfrak{L}(\mathfrak{B})$, the ϵ-S-essential pseudospectral radius of the operator A is of the form

$$r_{S,\epsilon}(A) = \sup_{n\geq 1} r_{S_n,\epsilon}(A_n).$$

Proof. By Theorem 2.2 it is known that

$$\|(\lambda S - A)^{-1}\| = \sup_{n\geq 1} \|(\lambda S_n - A_n)^{-1}\|.$$

Let $\lambda \in \sigma_{S,\epsilon}(A)$. Then, for any $\epsilon > 0$ we have

$$\sup_{n\geq 1} \|(\lambda S_n - A_n)^{-1}\| > \frac{1}{\epsilon}.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\|(\lambda S_{n_0} - A_{n_0})^{-1}\| > \frac{1}{\epsilon}.$$

This means that

$$\lambda \in \sigma_{S_{n_0},\epsilon}(A_{n_0}).$$

Consequently,

$$\sigma_{S,\epsilon}(A) \subset \bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n).$$

Conversely, if $\lambda \in \sigma_{S_{n_0},\epsilon}(A_{n_0})$ for any $n_0 \in \mathbb{N}$, then for $\epsilon > 0$ we have

$$\|(\lambda S_{n_0} - A_{n_0})^{-1}\| > \frac{1}{\epsilon}.$$

From the last relation we get

$$\sup_{n\geq 1} \| (\lambda S_n - A_n)^{-1} \| > \frac{1}{\epsilon}.$$

Then, by Theorem 2.2 it is established that

$$\bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n) \subset \sigma_{S,\epsilon}(A).$$

Finally, for $\epsilon > 0$ we obtain

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n).$$

Now, let us prove $r_{S,\epsilon}(A) = \sup_{n\geq 1} r_{S_n,\epsilon}(A_n)$. We have already proved that the ϵ-S-essential pseudospectrum of the operator A is of the form

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n).$$

Since $\sigma_{S_n,\epsilon}(A_n) \subset \sigma_{S,\epsilon}(A)$, $n \geq 1$, we have

$$r_{S_n,\epsilon}(A_n) \leq r_{S,\epsilon}(A).$$

Consequently,

$$\sup_{n\geq 1} r_{S_n,\epsilon}(A_n) \leq r_{S,\epsilon}(A).$$

On the contrary, in the case of $\sup_{n\geq 1} r_{S_n,\epsilon}(A_n) < r_{S,\epsilon}(A)$, we must obtain at least one element $\lambda_* \in \sigma_{S,\epsilon}(A)$ such that

$$\sup_{n\geq 1} r_{S_n,\epsilon}(A_n) < |\lambda_*| \leq r_{S,\epsilon}(A).$$

In this case, there is an integer $n_* \geq 1$ such that

$$\lambda_* \in \sigma_{S_{n_*},\epsilon}(A_{n_*}).$$

Hence, we have

$$r_{S_{n_*},\epsilon}(A_{n_*}) < |\lambda_*|.$$

However, this is a contradiction. Thus, we have that

$$r_{S,\epsilon}(A) = \sup_{n\geq 1} r_{S_n,\epsilon}(A_n).$$

□

In Theorems 3.1 and 3.2, even if the location of A and S blocks changes as desired, the results do not change. Thus, we can give the following theorem.

Theorem 3.3. *Let $f : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto function. Also, let $\mathfrak{B}_n$ be a Banach space, $A_n \in \mathfrak{C}(\mathfrak{B}_n)$ and $S_n \in \mathfrak{L}(\mathfrak{B}_n)$ for $n \geq 1$. Moreover, let $\mathcal{B} = \bigoplus_{n=1}^{\infty} \mathfrak{B}_n$ be the direct sum of $\mathcal{B}_n$, $n \geq 1$, $A = \bigoplus_{n=1}^{\infty} A_{f(n)} \in \mathfrak{C}(\mathcal{B})$, $\sup_{n\geq 1} \|S_{f(n)}\| < \infty$ and $S := \bigoplus_{n=1}^{\infty} S_{f(n)}$. Then, for each $\epsilon > 0$ the ϵ-S-essential pseudospectrum set is of the form*

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \sigma_{S_n,\epsilon}(A_n).$$

In the case of $A_n \in \mathfrak{L}(\mathfrak{B}_n)$, $n \geq 1$ and $A \in \mathfrak{L}(\mathfrak{B})$, the ϵ-S-essential pseudospectral radius of the operator A is of the form

$$r_{S,\epsilon}(A) = \sup_{n\geq 1} r_{S_n,\epsilon}(A_n).$$

4. Applications

In this section, we will provide some examples as applications of our theorems.

Example 4.1. Let $\mathfrak{B}_n = \mathbb{C},\ n \geq 1$ one-dimensional Euclidian space, $\mathfrak{B} = \left(\bigoplus\limits_{n=1}^{\infty} \mathbb{C}\right)_p$, $1 \leq p < \infty$,

$$S_n = s_n I : \mathbb{C} \to \mathbb{C},\ s_n \neq 0,\ n \geq 1,\ (s_n) \in l_p(\mathbb{C}),\ 1 \leq p < \infty,$$
$$A_n = a_n I : \mathbb{C} \to \mathbb{C},\ a_n \neq 0,\ n \geq 1,\ (a_n) \in l_p(\mathbb{C}),\ 1 \leq p < \infty,$$

and $S = \bigoplus\limits_{n=1}^{\infty} S_n,\ A = \bigoplus\limits_{n=1}^{\infty} A_n$. In this case, $S,\ A \in \mathfrak{L}(\mathfrak{B})$. For $\lambda \in \mathbb{C},\ \lambda \neq \dfrac{a_n}{s_n}$ and $n \geq 1$ we have

$$(\lambda S_n - A_n)^{-1} = \frac{1}{\lambda s_n - a_n}$$

and

$$\| (\lambda S_n - A_n)^{-1} \| = \frac{1}{|\lambda s_n - a_n|}.$$

Consequently, for any $n \geq 1$ we obtain

$$\sigma_{S_n}(A_n) = \left\{\frac{a_n}{s_n}\right\} \text{ and } r_{S_n}(A_n) = \left|\frac{a_n}{s_n}\right|.$$

Hence, by Theorem 3.1 we have

$$\sigma_S(A) = \overline{\bigcup_{n=1}^{\infty} \left\{\frac{a_n}{s_n}\right\}} \text{ and } r_S(A) = \sup_{n\geq 1}\left|\frac{a_n}{s_n}\right|.$$

On the other hand, for any $n \geq 1$ and $\epsilon > 0$

$$\begin{aligned}\sigma_{S_n,\epsilon}(A_n) &= \sigma_{S_n}(A_n) \cup \left\{\lambda \in \rho_{S_n}(A_n) : \|R_{S_n}(\lambda, A_n)\| > \frac{1}{\epsilon}\right\}\\ &= \left\{\frac{a_n}{s_n}\right\} \cup \left\{\lambda \in \rho_{S_n}(A_n) : \frac{1}{|\lambda s_n - a_n|} > \frac{1}{\epsilon}\right\}\\ &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{a_n}{s_n}\right| < \frac{\epsilon}{|s_n|}\right\}\end{aligned}$$

and

$$r_{S_n,\epsilon}(A_n) = \sup\left\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \left|\lambda - \frac{a_n}{s_n}\right| < \frac{\epsilon}{|s_n|}\right\}.$$

Hence, by Theorem 3.2 for $\epsilon > 0$ we have

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty}\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{a_n}{s_n}\right| < \frac{\epsilon}{|s_n|}\right\}$$

and

$$r_{S,\epsilon}(A) = \sup_{n\geq 1}\sup\left\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \left|\lambda - \frac{a_n}{s_n}\right| < \frac{\epsilon}{|s_n|}\right\}.$$

Example 4.2. Let $\mathfrak{B}_n = \mathbb{C}^2,\ n \geq 1$ two-dimensional Euclidian space, $\mathfrak{B} = \left(\bigoplus\limits_{n=1}^{\infty} \mathbb{C}^2\right)_p$, $1 \leq p < \infty$ and

$$S_n : \mathbb{C}^2 \to \mathbb{C}^2,\ S_n = \begin{pmatrix} 0 & \alpha_n \\ \alpha_n & 0 \end{pmatrix},\ \alpha_n \in \mathbb{C},\ \alpha_n \neq 0,\ n \geq 1,\ \sup_{n\geq 1}|\alpha_n| < \infty,$$
$$A_n : \mathbb{C}^2 \to \mathbb{C}^2,\ A_n = \begin{pmatrix} 0 & \beta_n \\ \beta_n & 0 \end{pmatrix},\ \beta_n \in \mathbb{C},\ \beta_n \neq 0,\ n \geq 1,\ \sup_{n\geq 1}|\beta_n| < \infty.$$

Then, $S = \bigoplus_{n=1}^{\infty} S_n$ and $A = \bigoplus_{n=1}^{\infty} A_n$ are the infinite diagonal block operator matrices. For $\lambda \in \mathbb{C}$ and $n \geq 1$ we have

$$\lambda S_n - A_n = \begin{pmatrix} 0 & \lambda\alpha_n - \beta_n \\ \lambda\alpha_n - \beta_n & 0 \end{pmatrix}.$$

Thus, for $\lambda \neq \dfrac{\beta_n}{\alpha_n}$ and $n \geq 1$ we get

$$(\lambda S_n - A_n)^{-1} = \begin{pmatrix} 0 & \dfrac{1}{\lambda\alpha_n - \beta_n} \\ \dfrac{1}{\lambda\alpha_n - \beta_n} & 0 \end{pmatrix} \in \mathfrak{L}\left(\mathbb{C}^2\right)$$

and

$$\| (\lambda S_n - A_n)^{-1} \| = \frac{1}{|\lambda\alpha_n - \beta_n|}.$$

Consequently, for any $n \geq 1$ we obtain

$$\sigma_{S_n}(A_n) = \left\{\frac{\beta_n}{\alpha_n}\right\} \text{ and } r_{S_n}(A_n) = \left|\frac{\beta_n}{\alpha_n}\right|.$$

Hence, by Theorem 3.1 we have

$$\sigma_S(A) = \overline{\bigcup_{n=1}^{\infty} \left\{\frac{\beta_n}{\alpha_n}\right\}} \text{ and } r_S(A) = \sup_{n\geq 1}\left|\frac{\beta_n}{\alpha_n}\right|.$$

On the other hand, for any $n \geq 1$ and $\epsilon > 0$

$$\begin{aligned} \sigma_{S_n,\epsilon}(A_n) &= \sigma_{S_n}(A_n) \cup \left\{\lambda \in \rho_{S_n}(A_n) : \|R_{S_n}(\lambda, A_n)\| > \frac{1}{\epsilon}\right\} \\ &= \left\{\frac{\beta_n}{\alpha_n}\right\} \cup \left\{\lambda \in \rho_{S_n}(A_n) : \frac{1}{|\lambda\alpha_n - \beta_n|} > \frac{1}{\epsilon}\right\} \\ &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{\beta_n}{\alpha_n}\right| < \frac{\epsilon}{|\alpha_n|}\right\} \end{aligned}$$

and

$$r_{S_n,\epsilon}(A_n) = \sup\left\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \left|\lambda - \frac{\beta_n}{\alpha_n}\right| < \frac{\epsilon}{|\alpha_n|}\right\}.$$

Hence, by Theorem 3.2 for $\epsilon > 0$ we have

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{\beta_n}{\alpha_n}\right| < \frac{\epsilon}{|\alpha_n|}\right\}$$

and

$$r_{S,\epsilon}(A) = \sup_{n\geq 1} \sup\left\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \left|\lambda - \frac{\beta_n}{\alpha_n}\right| < \frac{\epsilon}{|\alpha_n|}\right\}.$$

Example 4.3. Let $\mathfrak{B}_n = \mathbb{C}^2,\ n \geq 1$ two-dimensional Euclidian space, $\mathfrak{B} = \left(\bigoplus_{n=1}^{\infty} \mathbb{C}^2\right)_p$, $1 \leq p < \infty$ and

$$S_n : \mathbb{C}^2 \to \mathbb{C}^2,\ S_n = \begin{pmatrix} \alpha_n & -1 \\ \alpha_n & 0 \end{pmatrix},\ \alpha_n \in \mathbb{C},\ \alpha_n \neq 0,\ n \geq 1,\ \sup_{n\geq 1}|\alpha_n| < \infty,$$

$$A_n : \mathbb{C}^2 \to \mathbb{C}^2,\ A_n = \begin{pmatrix} \beta_n & 0 \\ \beta_n & 0 \end{pmatrix},\ \beta_n \in \mathbb{C},\ \beta_n \neq 0,\ n \geq 1,\ \sup_{n\geq 1}|\beta_n| < \infty.$$

Then, $S = \bigoplus_{n=1}^{\infty} S_n$ and $A = \bigoplus_{n=1}^{\infty} A_n$ are the infinite diagonal block operator matrices. For $\lambda \in \mathbb{C}$ and $n \geq 1$ we have

$$\lambda S_n - A_n = \begin{pmatrix} \lambda\alpha_n - \beta_n & -\lambda \\ \lambda\alpha_n - \beta_n & 0 \end{pmatrix}.$$

Thus, for $\lambda \neq \dfrac{\beta_n}{\alpha_n}$, $\lambda \neq 0$ and $n \geq 1$ we get

$$(\lambda S_n - A_n)^{-1} = \begin{pmatrix} 0 & \dfrac{1}{\lambda\alpha_n - \beta_n} \\ -\dfrac{1}{\lambda} & \dfrac{1}{\lambda} \end{pmatrix} \in \mathfrak{L}\left(\mathbb{C}^2\right)$$

and

$$\| (\lambda S_n - A_n)^{-1} \| = \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2}.$$

Consequently, for any $n \geq 1$ we obtain

$$\sigma_{S_n}(A_n) = \left\{ 0, \frac{\beta_n}{\alpha_n} \right\} \text{ and } r_{S_n}(A_n) = \left| \frac{\beta_n}{\alpha_n} \right|.$$

Hence, by Theorem 3.1 we have

$$\sigma_S(A) = \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \frac{\beta_n}{\alpha_n} \right\}} \text{ and } r_S(A) = \sup_{n\geq 1} \left| \frac{\beta_n}{\alpha_n} \right|.$$

On the other hand, for any $n \geq 1$ and $\epsilon > 0$

$$\begin{aligned}
&\sigma_{S_n,\epsilon}(A_n) \\
&= \sigma_{S_n}(A_n) \cup \left\{ \lambda \in \rho_{S_n}(A_n) : \|R_{S_n}(\lambda, A_n)\| > \frac{1}{\epsilon} \right\} \\
&= \left\{ 0, \frac{\beta_n}{\alpha_n} \right\} \cup \left\{ \lambda \in \rho_{S_n}(A_n) : \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2} > \frac{1}{\epsilon} \right\} \\
&= \left\{ \lambda \in \mathbb{C} : \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2} > \frac{1}{\epsilon} \right\}
\end{aligned}$$

and

$$\begin{aligned}
&r_{S_n,\epsilon}(A_n) \\
&= \sup \left\{ |\lambda| : \lambda \in \mathbb{C} \text{ and } \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2} > \frac{1}{\epsilon} \right\}.
\end{aligned}$$

Hence, by Theorem 3.2 for $\epsilon > 0$ we have

$$\sigma_{S,\epsilon}(A) = \bigcup_{n=1}^{\infty} \left\{ \lambda \in \mathbb{C} : \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2} > \frac{1}{\epsilon} \right\}$$

and

$$\begin{aligned}
&r_{S,\epsilon}(A) \\
&= \sup_{n\geq 1} \sup \left\{ |\lambda| : \lambda \in \mathbb{C} \text{ and } \left(\frac{1}{|\lambda|^2} + \frac{1}{2|\lambda\alpha_n - \beta_n|^2} + \frac{1}{2} \left(\frac{4}{|\lambda|^4} + \frac{1}{|\lambda\alpha_n - \beta_n|^4} \right)^{1/2} \right)^{1/2} > \frac{1}{\epsilon} \right\}.
\end{aligned}$$

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